Plasma 2
Lecture 8: Whistler Waves and Cyclotron Heating

APPH E6102y
Columbia University

Outline

- Review: electromagnetic waves in a **cold** magnetized plasma (Ch. 4)
- Kinetic electromagnetic waves in a **warm** magnetic plasma plasma (Ch. 10.3)
- Kinetic whistler waves and cyclotron damping (Ch. 10.3.2)
- Measurement of electron cyclotron damping
  (1984; https://doi.org/10.1063/1.864605)
- Measurement of Whistler Instabilities
  (1987; https://doi.org/10.1103/PhysRevLett.59.1821)

Ch. 4: Review of Cold Plasma Waves

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \epsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \cdot \mathbf{E} = \frac{\rho_q}{\epsilon_0}$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\tilde{\mathbf{J}} = \sum_s n_s e_s \tilde{\mathbf{v}}_s,$$

$$\tilde{\mathbf{J}} = \overleftrightarrow{\sigma} \cdot \tilde{\mathbf{E}}.$$

Ampere's Law (σ ~ perturbed plasma current)

$$\mathrm{i}\mathbf{k} \times \tilde{\mathbf{B}} = \epsilon_0 \mu_0 (-\mathrm{i}\omega) \overleftrightarrow{\mathbf{K}} \cdot \tilde{\mathbf{E}}$$

$$\overleftrightarrow{\mathbf{K}} = \overleftrightarrow{\mathbf{1}} - \frac{\overleftrightarrow{\boldsymbol{\sigma}}}{\mathrm{i}\omega\epsilon_0}$$

Dispersion Tensor

$$\mathbf{k} \times (\mathbf{k} \times \tilde{\mathbf{E}}) + \frac{\omega^2}{c^2} \overleftrightarrow{\mathbf{K}} \cdot \tilde{\mathbf{E}} = 0$$

$$\overleftrightarrow{\mathbf{Đ}} \cdot \tilde{\mathbf{E}} = 0$$

Magnetized Plasma

$$\begin{bmatrix} \tilde{J}_x \\ \tilde{J}_y \\ \tilde{J}_z \end{bmatrix} = \sum_s \frac{n_{s0}e_s^2}{m_s} \begin{bmatrix} \dfrac{-\mathrm{i}\omega}{\omega_{\mathrm{c}s}^2 - \omega^2} & \dfrac{\omega_{\mathrm{c}s}}{\omega_{\mathrm{c}s}^2 - \omega^2} & 0 \\ \dfrac{-\omega_{\mathrm{c}s}}{\omega_{\mathrm{c}s}^2 - \omega^2} & \dfrac{-\mathrm{i}\omega}{\omega_{\mathrm{c}s}^2 - \omega^2} & 0 \\ 0 & 0 & \dfrac{\mathrm{i}}{\omega} \end{bmatrix} \begin{bmatrix} \tilde{E}_x \\ \tilde{E}_y \\ \tilde{E}_z \end{bmatrix}. \tag{4.4.5}$$

Magnetized Plasma

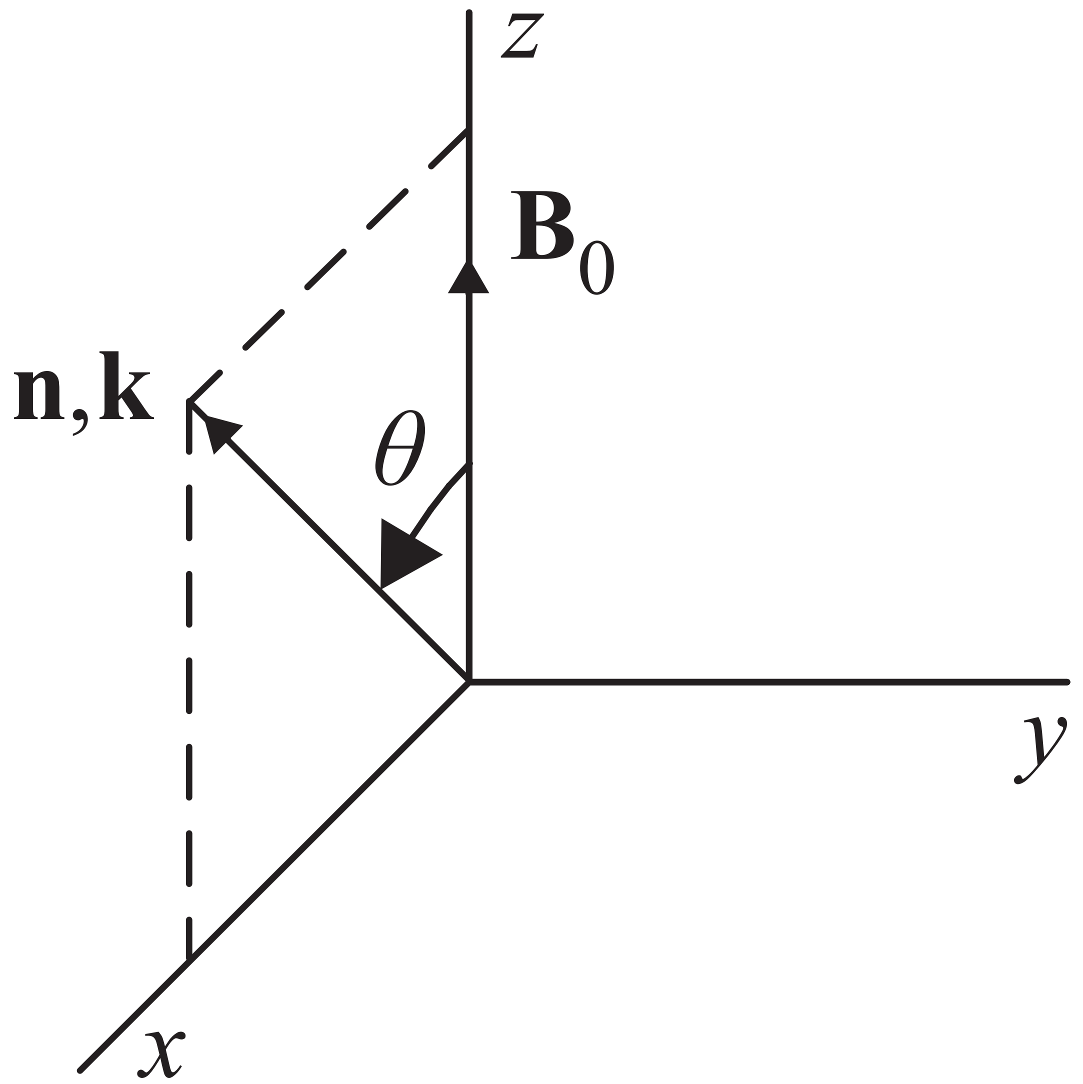
$$\overleftrightarrow{\sigma} = \sum_s \frac{n_{s0}e_s^2}{m_s}\begin{bmatrix} \dfrac{-\mathrm{i}\omega}{\omega_{cs}^2-\omega^2} & \dfrac{\omega_{cs}}{\omega_{cs}^2-\omega^2} & 0 \\ \dfrac{-\omega_{cs}}{\omega_{cs}^2-\omega^2} & \dfrac{-\mathrm{i}\omega}{\omega_{cs}^2-\omega^2} & 0 \\ 0 & 0 & \dfrac{\mathrm{i}}{\omega} \end{bmatrix} \qquad \overleftrightarrow{\mathbf{K}} = \begin{bmatrix} S & -\mathrm{i}D & 0 \\ -\mathrm{i}D & S & 0 \\ 0 & 0 & P \end{bmatrix}, \tag{4.4.7}$$

$$S = 1 - \sum_s \frac{\omega_{ps}^2}{\omega^2-\omega_{cs}^2}, \qquad D = \sum_s \frac{\omega_{cs}\,\omega_{ps}^2}{\omega(\omega^2-\omega_{cs}^2)}, \qquad P = 1 - \sum_s \frac{\omega_{ps}^2}{\omega^2}.$$

Propagation Parallel to B ($\theta = 0$; $k = k_{\|}$)

$$\begin{bmatrix} S-n^2 & -\mathrm{i}D & 0 \\ \mathrm{i}D & S-n^2 & 0 \\ 0 & 0 & P \end{bmatrix} \begin{bmatrix} \widetilde{E}_x \\ \widetilde{E}_y \\ \widetilde{E}_z \end{bmatrix} = 0. \quad (4.4.20)$$

$$n \equiv |k|c/\omega$$



Whistler Mode

$$n^2 = R = 1 - \sum_s \frac{\omega_{ps}^2}{\omega(\omega + \omega_{cs})}$$

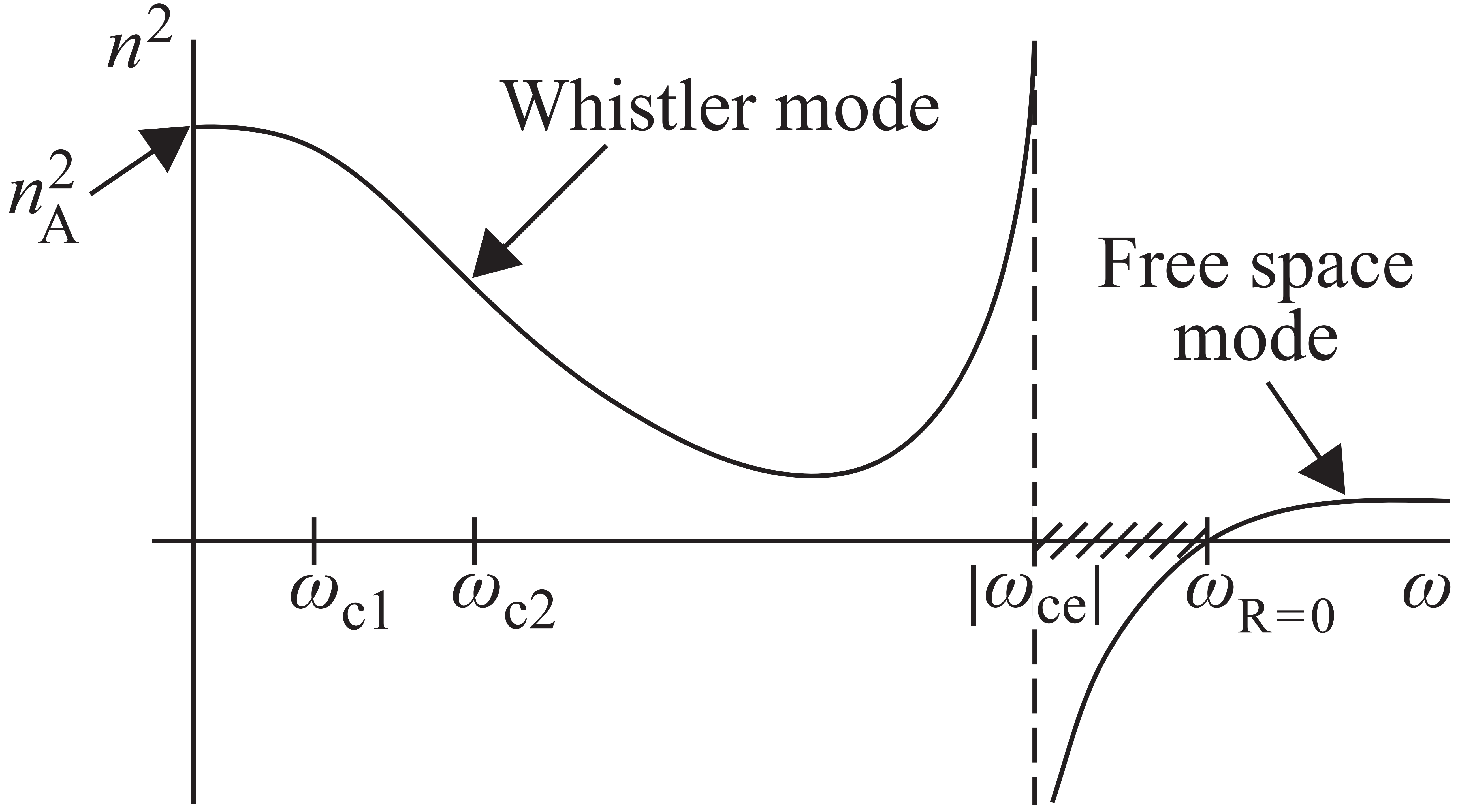
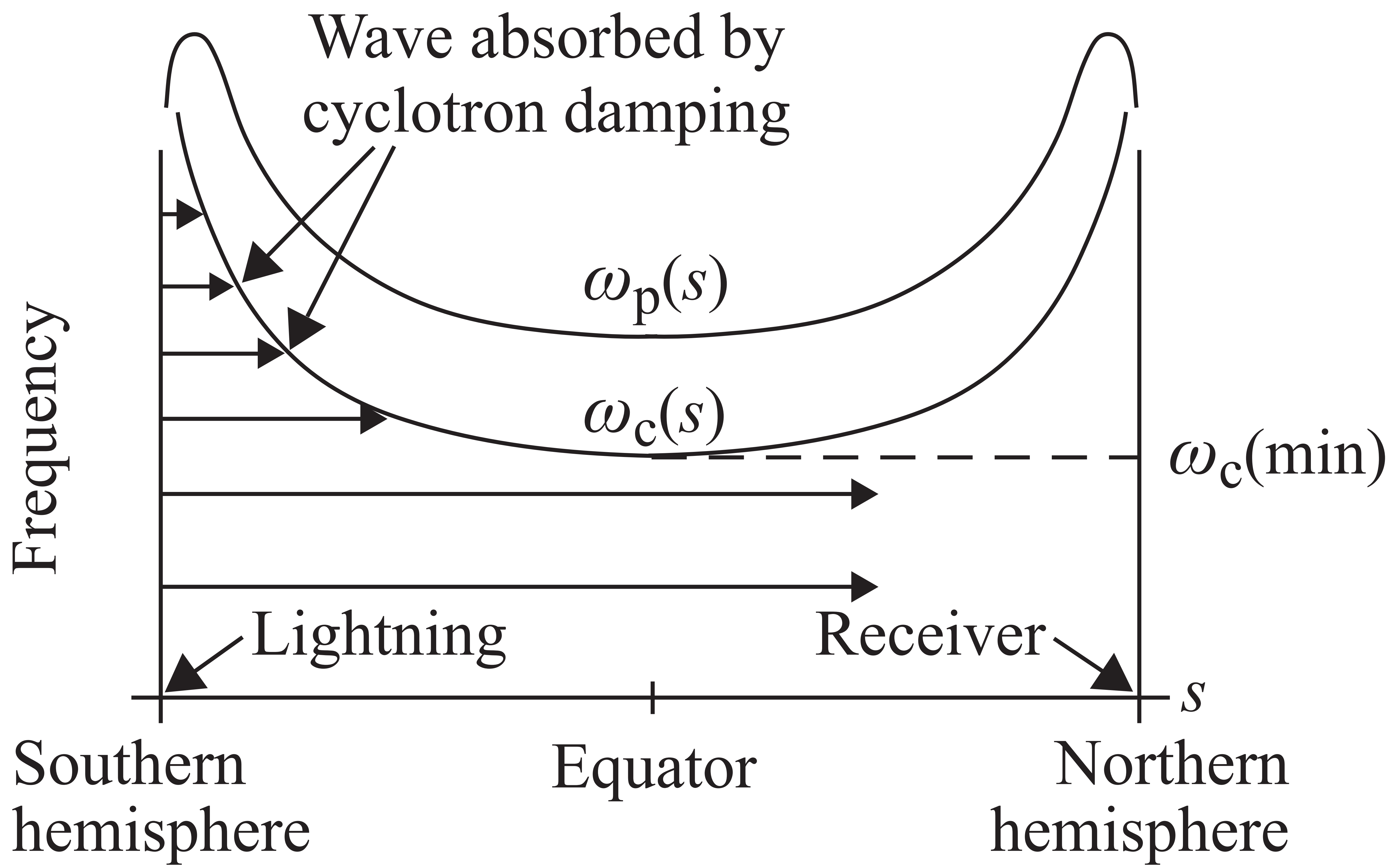


Figure 4.15 A plot of $n^2 = R$ as a function of frequency.

Whistler Mode

$$n^2 = R = 1 - \sum_s \frac{\omega_{ps}^2}{\omega(\omega + \omega_{cs})}$$



$$t(\omega) = \frac{1}{2c\,\omega^{1/2}} \int \frac{\omega_p \omega_c}{(\omega_c - \omega)^{3/2}} \mathrm{d}s, \qquad (4.4.33)$$

(WKB) Ray Paths

For an inhomogeneous time-dependent medium, Maxwell's equations can be written as a set of linear homogeneous partial differential equations of the form

$$\overleftrightarrow{\mathbf{Đ}}(\boldsymbol{\nabla}, \partial/\partial t, \mathbf{r}, t) \cdot \mathbf{f} = 0, \tag{4.5.1}$$

$$\overleftrightarrow{\mathbf{Đ}}(\mathrm{i}\mathbf{k}, -\mathrm{i}\,\omega, \mathbf{r}, t) \cdot \mathbf{f} = 0,$$

$$\delta Đ = \boldsymbol{\nabla}_{\mathbf{k}} Đ \cdot \frac{\mathrm{d}\mathbf{k}}{\mathrm{d}\tau}\mathrm{d}\tau + \frac{\partial Đ}{\partial \omega}\frac{\mathrm{d}\omega}{\mathrm{d}\tau}\mathrm{d}\tau + \boldsymbol{\nabla} Đ \cdot \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}\tau}\mathrm{d}\tau + \frac{\partial Đ}{\partial t}\frac{\mathrm{d}t}{\mathrm{d}\tau}\mathrm{d}\tau = 0. \tag{4.5.4}$$

WAVE ENERGY DENSITY

GOAL: DERIVE AN EQUATION THAT DESCRIBES THE "DYNAMICS" OF WAVES. WE WILL DERIVE THE GENERALIZATION OF POYNTING'S THEOREM TO DISPERSIVE MEDIA. (WE ARE, IN A MATTER OF SPEAKING, PERFORMING A WKB ANALYSIS OF THE MAXWELL-PLASMA SYSTEM.)

$$-i\omega\,\bar{\bar{D}}(\omega,\bar{k})\cdot\bar{E}(\omega,\bar{k}) = -4\pi\,\bar{J}_{EXT}(\bar{k},\omega)$$

$$\bar{\bar{D}}(\omega,\bar{k}) = \left(1 - \frac{k^2c^2}{\omega^2}\right)\bar{\bar{I}} + \frac{\bar{k}c\,\bar{k}c}{\omega^2} + \frac{4\pi i\,\bar{\bar{\sigma}}}{\omega}$$

OR $\langle w_k \rangle$ = TIME AVERAGED ENERGY DENSITY

$$= \frac{1}{2} \frac{|u|^2}{8\pi} \frac{\partial}{\partial \omega}\left(\omega_R \bar{\xi}_0^* \cdot \bar{\bar{D}}_R \cdot \bar{\xi}_0 \right)$$ (SINCE THIS IS REAL)

THEN WE CAN WRITE..

$$\frac{\partial}{\partial t}\langle w_k \rangle + \frac{\partial}{\partial \bar{r}} \cdot \left(\bar{V}_g \langle w_k \rangle \right) - 2\omega_I \langle w_k \rangle$$

$$= -\frac{1}{2} \mathrm{Re} \left\{ E^* \cdot J_{EXT} \right\}$$

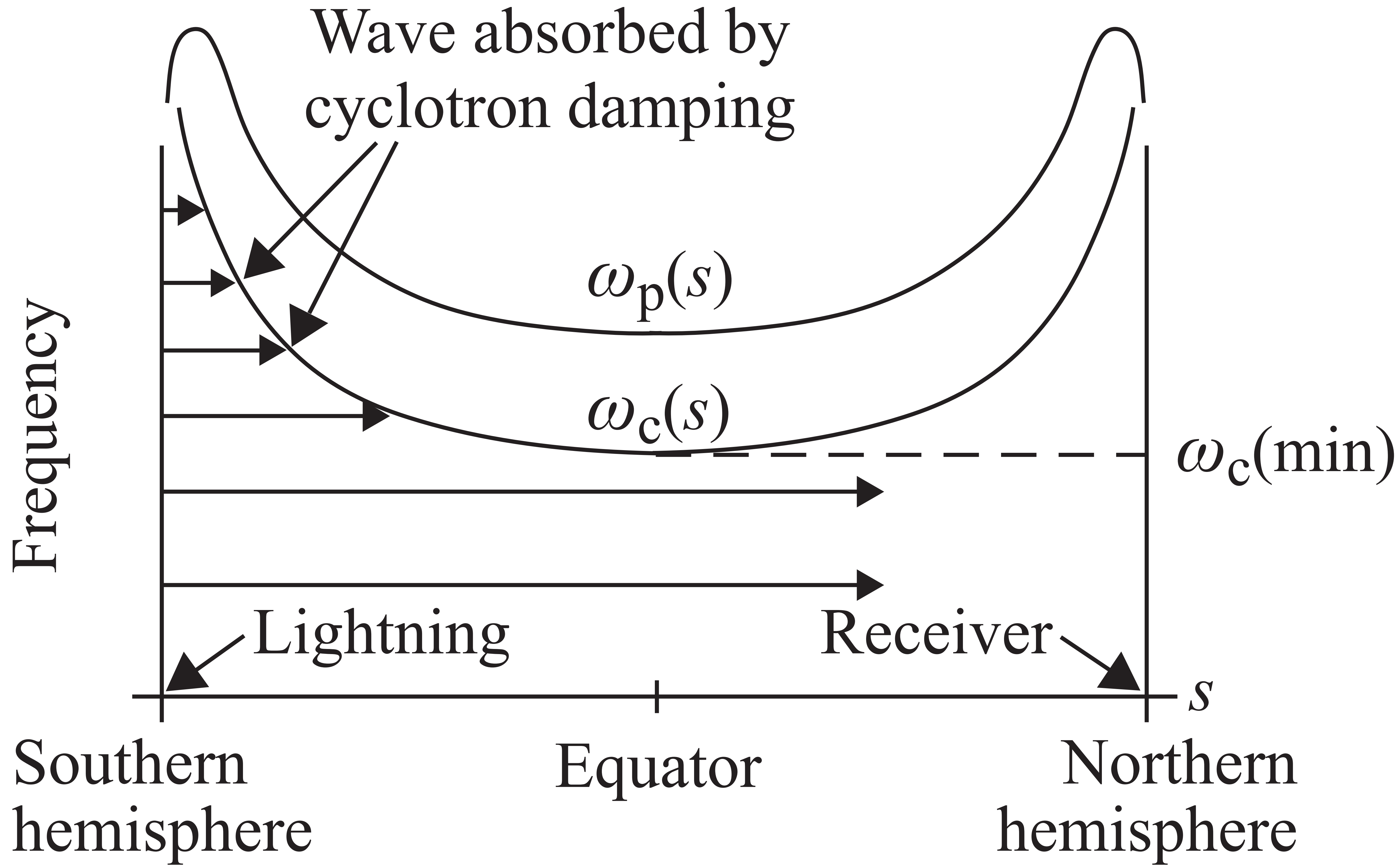
WHERE $$\bar{V}_g \equiv - \frac{\frac{\partial}{\partial k}\left(\omega_R \bar{\xi}_0^* \cdot \bar{\bar{D}}_R \cdot \bar{\xi}_0 \right)}{\frac{\partial}{\partial \omega}\left(\omega_R \bar{\xi}_0^* \cdot \bar{\bar{D}}_R \cdot \bar{\xi}_0 \right)}$$ = GROUP VELOCITY

$$\omega_I = - \frac{\omega_R \bar{\xi}_0^* \cdot \bar{\bar{D}}_I \cdot \bar{\xi}_0}{\frac{\partial}{\partial \omega}\left(\omega_R \bar{\xi}_0^* \cdot \bar{\bar{D}}_R \cdot \bar{\xi}_0 \right)} = -\gamma$$

γ = DAMPING RATE

Whistler Mode

$$n^2 = R = 1 - \sum_s \frac{\omega_{ps}^2}{\omega(\omega + \omega_{cs})}$$



$$t(\omega) = \frac{1}{2c\,\omega^{1/2}} \int \frac{\omega_p \omega_c}{(\omega_c - \omega)^{3/2}} \mathrm{d}s, \qquad (4.4.33)$$

10.3 Electromagnetic Waves

$$\mathbf{k} \times (\mathbf{k} \times \tilde{\mathbf{E}}) + \frac{\omega^2}{c^2} \overleftrightarrow{\mathbf{K}} \cdot \tilde{\mathbf{E}} = 0, \qquad (10.3.1)$$

$$\tilde{\mathbf{J}} = \sum_s n_s e_s \tilde{\mathbf{v}}_s,$$

$$\tilde{\mathbf{J}} = \sum_s e_s \int_0^{2\pi} \int_{-\infty}^{\infty} \int_0^{\infty} \mathbf{v} \tilde{f}_s \upsilon_\perp \mathrm{d}\upsilon_\perp \, \mathrm{d}\upsilon_\parallel \, \mathrm{d}\phi, \qquad (10.3.15)$$

$$\tilde{\mathbf{J}} = \overleftrightarrow{\boldsymbol{\sigma}} \cdot \tilde{\mathbf{E}}$$

$$\begin{bmatrix} \widetilde{J}_x \\ \widetilde{J}_y \\ \widetilde{J}_z \end{bmatrix} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} \begin{bmatrix} \widetilde{E}_x \\ \widetilde{E}_y \\ \widetilde{E}_z \end{bmatrix}, \qquad (10.3.16)$$

Parallel Propagation

$$\begin{bmatrix} S-n^2 & -\mathrm{i}D & 0 \\ \mathrm{i}D & S-n^2 & 0 \\ 0 & 0 & P \end{bmatrix} \begin{bmatrix} \widetilde{E}_x \\ \widetilde{E}_y \\ \widetilde{E}_z \end{bmatrix} = 0. \qquad \begin{bmatrix} K_{xx} - \frac{c^2k^2}{\omega^2} & K_{xy} & 0 \\ K_{yx} & K_{yy} - \frac{c^2k^2}{\omega^2} & 0 \\ 0 & 0 & K_{zz} \end{bmatrix} \begin{bmatrix} \widetilde{E}_x \\ \widetilde{E}_y \\ \widetilde{E}_z \end{bmatrix} = 0.$$

for example:

$$K_{xx} = 1 - \sum_s \frac{\omega_{\mathrm{p}s}^2}{\omega} \sum_n \int_{-\infty}^{\infty} \int_0^{\infty} \frac{n^2 J_n^2(\beta_s)}{\beta_s^2 \boxed{(k_{\|}v_{\|} - \omega + n\,\omega_{\mathrm{c}s})}}$$

$$\times \left[\left(1 - \frac{k_{\|}v_{\|}}{\omega}\right) \frac{\partial F_{s0}}{\partial v_{\perp}} + \frac{k_{\|}v_{\perp}}{\omega} \frac{\partial F_{s0}}{\partial v_{\|}} \right] 2\pi v_{\perp}^2 \, \mathrm{d}v_{\perp} \, \mathrm{d}v_{\|},$$

resonance

Cyclotron Resonance

$$Đ(k,\omega) = K_{xx} - \frac{c^2 k_\parallel^2}{\omega^2} \pm \mathrm{i} K_{xy} = 0,$$

$$Đ(k,\omega) = 1 - \frac{c^2 k_\parallel^2}{\omega^2} - \sum_s \frac{\omega_{\mathrm{p}s}^2}{\omega} \times \int_{-\infty}^{\infty} \int_0^{\infty} \frac{\frac{\partial F_{s0}}{\partial \upsilon_\perp} + \frac{k_\parallel}{\omega}\left(\upsilon_\perp \frac{\partial F_{s0}}{\partial \upsilon_\parallel} - \upsilon_\parallel \frac{\partial F_{s0}}{\partial \upsilon_\perp}\right)}{k_\parallel \upsilon_\parallel - \omega \pm \omega_{\mathrm{c}s}} \pi \upsilon_\perp^2 \mathrm{d}\upsilon_\perp \mathrm{d}\upsilon_\parallel = 0. \quad (10.3.32)$$

Cyclotron Resonance

$$Đ(k,\omega) = 1 - \frac{c^2 k_{\|}^2}{\omega^2} - \sum_s \frac{\omega_{ps}^2}{\omega^2} \int_{-\infty}^{\infty} \frac{G_{s0}(\upsilon_{\|})}{\upsilon_{\|} - \dfrac{\omega^2 \pm \omega_{cs}}{k_{\|}}} d\upsilon_{\|} = 0, \quad (10.3.33)$$

$$G_{s0}(\upsilon_{\|}) = \int_0^{\infty} \left[\left(\frac{\omega}{k_{\|}} - \upsilon_{\|} \right) \frac{\partial F_{s0}}{\partial \upsilon_{\perp}} + \upsilon_{\perp} \frac{\partial F_{s0}}{\partial \upsilon_{\|}} \right] \pi \upsilon_{\perp}^2 d\upsilon_{\perp}. \quad (10.3.34)$$

Cyclotron Damping

$$Đ_\mathrm{r} = 1 - \frac{c^2 k_\parallel^2}{\omega^2} - \sum_s \frac{\omega_{\mathrm{p}s}^2}{\omega^2} P \int_{-\infty}^{\infty} \frac{G_{s0}(v_\parallel)}{v_\parallel - \dfrac{\omega \pm \omega_{\mathrm{c}s}}{k_\parallel}} \mathrm{d}v_\parallel = 0 \qquad (10.3.38)$$

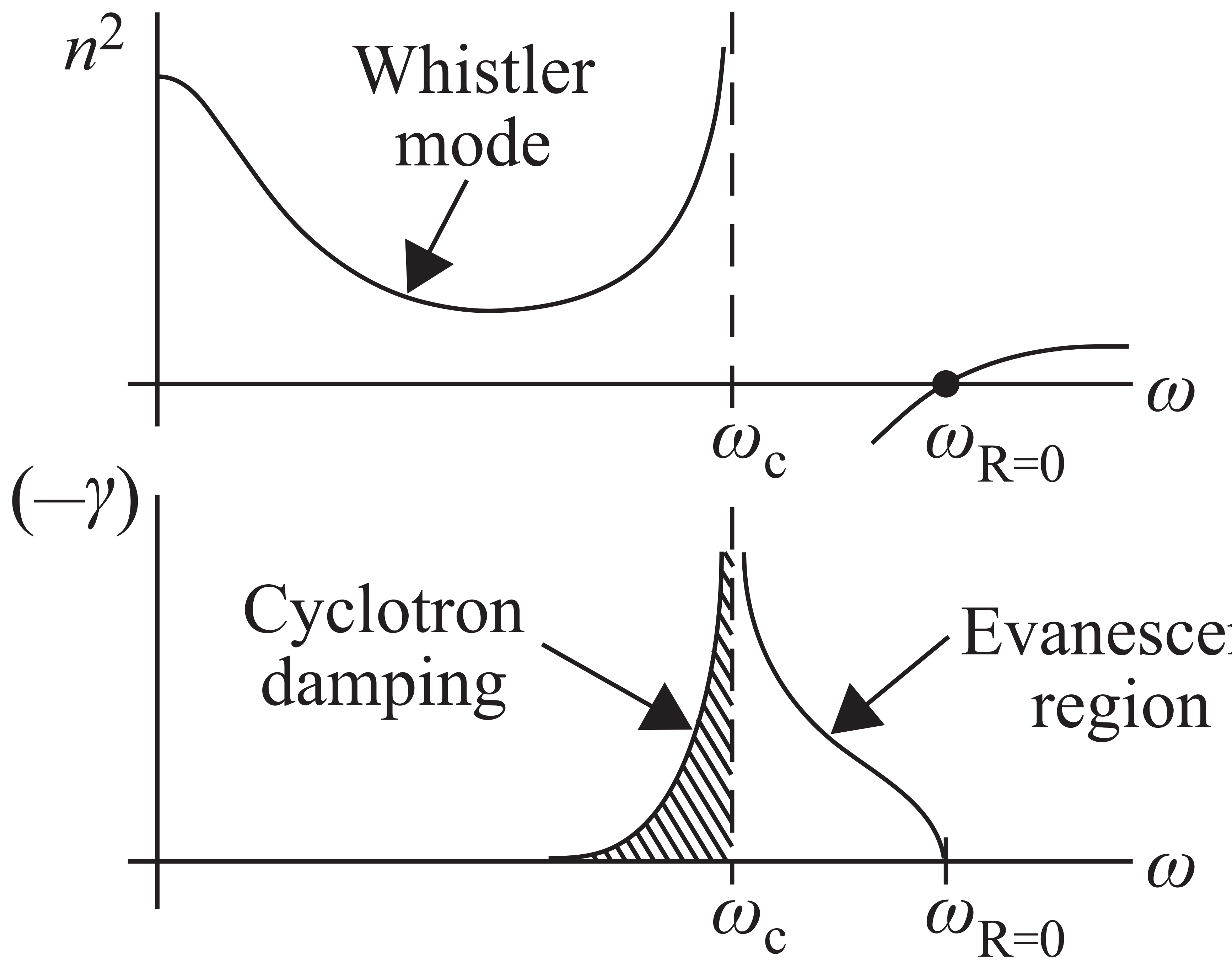
$$\gamma = \frac{-Đ_\mathrm{i}}{\partial Đ_\mathrm{r}/\partial\omega} = \pi \frac{k_\parallel/|k_\parallel|}{\partial Đ_\mathrm{r}/\partial\omega} \sum_s \frac{\omega_{\mathrm{p}s}^2}{\omega^2} G_{s0}(v_\parallel)\Big|_{v_\parallel = v_{\parallel\mathrm{Res}}}, \qquad (10.3.39)$$

$$G_{s0}(v_\parallel) = \frac{\omega}{k_\parallel} \int_0^\infty \frac{\partial F_{s0}}{\partial v_\perp} \pi v_\perp^2 \mathrm{d}v_\perp - \int_0^\infty \left(v_\parallel \frac{\partial F_{s0}}{\partial v_\perp} - v_\perp \frac{\partial F_{s0}}{\partial v_\parallel} \right) \pi v_\perp^2 \mathrm{d}v_\perp. \qquad (10.3.40)$$

For Isotropic F(v)

Cyclotron Damping

$$(-\gamma) = \pi \frac{\omega/|k_{||}|}{\partial Đ_r/\partial \omega} \frac{\omega_p^2}{\omega^2} \left(\frac{m}{2\pi\kappa T}\right)^{1/2} \exp\left(-\frac{m v_{||\mathrm{Res}}^2}{2\kappa T}\right)\Bigg|_{v_{||\mathrm{Res}}=\frac{\omega-\omega_c}{k_{||}}} . \qquad (10.3.45)$$



Whistler (Cyclotron) Instability

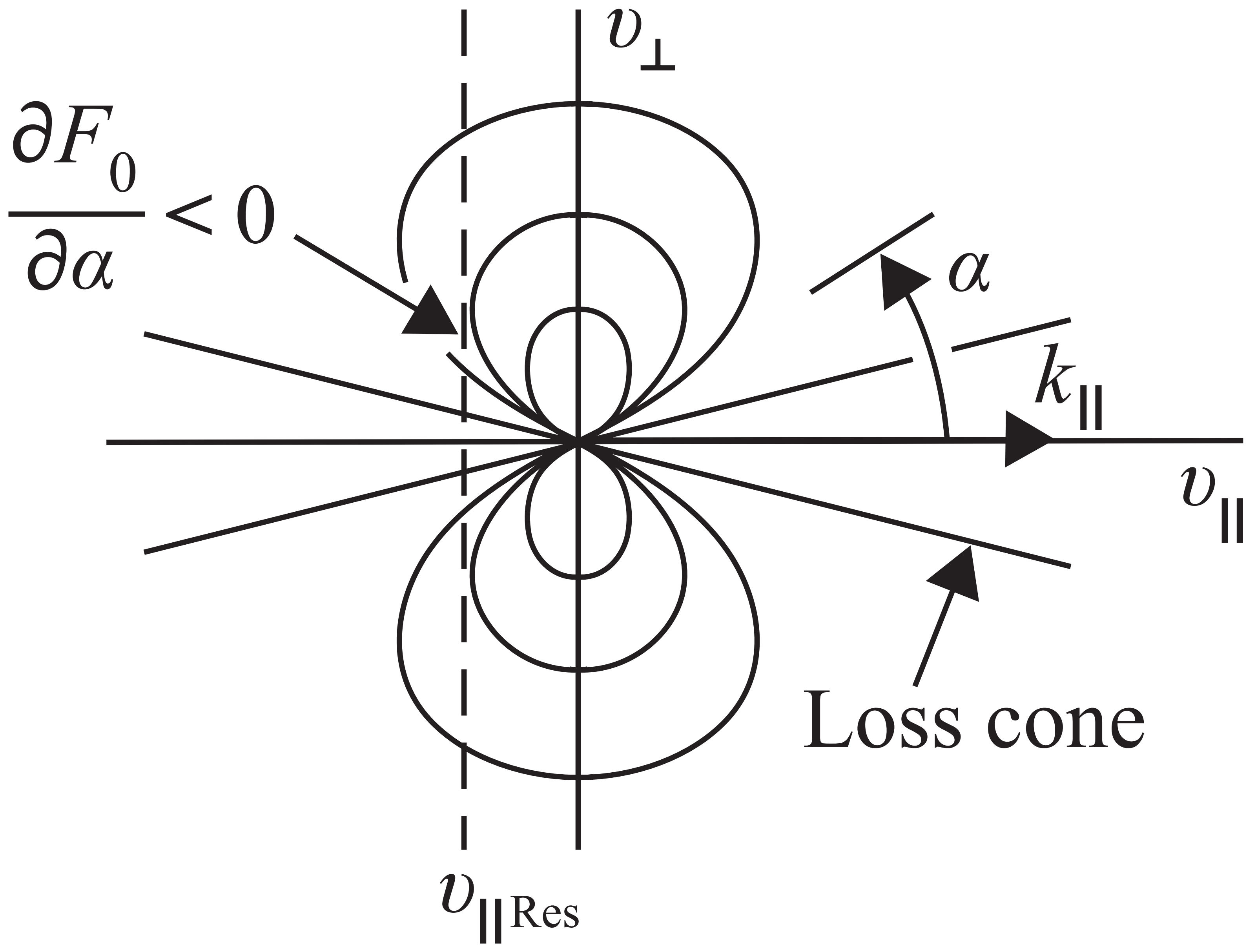


Figure 10.25 A loss-cone electron velocity distribution showing the cyclotron resonance velocity, $v_{||\text{Res}}$, for a whistler-mode wave propagating in the $k_{||}$ direction.

Next Week

Electron-cyclotron heating in a pulsed mirror experiment

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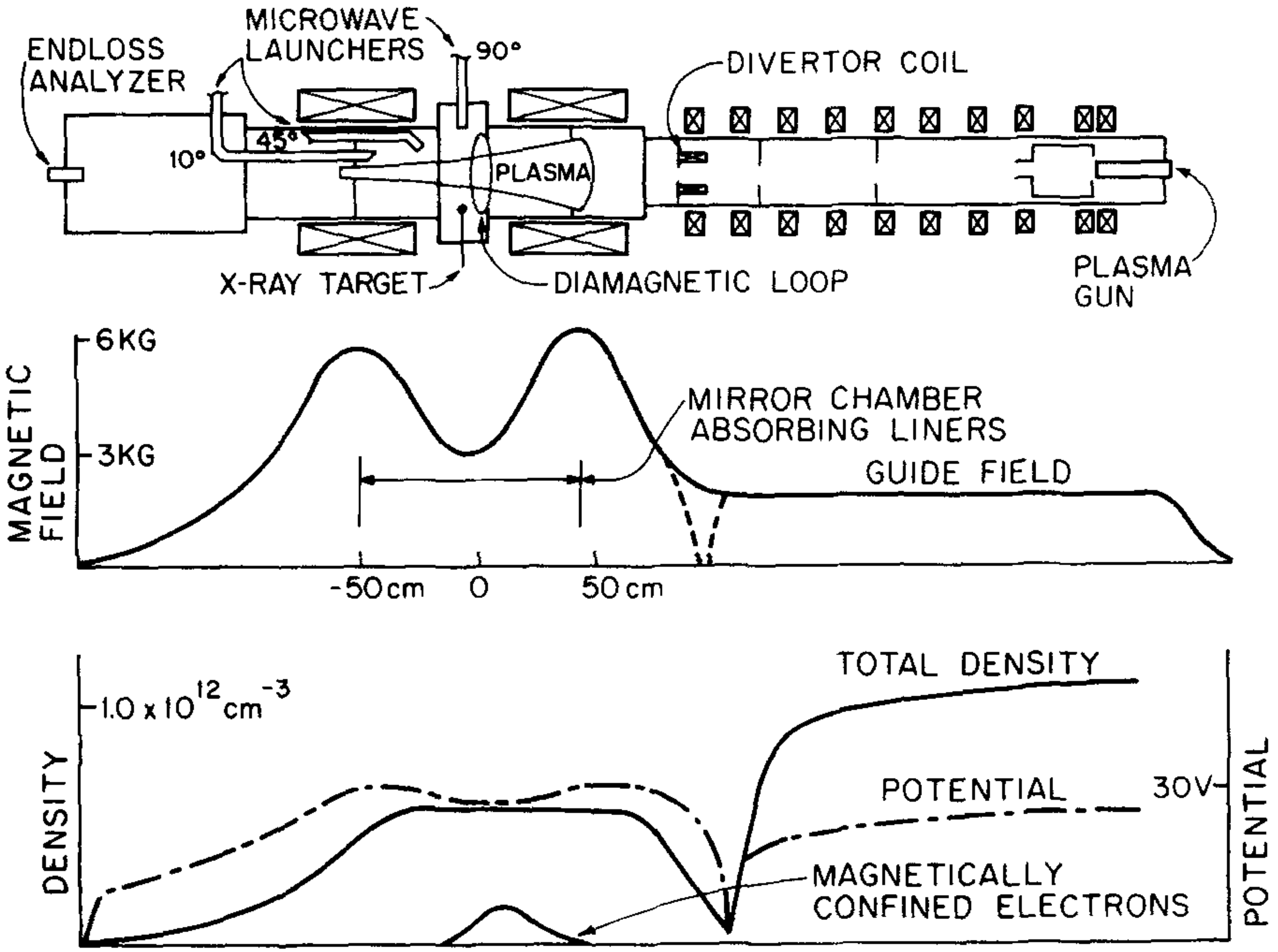
(Received 5 December 1983; accepted 6 August 1984)

Experimental measurements of electron-cyclotron resonance heating (ECRH) of a highly ionized plasma in mirror geometry is compared to a two-dimensional, time-dependent, Fokker–Planck simulation. Measurements of the absorption strength of the electrons and of the energy confinement of the ions helped to specify the parameters of the code. The electron energy distribution is measured with an end-loss analyzer and a target x-ray detector. These characterize a non-Maxwellian distribution consisting of "passing" (10 eV $< T_{e,p} <$ 30 eV), "warm" (50 eV $< T_{e,w} <$ 300 eV), and "hot" (1.2 keV $< T_{e,h} <$ 4.0 keV) electron populations. The temperature and fractional densities of the warm and hot populations depend on the absorbed power and total density. A similar distribution is calculated with the simulation program that reproduces the end-loss and x-ray signals. Both the experimental measurements and the simulation are described.



Next Week

VOLUME 59, NUMBER 16 PHYSICAL REVIEW LETTERS 19 OCTOBER 1987

Warm Electron-Driven Whistler Instability in an Electron-Cyclotron–Resonance Heated, Mirror-Confined Plasma

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(Received 8 May 1986; revised manuscript received 24 February 1987)

The whistler electron microinstability has been observed in the Constance-*B* quadrupole-mirror, electron-cyclotron–resonance heated plasma. Experimental evidence indicates that the warm-electron component (2 keV) drives the instability while the hot-electron component (400 keV) is stable. Dispersion-relation calculations using a new distribution function (electron-cyclotron–resonance heated distribution) to model the warm-electron component are in agreement with this experimental result.

"Whistler instability in an electron-cyclotron-resonance-heated, mirror-confined plasma," Garner, Mauel, Hokin, Post, and Smatlak, *Physics of Fluids B*, **2**, 242 (1990); [https://doi.org/10.1063/1.859234]

Next Week

Tokamak Plasma:
A Complex Physical System
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$$m_i n \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} + \nabla p = \frac{1}{c} \boldsymbol{j} \times \boldsymbol{B} \qquad (88)$$

$$\boldsymbol{j} = \frac{c}{4\pi} \nabla \times \boldsymbol{B} \qquad (89)$$

$$\operatorname{div} \boldsymbol{B} = 0 \qquad (90)$$

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times \boldsymbol{v} \times \boldsymbol{B} \qquad (91)$$

$$\frac{\partial n}{\partial t} + \operatorname{div}(n\boldsymbol{v}) = 0. \qquad (92)$$

$$m_i n \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} + \nabla_{\perp} P = \frac{1}{4\pi} (\boldsymbol{B} \cdot \nabla) \boldsymbol{B}_{\perp} \qquad (95)$$

$$\frac{\partial \boldsymbol{B}_{\perp}}{\partial t} = \nabla \times (\boldsymbol{v} \times \boldsymbol{B}_{\perp}) + B_T \frac{\partial \boldsymbol{v}}{\partial z} \qquad (96)$$