Plasma 2
Lecture 5: Quasilinear Theory

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NON-LINEAR STABILITY OF PLASMA OSCILLATIONS*

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The collective behavior of a fully ionized plasma in which the number of particles in a sphere of radius a, the Debye length, is very large compared to one is governed by the collisionless Boltzmann or Vlasov equation. In an infinite homogeneous plasma of this type, it is well known that in the "linearized" theory a velocity distribution $f_0(v)$ consisting of a main part that is a monotonically decreasing function of energy plus a small gentle bump on the tail of the main part (e.g. a Maxwellian plus runaway electrons) leads to unstable (growing) plasma oscillations, and that the unstable oscillations are those for which $v\, \partial f_0(v)/\partial v > 0$ for $v = \omega/k$ (ω is the frequency and k the wave number).

After a sufficient time these waves grow to such an amplitude that the non-linear terms in the Vlasov equation are important and the linearization is no longer valid. The question then arises as to the behavior of these waves in the non-linear region and it is this question which we consider.

The method is to divide the non-linear terms into two groups, one of which combined with the linear terms yields a non-linear dispersion relation, while the other provides a weak coupling between the different modes. The non-linear dispersion relation leads to the establishment of an equilibrium spectrum, which then decays slowly to zero due to the mode-coupling terms. The limiting of the wave amplitudes to the equilibrium spectrum is due to flattening of the bump in the velocity distribution by non-linear effects. The slow decay of the equilibrium spectrum leads to further changes in the velocity distribution so that asymptotically the distribution function is a monotonically decreasing function of energy and hence stable. Analytic expressions for the equilibrium spectrum and the equilibrium velocity distribution are obtained. An approximate value for the maximum energy in the equilibrium electric field is given by the geometric mean of the thermal energy and the drift energy of the particles in the bump.

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Velocity Space Diffusion from Weak Plasma Turbulence in a Magnetic Field

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The quasi-linear velocity space diffusion is considered for waves of any oscillation branch propagating at an arbitrary angle to a uniform magnetic field in a spatially uniform plasma. The space-averaged distribution function is assumed to change slowly compared to a gyroperiod and characteristic times of the wave motion. Nonlinear mode coupling is neglected. An H-like theorem shows that both resonant and nonresonant quasi-linear diffusion force the particle distributions towards marginal stablity. Creation of the marginally stable state in the presence of a sufficiently broad wave spectrum in general involves diffusing particles to infinite energies, and so the marginally stable plateau is not accessible physically, except in special cases. Resonant particles with velocities much larger than typical phase velocities in the excited spectrum are scattered primarily in pitch angle about the magnetic field. Only particles with velocities the order of the wave phase velocities or less are scattered in energy at a rate comparable with their pitch angle scattering rate.

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The Slow Evolution of the Average

linear waves

$$f_s(z, v_z, t) = \langle f_s \rangle (v_z, t) + f_{s1}(z, v_z, t). \tag{11.1.7}$$

$$\langle f_s \rangle (v_z, t) = \frac{1}{2L} \int_{-L}^{L} f_s(z, v_z, t) \mathrm{d}z, \tag{11.1.5}$$

Quasilinear Vlasov-Poisson

which all of the spatial variations are in the z direction. The governing equations for such a system are the Vlasov equation,

$$\frac{\partial f_s}{\partial t} + v_z \frac{\partial f_s}{\partial z} - \frac{e_s}{m_s} \frac{\partial \Phi}{\partial z} \frac{\partial f_s}{\partial v_z} = 0, \tag{11.1.1}$$

and Poisson's equation,

$$\frac{\partial^2 \Phi}{\partial z^2} = -\sum_s \frac{e_s}{\epsilon_0} \int_{-\infty}^{\infty} f_s \, \mathrm{d}v_z, \tag{11.1.2}$$

Quasilinear Vlasov-Poisson

$$\frac{\partial}{\partial t}\langle f_s \rangle + \left\langle v_z \frac{\partial f_s}{\partial z} \right\rangle = \frac{e_s}{m_s}\left\langle \frac{\partial \Phi}{\partial z}\frac{\partial f_s}{\partial v_z} \right\rangle. \tag{11.1.11}$$

What is the (slow) evolution of the average distribution?

Quasilinear Vlasov-Poisson

$$\frac{\partial}{\partial t}\langle f_s\rangle + \left\langle v_z \frac{\partial f_s}{\partial z}\right\rangle = \frac{e_s}{m_s}\left\langle \frac{\partial \Phi}{\partial z}\frac{\partial f_s}{\partial v_z}\right\rangle. \tag{11.1.11}$$

$$\frac{e_s}{m_s}\left\langle \frac{\partial \Phi}{\partial z}\frac{\partial f_s}{\partial v_z}\right\rangle = \frac{e_s}{m_s}\left\langle \left(\frac{\partial \Phi_0}{\partial z} + \frac{\partial \Phi_1}{\partial z}\right)\left(\frac{\partial \langle f_s\rangle}{\partial v_z} + \frac{\partial f_{s1}}{\partial v_z}\right)\right\rangle. \tag{11.1.13}$$

Quasilinear Vlasov-Poisson

Equation for Average

$$\frac{\partial}{\partial t}\langle f_s\rangle = \frac{e_s}{m_s}\frac{\partial}{\partial v_z}\left\langle f_{s1}\frac{\partial \Phi_1}{\partial z}\right\rangle. \tag{11.1.15}$$

$$\left(\frac{\partial}{\partial t} + v_z\frac{\partial}{\partial z}\right) f_{s1}(z, v_z, t) = \frac{e_s}{m_s}\frac{\partial \Phi_1}{\partial z}\frac{\partial\langle f_s\rangle}{\partial v_z} + \frac{e_s}{m_s}\frac{\partial}{\partial v_z}\left[\frac{\partial \Phi_1}{\partial z} f_{s1} - \left\langle\frac{\partial \Phi_1}{\partial z} f_{s1}\right\rangle\right]. \tag{11.1.16}$$

Equation for Fluctuations

Quasilinear Vlasov-Poisson

Linear part…

$$\hat{f}_{s1}(k, v_z) = -\frac{e_s}{m_s}\frac{k\hat{\Phi}_1(k)}{\omega - k v_z}\frac{\partial\langle f_s\rangle}{\partial v_z} \tag{11.1.25}$$

$$k^2\hat{\Phi}_1(k) = \sum_s \frac{e_s}{\epsilon_0}\int_{-\infty}^{\infty}\hat{f}_{s1}(k, v_z)\,\mathrm{d}v_z. \tag{11.1.26}$$

Quasilinear Vlasov-Poisson

Quasi-linear part…

$$\frac{\partial}{\partial v_z}\left\langle f_{s1}\frac{\partial \Phi_1}{\partial z}\right\rangle = \frac{\partial}{\partial v_z}\frac{1}{2L}\int_{-L}^{L}\frac{\partial \Phi_1}{\partial z} f_{s1}\,\mathrm{d}z$$

$$= \frac{\partial}{\partial v_z}\frac{1}{2L}\int_{-L}^{L}\left[\left\{\frac{\partial}{\partial z}\int_{-\infty}^{\infty}\tilde{\Phi}_1(k,t)\,\mathrm{e}^{\mathrm{i}kz}\mathrm{d}k\right\}\right.$$

$$\left.\times\int_{-\infty}^{\infty}\tilde{f}_{s1}(k',v_z,t)\,\mathrm{e}^{\mathrm{i}k'z}\,\mathrm{d}k'\right]\mathrm{d}z, \qquad (11.1.33)$$

Quasilinear Vlasov-Poisson

Quasi-linear part…

$$\frac{\partial}{\partial v_z}\left\langle f_{s1}\frac{\partial\Phi_1}{\partial z}\right\rangle = \frac{\partial}{\partial v_z}\frac{1}{2L}\int_{-L}^{L}\frac{\partial\Phi_1}{\partial z}f_{s1}\,\mathrm{d}z$$

$$= \frac{\partial}{\partial v_z}\frac{1}{2L}\int_{-L}^{L}\left[\left\{\frac{\partial}{\partial z}\int_{-\infty}^{\infty}\tilde{\Phi}_1(k,t)\,\mathrm{e}^{\mathrm{i}kz}\mathrm{d}k\right\}\right.$$

$$\left.\times\int_{-\infty}^{\infty}\tilde{f}_{s1}(k',v_z,t)\,\mathrm{e}^{\mathrm{i}k'z}\,\mathrm{d}k'\right]\mathrm{d}z, \tag{11.1.33}$$

$$\lim_{L\to\infty}\int_{-L}^{L}\mathrm{e}^{\mathrm{i}(k+k')z}\mathrm{d}z = 2\pi\delta(k+k'), \tag{11.1.34}$$

Quasilinear Vlasov-Poisson

Quasi-linear part…

$$\frac{\partial}{\partial v_z}\left\langle f_{s1}\frac{\partial \Phi_1}{\partial z}\right\rangle = \frac{\partial}{\partial v_z}\frac{1}{2L}\int_{-L}^{L}\frac{\partial \Phi_1}{\partial z} f_{s1}\,\mathrm{d}z$$

$$= \frac{\partial}{\partial v_z}\frac{1}{2L}\int_{-L}^{L}\left[\left\{\frac{\partial}{\partial z}\int_{-\infty}^{\infty}\tilde{\Phi}_1(k,t)\,\mathrm{e}^{\mathrm{i}kz}\mathrm{d}k\right\}\right.$$

$$\left.\times\int_{-\infty}^{\infty}\tilde{f}_{s1}(k',v_z,t)\,\mathrm{e}^{\mathrm{i}k'z}\,\mathrm{d}k'\right]\mathrm{d}z, \qquad (11.1.33)$$

$$\lim_{L\to\infty}\int_{-L}^{L}\mathrm{e}^{\mathrm{i}(k+k')z}\mathrm{d}z = 2\pi\delta(k+k'), \qquad (11.1.34)$$

Quasilinear Vlasov-Poisson

Quasi-linear part…

$$\frac{\partial}{\partial v_z}\left\langle f_{s1}\frac{\partial \Phi_1}{\partial_z}\right\rangle = -\frac{\pi}{L}\frac{\partial}{\partial v_z}\int_{-\infty}^{\infty} \mathrm{i}k\tilde{\Phi}_1(-k,t)\tilde{f}_{s1}(k,v_z,t)\,\mathrm{d}k. \qquad (11.1.35)$$

Quasilinear Vlasov-Poisson

$$\tilde{\Phi}(x,t) = \int_L \frac{d\omega}{2\pi} \sum_k \Phi_k e^{-j\omega t} e^{+jkx}$$

$\omega(k)$ ~ DISPERSION RELATION

$$\tilde{\Phi}^*(x,t) = \tilde{\Phi}(x,t) \sim \text{A REAL NUMBER}$$

$$= \int_L \frac{d\omega}{2\pi} \sum_k \Phi_k^* e^{+j\omega^* t} e^{-jkx}$$

$k \to -k$

$$\text{or} \quad \Phi_k^* = \Phi_{-k}$$

$$\omega^*(k) = -\omega(-k)$$

$$\omega_R^*(k) = -\omega_R(-k)$$

$$\omega_I^*(k) = +\omega_I(-k)$$

$$f_k = \frac{e}{m} \frac{k\Phi_k}{\omega - kv} \frac{2\langle f\rangle}{2v}$$

$$\Phi_{-k} f_k = \frac{ek}{m} \frac{|\Phi_k|^2}{\omega - kv} \frac{2\langle f\rangle}{2v}$$

$$\sum_k \frac{|\Phi_k|^2}{\omega(k) - kv} = \sum_k \frac{|\Phi_k|^2}{\omega_R(k) - kv + i\omega_I(k)} = \sum_k \frac{|\Phi_k|^2\left[(\omega_R - kv) - i\,\omega_I(k)\right]}{(\omega_R - kv)^2 + \omega_I^2}$$

REAL PART IS ODD IN k

IMAGINARY PART IS EVEN

Quasilinear Velocity-Space Diffusion

$$\frac{\partial}{\partial t}\langle f_s\rangle(v_z,t) = \frac{\partial}{\partial v_z}\left[D_{\mathrm{q}}(v_z,t)\frac{\partial}{\partial v_z}\langle f_z\rangle(v_z,t)\right], \qquad (11.1.44)$$

~ $(e/m)^2\ \tau_{cor}\ |E^2|$

$$D_{\mathrm{q}}(v_z,t) = \frac{2}{\epsilon_0}\left(\frac{e_s}{m_s}\right)^2 \int_{-\infty}^{\infty} \frac{\mathrm{i}\mathscr{E}(k,t)}{\omega - k v_z}\,\mathrm{d}k \qquad (11.1.45)$$

where $\mathscr{E}(k,t) = (\pi\epsilon_0/2L)|\tilde{E}_1(k,t)|^2$ is called the *spectral density* of the electric field.

$$\frac{\partial \mathscr{E}(k,t)}{\partial t} = 2\gamma(k,t)\mathscr{E}(k,t), \qquad (11.1.43)$$

Quasilinear Velocity-Space Diffusion

$$\frac{\partial}{\partial t}\langle f_s\rangle(v_z,t) = \frac{\partial}{\partial v_z}\left[D_{\mathrm{q}}(v_z,t)\frac{\partial}{\partial v_z}\langle f_z\rangle(v_z,t)\right], \tag{11.1.44}$$

$$D_{\mathrm{q}}(v_z,t) = \frac{2}{\epsilon_0}\left(\frac{e_s}{m_s}\right)^2 \int_{-\infty}^{\infty} \frac{\mathscr{E}(k,t)\gamma(k,t)}{[\omega_{\mathrm{r}}(k,t) - kv_z]^2 + \gamma^2(k,t)}\,\mathrm{d}k, \tag{11.1.47}$$

$$\frac{\partial \mathscr{E}(k,t)}{\partial t} = 2\gamma(k,t)\mathscr{E}(k,t), \tag{11.1.43}$$

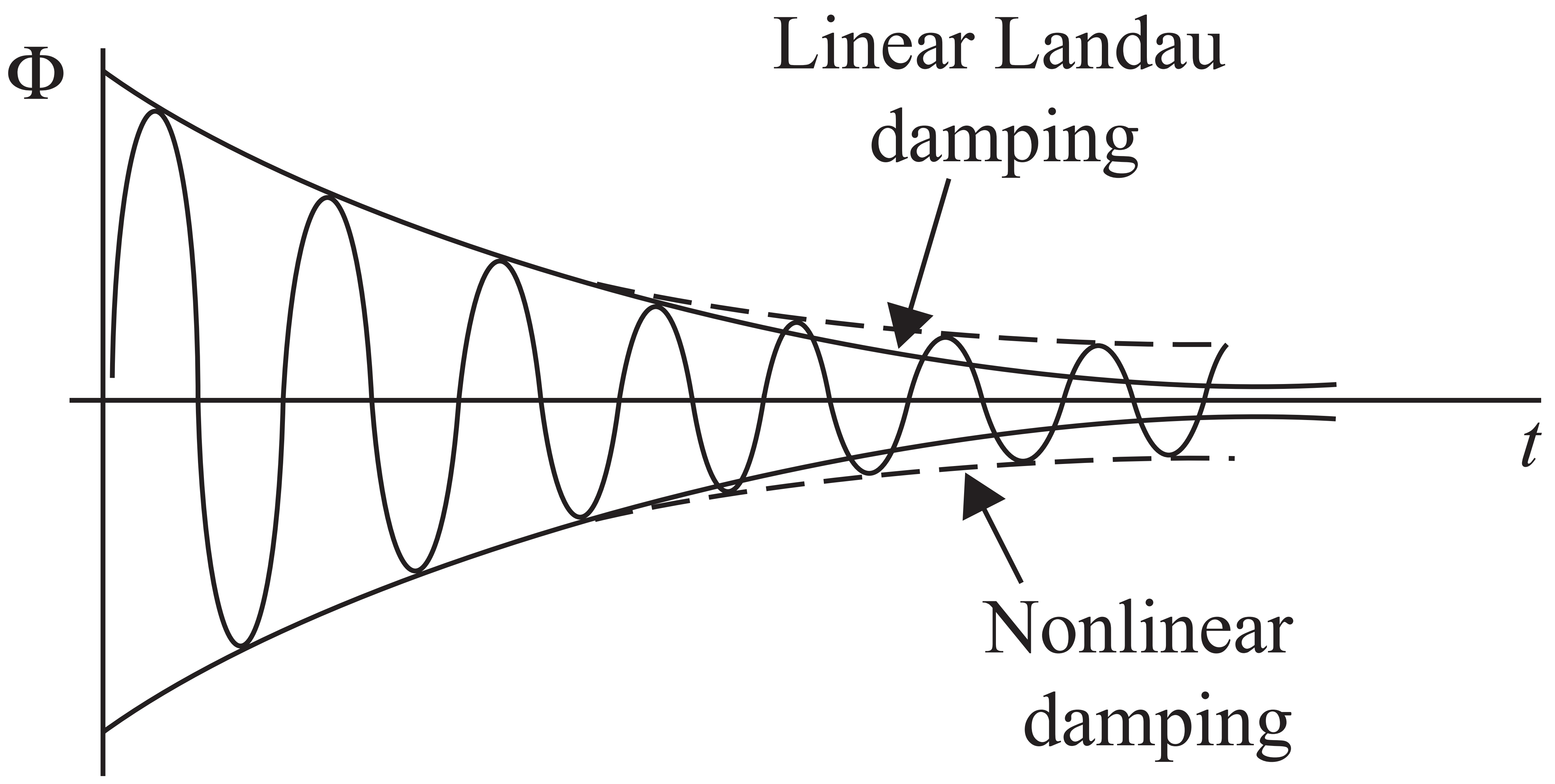
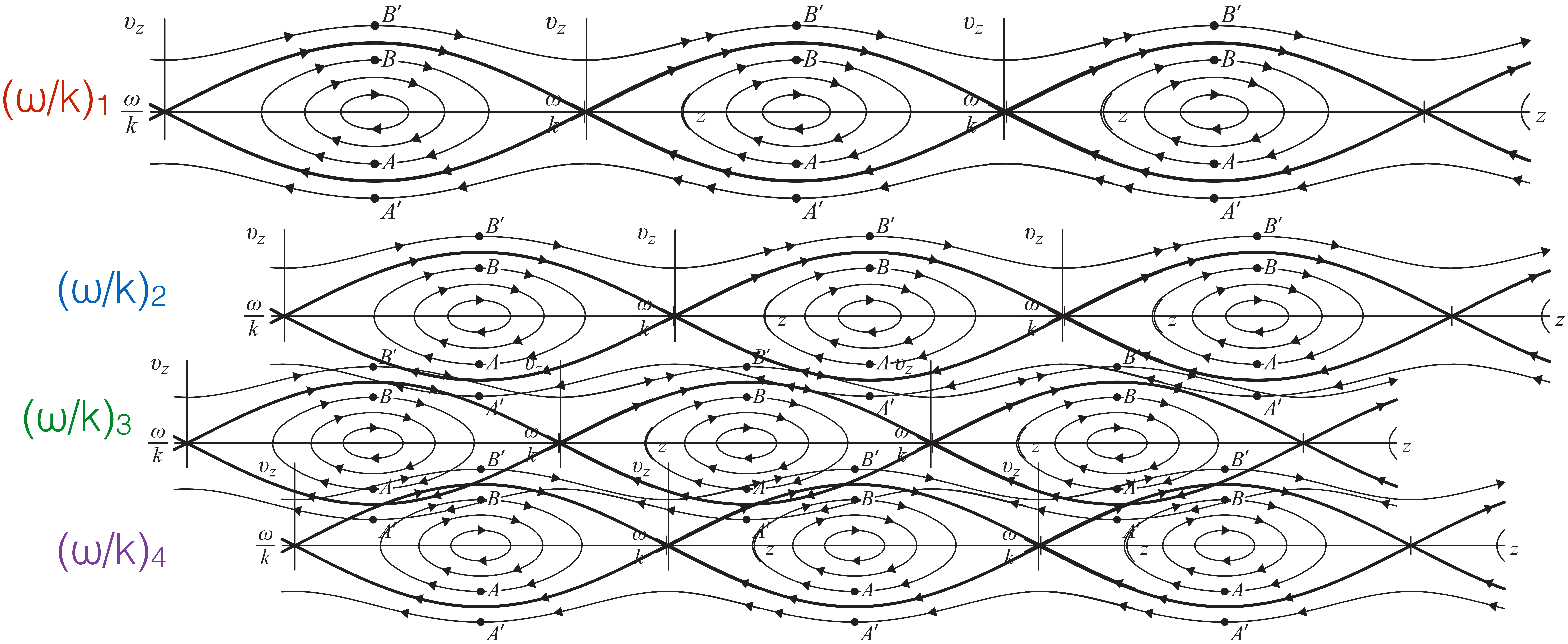
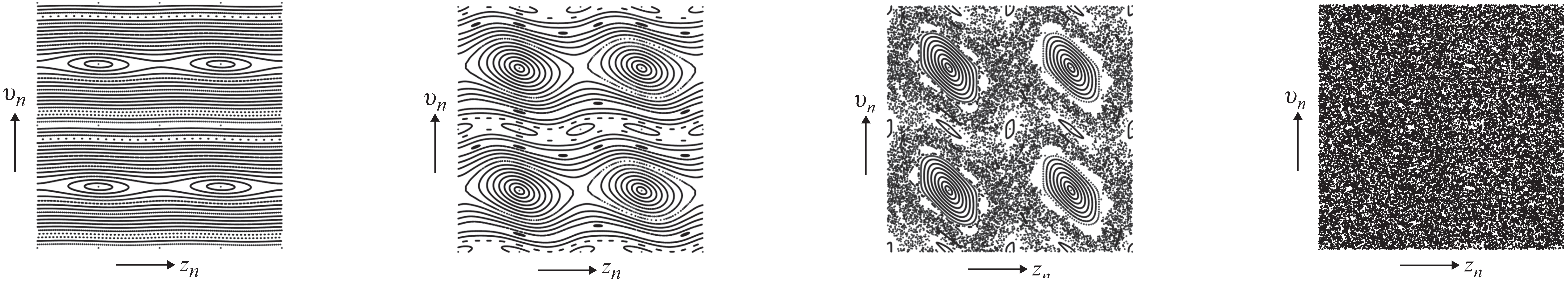


Figure 9.15 The nonlinear effects of particle trapping tend to increase the wave amplitude relative to the predictions of linear Landau damping.

Many Wave-Particle Resonances



Quasilinear Velocity-Space Diffusion



$$z_{m+1} = z_m + v_m \tag{11.1.56}$$

$$v_{m+1} = v_m - 2\pi\epsilon^2 \sin(2\pi z_{m+1}). \tag{11.1.57}$$

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Nonlinear Development of the Beam–Plasma Instability

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The nonlinear limit of wave growth induced by a low density cold electron beam in a collisionless plasma is calculated from a simple physical model. The bandwidth of the growing "noise" is so small that the beam interacts with a nearly sinusoidal electric field.

Experimental Test of Quasilinear Theory*

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The shape and amplitude of the electron-plasma wave spectrum resulting from a "gentle bump" on the tail of the electron velocity distribution of a plasma is measured and found to be in good agreement with quasilinear theory.

In this Letter we report an experiment designed to test the validity of this theory by measuring the electron-plasma wave spectrum resulting from the injection of an electron beam of sufficiently low density and large velocity spread to satisfy the assumptions of quasilinear theory. In prior beam-plasma experiments the initial velocity spread of the beam electrons was not sufficient to meet the requirements.[3-5]

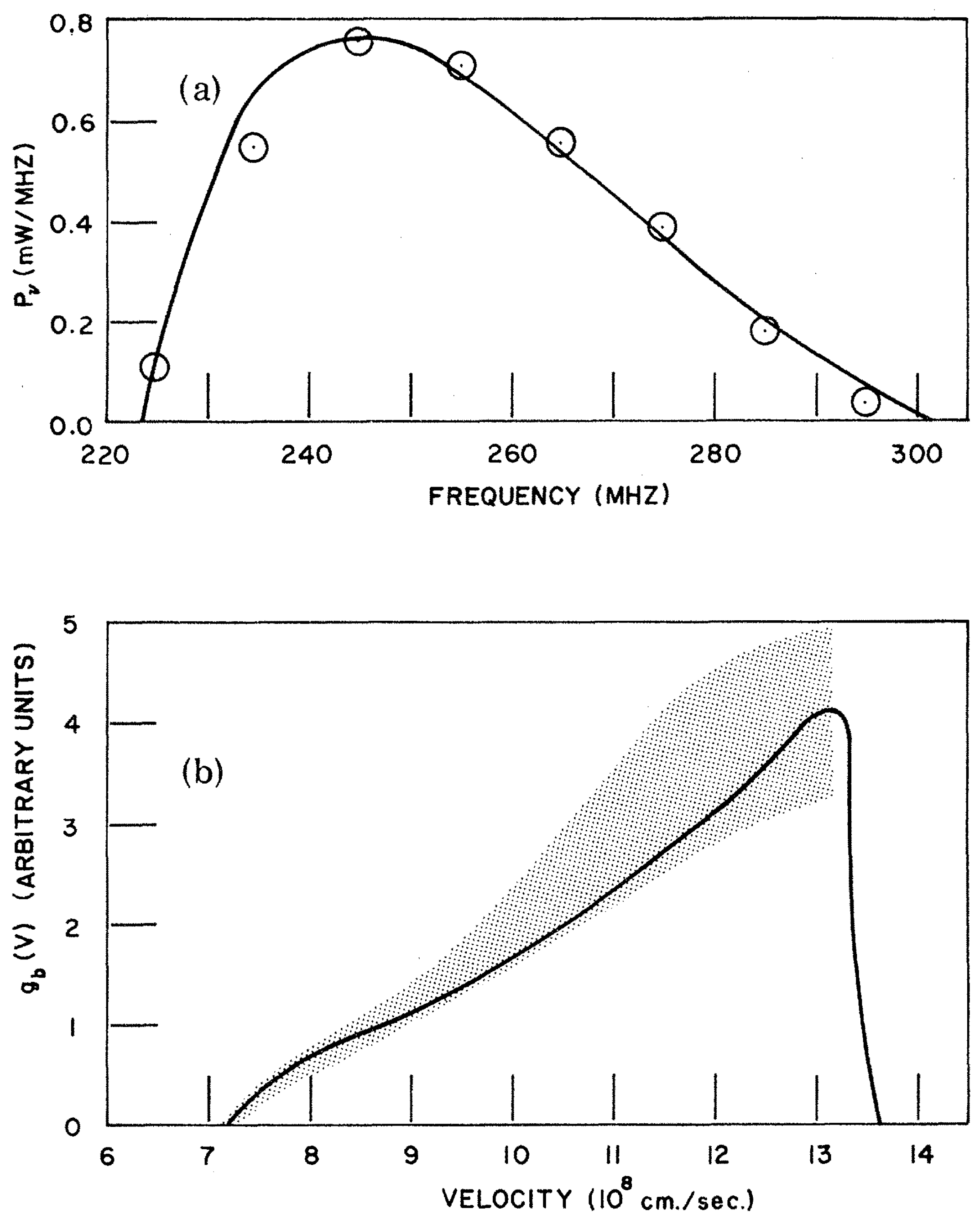

FIG. 3. (a) Shape of wave spectrum. Solid line is theory, circles are experimental values. Beam current is 2 mA. (b) Inferred beam-velocity distribution versus energy analyzer distribution. The solid curve is obtained from an electronically differentiated output of the analyzer.