WAVE ENERGY DENSITY

GOAL: DERIVE AN EQUATION THAT DESCRIBES THE "DYNAMICS" OF WAVES. WE WILL DERIVE THE GENERALIZATION OF POYNTING'S THEOREM TO DISPERSIVE MEDIA. (WE ARE, IN A MATTER OF SPEAKING, PERFORMING A WKB ANALYSIS OF THE MAXWELL-PLASMA SYSTEM.)

WE BEGIN BY FOURIER-LAPLACE TRANSFORMING THE LINEARIZED EQUATIONS FOR THE WAVE. THESE ARE DEFINED AS

$$f(\bar{r},t) = \int_L \frac{d\omega}{2\pi} \iiint_F \frac{d^3k}{(2\pi)^3}\, \underset{\sim}{f}(\bar{k},\omega)\, e^{-j(\omega t - \bar{k}\cdot\bar{r})}$$

$$\underset{\sim}{f}(\bar{k},\omega) = \int dt \iiint d^3r\, f(\bar{r},t)\, e^{+j(\omega t - \bar{k}\cdot\bar{r})}$$

WE HAVE ALREADY SHOWN THAT THE RESULTS BECOME

$$-j\omega\, \bar{\bar{D}}(\omega,\bar{k})\cdot\bar{E}(\omega,\bar{k}) = -4\pi\, \bar{J}_{EXT}(\bar{k},\omega) \qquad (1)$$

WHERE $\bar{J} = \bar{J}_{INT} + \bar{J}_{EXT}$

$\bar{J}_{EXT}$ ← ANTENNA, DRIVE, ETC

$\bar{J}_{INT}$ ← PLASMA CURRENTS

$$\bar{J}_{INT} = \bar{\bar{\sigma}}\cdot\bar{E}$$

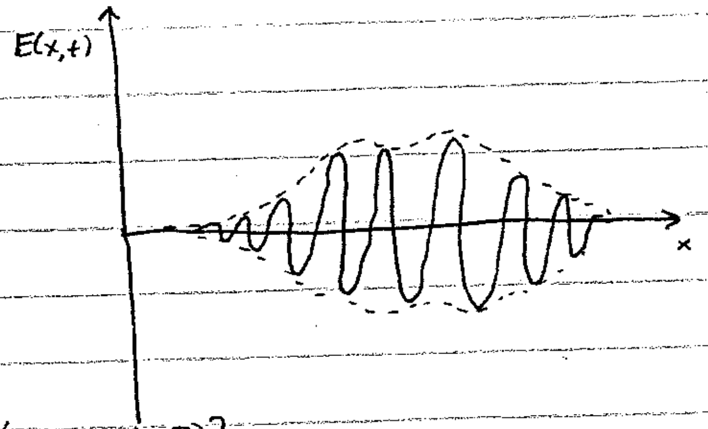
$$\bar{\bar{D}}(\omega,\bar{k}) = \left(1 - \frac{k^2c^2}{\omega^2}\right)\bar{\bar{I}} + \frac{\bar{k}c\,\bar{k}c}{\omega^2} + \frac{4\pi i\,\bar{\bar{\sigma}}}{\omega}$$

IF WE INVERT THIS EQUATION, WE FIND

$$\bar{E}(\omega,\bar{k}) = -\frac{4\pi j\ \bar{\bar{D}}^{ADJ}\cdot\bar{J}_{EXT}}{\omega\, \mathrm{DET}|\bar{\bar{D}}|}$$

THAT THE ZEROS OF $\omega\,\mathrm{DET}|\bar{\bar{D}}|$ ARE THE SINGULARITIES OF $\bar{E}(\omega,\bar{k})$. THEY REPRESENT THE <u>NORMAL</u> <u>MODES</u>.

LET'S USE THE CONCEPT OF <u>NORMAL MODES</u> TO DESCRIBE HOW A WAVE-PACKET OF A NEARLY "PURE" MODE "EVOLVES" OR "MOVES" IN SPACE AND TIME.



WE WILL SAY THAT THE PACKET IS <u>BIG</u>. THUS $\omega \approx$ REAL FOR A REAL $\bar{k}$. WE CAN WRITE

$$\bar{E}(\bar{r},t) = \mathrm{Re}\left\{\bar{\mathcal{E}}_0\, u(\bar{r},t)\, e^{-j(\omega_k t - \bar{k}\cdot\bar{r})}\right\}$$

WHERE $\bar{\mathcal{E}}_0$, u ARE COMPLEX. THE FUNCTION $u(\bar{r},t)$ REPRESENTS THE WAVE PACKET. WE ASSUME THAT $u(\bar{r},t)$ CHANGES VERY VERY SLOWLY COMPARED WITH $1/\omega$ AND $1/|\bar{k}|$.

THEN WE CAN USE EQ. (1) AND OUR ASSUMPTIONS ABOUT THE WAVE TO DERIVE OUR GENERALIZED POYNTING'S THEOREM.

REMEMBER HOW CONVOLUTIONS WORK? EXAMPLE: LET $f(k)$ AND $g(k)$ BE TRANSFORMS OF $f(x)$ AND $g(x)$. THEN

$$f(k)g(k) = \int dx \left[\int dx' f(x') g(x-x') \right] e^{-jkx}$$

$$= \int dx \left[\int dx' g(x-x') \int \frac{dk'}{2\pi} f(k') e^{jk'x'} \right] e^{-jkx}$$

$$= \int dx \left[\int \frac{dk'}{2\pi} f(k') \int dx' g(x-x') e^{-jk'(x-x')} \right] e^{-j(k-k')x}$$

$$= \int \frac{dk'}{2\pi} \int dx \, f(k') g(k') e^{-j(k-k')x}$$

$$= \int \frac{dk'}{2\pi} f(k') g(k') \; 2\pi \delta(k-k') \qquad \text{Q.E.D.}$$

THUS, WE CAN WRITE EQ. 1 AS

$$\int dt' \iiint d^3r' \, \bar{\bar{\mathcal{D}}}(r', t') \cdot \bar{E}(r-r', t-t') = -4\pi \bar{J}_{EXT}(\bar{r}, t)$$

WHERE $\bar{\bar{\mathcal{D}}}(\bar{r}, t')$ IS THE INVERSE TRANSFORM OF $-j\omega \bar{\bar{D}}(\bar{k}, \omega)$.

THE WAY WE WILL INCORPORATE OUR ASSUMPTION ABOUT HOW SLOWLY THE WAVE PACKET VARIES IS BY EXPANDING $\mathcal{U}(r-r', t-t')$ ABOUT THE LOCATION $(\bar{r}, t)$. IN THIS WAY, WE GET A <u>LOCAL</u> EQUATION FOR HOW THE WAVE PACKET EVOLVES — EVEN THOUGH WAVES ARE <u>NON-LOCAL</u> THINGS.

WE'LL EXPAND

$$\bar{E}(\bar{r}-\bar{r}', t-t') = \bar{\mathcal{E}}_0\left[u(\bar{r},t) - t'\frac{\partial u}{\partial t} - \bar{r}'\cdot\frac{\partial u}{\partial \bar{r}} + \cdots\right] e^{-j(\omega_R[t-t'] - \bar{k}\cdot[\bar{r}-\bar{r}'])}$$

USE THE IDENTITIES ...

$$-j\,\omega_R\,\bar{\bar{D}}(\omega_R,\bar{k}) = \int dt' \int d^3r'\, \bar{\bar{\mathcal{D}}}(\bar{r}',t')\, e^{+j(\omega_R t' - \bar{k}\cdot\bar{r}')}$$

$$\frac{\partial}{\partial \omega}\left(\omega_R \bar{\bar{D}}\right) = \int dt' \int d^3r' (-t')\, \bar{\bar{\mathcal{D}}}\, e^{+j(\omega_R t' - \bar{k}\cdot\bar{r}')}$$

$$-\frac{\partial}{\partial \bar{k}}\left(\omega \bar{\bar{D}}\right) = \int dt' \int d^3r' (-\bar{r}')\, \bar{\bar{\mathcal{D}}}\, e^{+j(\omega_R t' - \bar{k}\cdot\bar{r}')}$$

TO OBTAIN

$$e^{-j(\omega_R t - \bar{k}\cdot\bar{r})}\left[\left(-j\omega_R \bar{\bar{D}}\cdot\bar{\mathcal{E}}_0 u\right) + \frac{\partial}{\partial\omega}(\omega\bar{\bar{D}})\cdot\bar{\mathcal{E}}_0\frac{\partial u}{\partial t} - \frac{\partial}{\partial \bar{k}}(\omega\bar{\bar{D}})\cdot\bar{\mathcal{E}}_0\frac{\partial u}{\partial \bar{r}}\right]$$

$$= -4\pi \bar{J}_{EXT}(\bar{r},t) \qquad (2)$$

NOW, REMEMBER THAT

$$\bar{\bar{D}} = \bar{\bar{D}}_{REAL} + i\,\bar{\bar{D}}_{IMAG}$$ (ACTUALLY HERMETIAN PLUS ANTI HERMETIAN)

WHERE $\bar{\bar{D}}_{REAL}(\omega,k)$ JUST REPRESENTS THE REACTIVE ENERGY FLOW AND $\bar{\bar{D}}_{IMAG}$ REPRESENTS RESISTANCE.
WE NOTE THAT FOR A REAL NORMAL MODE THE RESISTIVE PART MUST BE SMALL. THIS IS SO THE WAVE DOES NOT DAMP TOO FAST.

THUS, WE CAN SEPARATE EQ (2) INTO BOTH REAL AND IMAGINARY PARTS.

FIRST, MULTIPLY BY $\frac{1}{8\pi}\bar{\xi}_0^{*} u^{*} e^{i(\omega_R t - k\cdot r)}$

THEN THE IMAGINARY PART IS

$$-\frac{1}{8\pi}\omega_R \bar{\xi}_0^{*}\cdot \bar{\bar{D}}_R \cdot \bar{\xi}_0 |u|^2 \approx 0 \tag{3}$$

THE REAL PART IS

$$\omega_R \bar{\xi}_0^{*}\cdot\bar{\bar{D}}_I\cdot\bar{\xi}_0 \frac{1}{8\pi}|u|^2 + \frac{\partial}{\partial\omega}\left(\omega_R \bar{\xi}_0^{*}\cdot \bar{\bar{D}}_R\cdot\bar{\xi}_0\right)\frac{\partial}{\partial t}\left(\frac{1}{2}|u|^2\right)\frac{1}{8\pi}$$

$$-\frac{\partial}{\partial k}\left(\omega_R \bar{\xi}_0^{*}\cdot\bar{\bar{D}}_R\cdot\bar{\xi}_0\right)\cdot\frac{\partial}{\partial r}\left(\frac{1}{2}|u|^2\right)\frac{1}{8\pi}$$

$$= -\frac{1}{2}Re\left\{E^{*}\cdot J_{EXT}\right\} \tag{4}$$

EQ. 4 IS OUR DISPERSION RELATION FOR OUR MODE... IT SIMPLY RELATES (ω_R, k)

EQ. 5 IS THE EQUATION FOR WAVE ENERGY DENSITY.

IF WE DEFINE

$$\langle W_k\rangle \equiv \frac{1}{2}Re\left\{\frac{|u|^2}{8\pi}\frac{\partial}{\partial\omega}\left(\omega_R\bar{\xi}_0^{*}\cdot\bar{\bar{D}}_R\cdot\bar{\xi}_0\right)\right\}$$

OR $\langle w_k \rangle$ = TIME AVERAGED ENERGY DENSITY

$$= \frac{1}{2}\frac{|u|^2}{8\pi}\frac{\partial}{\partial\omega}\left(\omega_R \bar{\xi}_0^* \cdot \bar{\bar{D}}_R \cdot \bar{\xi}_0\right)$$ (SINCE THIS IS REAL)

THEN WE CAN WRITE..

$$\frac{\partial}{\partial t}\langle w_k \rangle + \frac{\partial}{\partial \bar{r}}\cdot\left(\bar{V}_g \langle w_k \rangle\right) - 2\omega_I \langle w_k \rangle$$

$$= -\frac{1}{2}\,\mathrm{Re}\left\{E^* \cdot J_{EXT}\right\}$$

WHERE $$\bar{V}_g \equiv -\frac{\frac{\partial}{\partial k}\left(\omega_R \bar{\xi}_0^* \cdot \bar{\bar{D}}_R \cdot \bar{\xi}_0\right)}{\frac{\partial}{\partial\omega}\left(\omega_R \bar{\xi}_0^* \cdot \bar{\bar{D}}_R \cdot \bar{\xi}_0\right)}$$ = GROUP VELOCITY

$$\omega_I = -\frac{\omega_R \bar{\xi}_0^* \cdot \bar{\bar{D}}_I \cdot \bar{\xi}_0}{\frac{\partial}{\partial\omega}\left(\omega_R \bar{\xi}_0^* \cdot \bar{\bar{D}}_R \cdot \bar{\xi}_0\right)} = -\gamma$$

γ = DAMPING RATE

$\frac{1}{2}\mathrm{Re}\left\{E^* \cdot J_{EXT}\right\}$ = POWER DISSIPATED IN EXTERNAL CIRCUIT DUE TO WAVE.

NOTE: WITHOUT A PLASMA (i.e. $\omega_p^2 \to 0$) WE SHOULD GET BACK POYNTING'S THEOREM...

$$\bar{\bar{D}} = \begin{pmatrix} 1-\frac{k^2c^2}{\omega^2} & 0 & 0 \\ 0 & 1-\frac{k^2c^2}{\omega^2} & 0 \\ 0 & 0 & 0 \end{pmatrix} ; \quad \bar{\xi}_0 \perp \bar{k}$$

IF YOU SUBSTITUTE... YOU'LL FIND IT'S TRUE!