Plasma 2
Lecture 24:
Tearing Modes

APPH E6102y
Columbia University

Resistive MHD

"IDEAL" MHD DESCRIBES THE "FAST" TIME DYNAMICS OF A MAGNETIZED PLASMA, $\omega \sim \omega_A$.

BUT, ON LONGER TIME SCALES, RESISTIVITY (AND OTHER NON-IDEAL EFFECTS) ARE IMPORTANT.

MHD

$$\rho \frac{d\bar{v}}{dt} = -\nabla p + J \times B$$

$$\frac{\partial \rho}{\partial t} = -\bar{\nabla} \cdot \rho \bar{v}$$

$$\bar{E} = -\bar{v} \times \bar{B} + \eta \bar{J}$$

$$\nabla \times B = \mu_0 \bar{J}$$

$$\frac{\partial \bar{B}}{\partial t} = -\nabla \times \bar{E}$$

η (STAINLESS STEEL) $\sim 0.7\, \mu\Omega \cdot m$

η (HBT-EP) $\sim 4-10 \times \eta_{ss}$

$$\eta = \frac{m_e}{e^2 n \tau_e} \propto \frac{1}{T_e^{3/2}}$$

WHAT IS η?

FOR A CONSTANT $\bar{E}$, COLLISIONS LEAD TO A CONSTANT DRIFT VELOCITY.

$$0 \simeq q\bar{E} + m\bar{v}/\tau_{col} \Rightarrow \bar{v} = \frac{q\tau_{col}}{m}\bar{E}$$

$$\bar{J} = en(\bar{v}_i - \bar{v}_e) \simeq -en\bar{v}_e = \frac{e^2 n \tau_{col}}{m_e}\bar{E}$$

No “Equilibrium” in Resistive MHD

MHD:

$$\mu_0 \nabla p = (\nabla \times B) \times \bar{B}$$

FORCE BALANCE IS EASY TO FIND
i.e. GRAD-SHAFRANOV

INDUCTION

$$\frac{\partial B}{\partial t} \approx 0 \simeq \nabla \times (v \times \bar{B}) - \nabla \times \left(\frac{\eta}{\mu_0} \nabla \times B\right)$$

STATIC EQUILIBRIUM $\bar{v} \approx 0$

SO

$$0 \simeq \nabla \times \left(\frac{\eta}{\mu_0} \nabla \times B\right)$$

NO EQUILIBRIUM IN GENERAL

FINITE RESISTIVITY IMPLIES NO EQUILIBRIUM

PLASMA DISCHARGES (LIKE A TOKAMAK) MUST BE ACTIVELY SUSTAINED*

e.g. $\nabla \times J = 0 \Rightarrow$ NO TOROIDAL EQUILIBRIUM
NO POLOIDAL, DIAMAGNETIC CURRENT

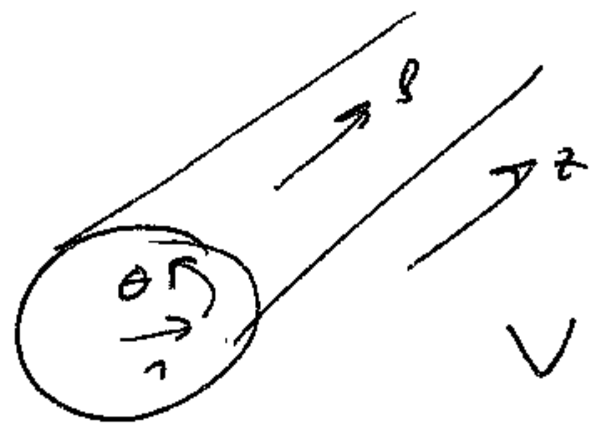
RESISTIVE EQUILIBRIUM NEEDS SOURCE/SINKS OF CURRENT
i.e. ELECTRODES

*NOT SO FOR SOME STELLARATORS AND THE LEVITATED DIPOLE

Resistivity Results in Diffusion

$$\frac{\partial \rho}{\partial t} = -\nabla\cdot\rho\bar{V}$$

$$\frac{\partial B}{\partial t} = \nabla\times(V\times B) - \nabla\times\left(\frac{\eta}{\mu_0}\nabla\times B\right)$$



$$V \sim \underbrace{\frac{E\times B}{B^2}}_{\text{MOSTLY POLOIDAL}} + \underbrace{\frac{\eta\bar{B}\times\bar{J}}{B^2}}_{\text{MOSTLY RADIAL}}$$

$$\frac{\partial \rho}{\partial t} = -\nabla\left(\frac{\rho\eta}{B^2}B\times J\right)$$

$$J \sim \frac{B\times\nabla p}{B^2}$$

$$= \nabla\cdot\left(\frac{\rho\eta}{B^2}\nabla_\perp p\right)$$

$$p = \frac{\rho kT}{m_i}$$

$$\cong \nabla\cdot\frac{\rho kT\eta}{B^2 m_i}\nabla_\perp\rho$$

$$D_\rho \sim \frac{\rho\eta}{B^2}V_{THi}^2 = \frac{\eta}{\mu_0}\left(\frac{V_{THi}}{V_A}\right)^2 \cong (\eta/\mu_0)\beta$$

$$\frac{\partial B}{\partial t} \cong -\nabla\times\left(\frac{\eta}{\mu_0}\nabla\times B\right)$$

$$= -\frac{\eta}{\mu_0}\left[\nabla(\nabla\cdot B) - \nabla^2 B\right] - \nabla\left(\frac{\eta}{\mu_0}\right)\times J$$

→ SMALL IF η WEAKLY OR CYLINDRICAL GEOMETRY

$$= \left(\frac{\eta}{\mu_0}\right)\nabla^2 B$$

→ MAGNETIC DIFFUSION COEFFICIENT

$$D_m = \frac{\eta}{\mu_0}$$

$$D_\rho \ll D_m$$

$$D_\rho \sim D_m\beta$$

Some Characteristic Parameters…

$D_m \sim \frac{\eta}{\mu_0}$ → CURRENT RELAXES ON THE TIME SCALE OF $\tau_m \approx a^2/2D_m$

DEVICE	SIZE, ⟨a⟩	T_e, (keV)	n (m^{-3})	η	τ_m (sec)
HBT-EP	0.14	0.05	1×10^{19}	3×10^{-6}	8 msec
T-3	0.13	1.0	2×10^{19}	3×10^{-8}	600 msec
DIII-D	0.85	6	8×10^{19}	2×10^{-9}	400 sec
JET	1.6	15	1×10^{20}	6×10^{-10}	1.5 Hours
ITER	2.6	10	1×10^{20}	1×10^{-9}	2.2 Hours

η (copper) = $1.7\times10^{-8}\ \Omega\,m$

(Review) MHD for Toroidal Plasma

REVIEW

VORTICITY

AXIAL CURRENT

REDUCED MHD

$$\rho \frac{d\bar{v}}{dt} = -\nabla p + J \times B \longrightarrow \rho \frac{d}{dt} \nabla_\perp^2 \chi = (B \cdot \nabla) \nabla_\perp^2 \psi$$

$$\bar{E} = -\bar{v} \times \bar{B}$$

$\bar{B} \cdot \bar{\nabla}$

INDUCTION

$$\frac{\partial \bar{B}}{\partial t} = -\nabla \times E \longrightarrow \frac{\partial \bar{B}}{\partial t} = \nabla \times (v \times B) - \nabla \times (\eta \bar{J})$$

$$\frac{d\bar{B}}{dt} = (B \cdot \bar{\nabla}) \bar{v} - \nabla \times (\eta \bar{J})$$

$\bar{B} \cdot \bar{\nabla}$

FOR EXTERNAL MODES

$\bar{B} \cdot \bar{\nabla} \neq 0$ ANYWHERE INSIDE PLASMA

FOR INTERNAL MODES

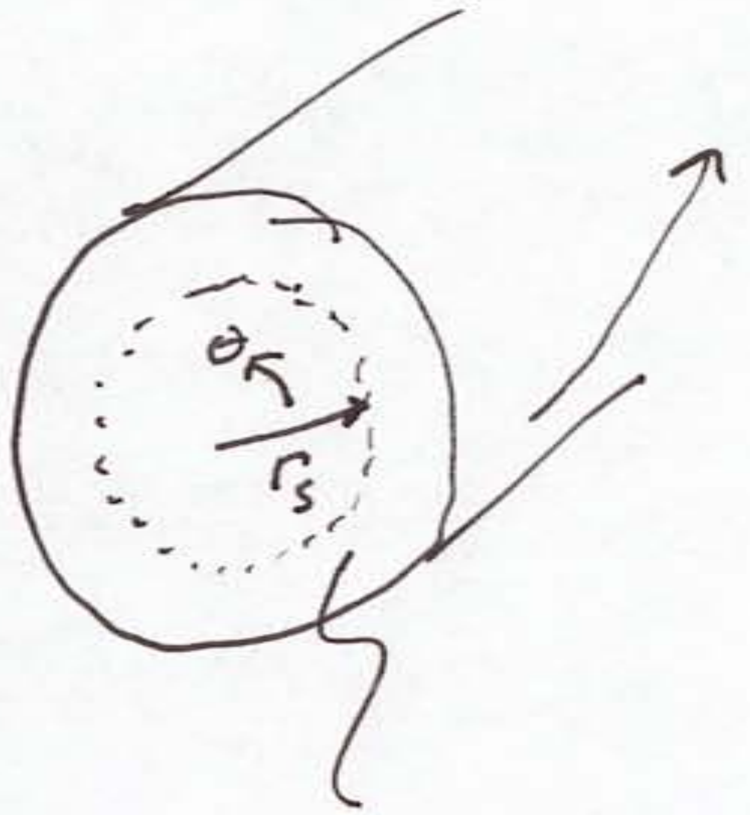
$\bar{B} \cdot \bar{\nabla} = 0$ INSIDE PLASMA

$$\bar{B} \cdot \bar{\nabla} \sim \frac{B_\theta}{r} (m - nq(r))$$

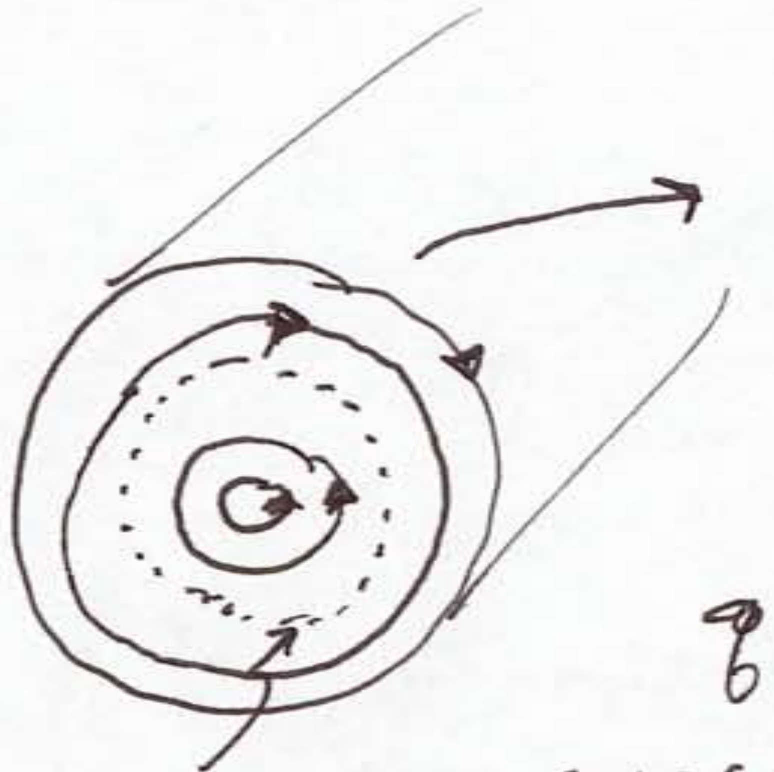
HELICAL INTERNAL RESONANCE

Magnetic Islands

INTERNAL RESONANT CURRENTS CREATE "ISLANDS"



$m - nq(r_s) = 0$



$q(0) < q(r_s) < q(a)$

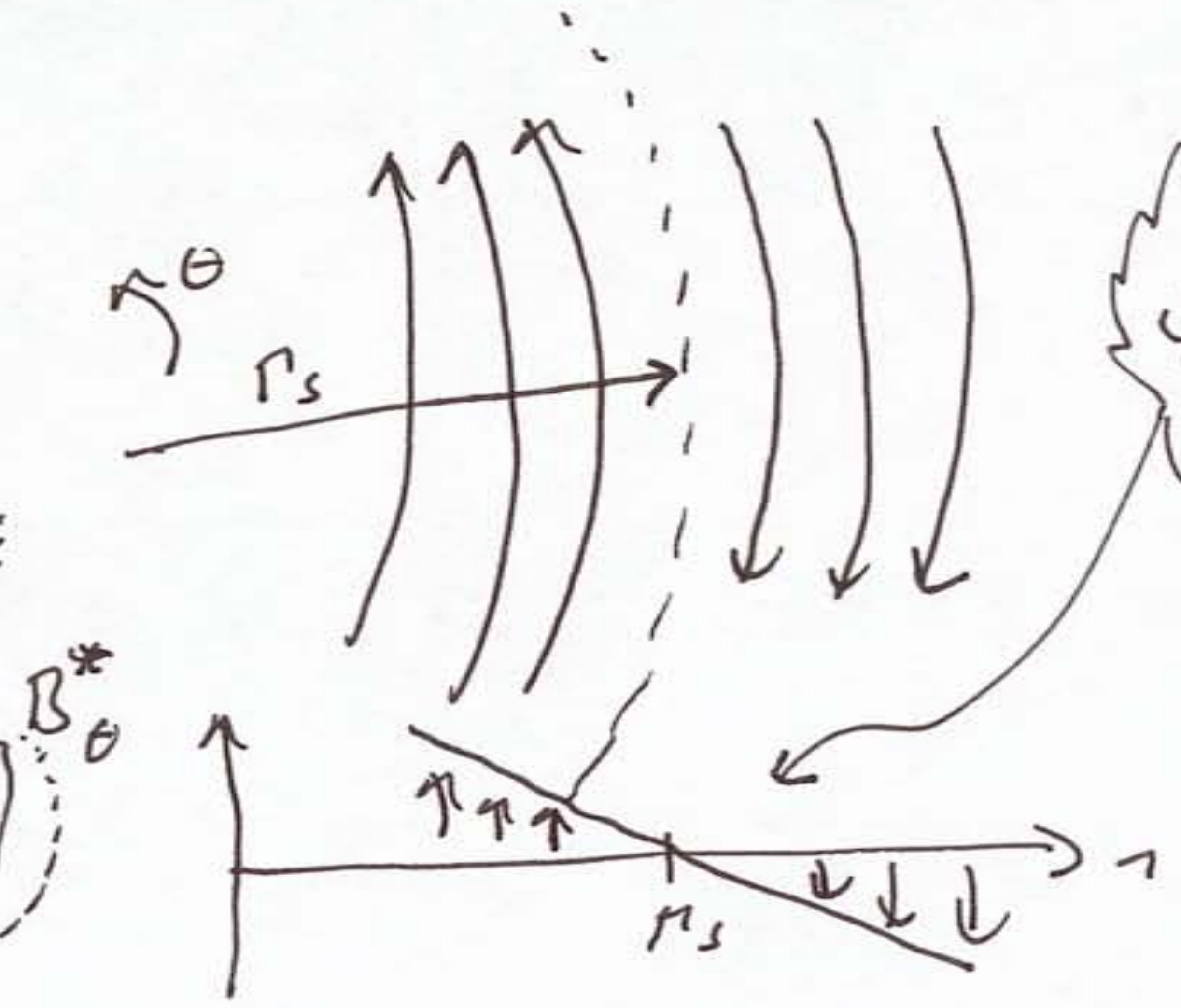
RESONANT SURFACE $B^*_\theta = 0$

$\bar{B}^* = \hat{z} B_z + \hat{z} \times \nabla \psi^*$

$\bar{B} = \hat{z} B_z + \hat{z} \times \nabla \psi$

$$B^*_\theta = B_\theta - B_z \frac{n r}{m R} = \frac{B_\theta}{m}\left(m - n q(r)\right)$$

$$\frac{\partial B^*_\theta}{\partial r} = \frac{\partial B_\theta}{\partial r} - \frac{B_z n}{m R} = \frac{B_z}{R q}\left[1 - s - \frac{n q}{m}\right] = -\frac{B_z}{R q} s \; @ \; r = r_s$$



SLOPE: $\frac{\partial B^*_\theta}{\partial r} = -\frac{B_z}{qR} s$

$s \equiv \frac{r}{q}\frac{dq}{dr}$ MAGNETIC SHEAR

How Big are Magnetic Islands?

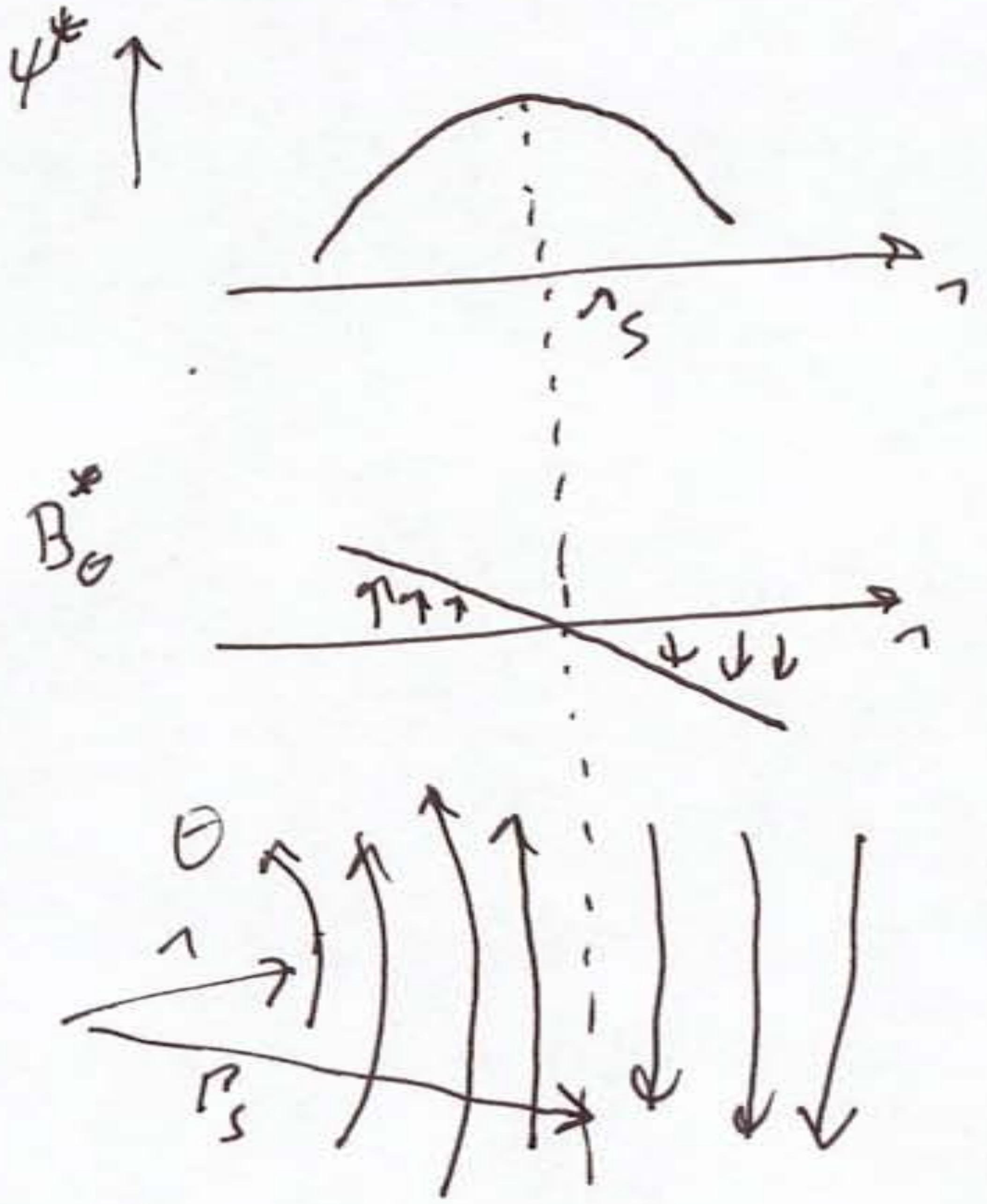
$$B^* = \hat{z} B_z + \hat{z} \times \nabla \psi^* = \hat{z} B_z + \hat{\theta} \frac{\partial \psi^*}{\partial r} - \frac{\hat{r}}{r} \frac{\partial \psi^*}{\partial \theta}$$

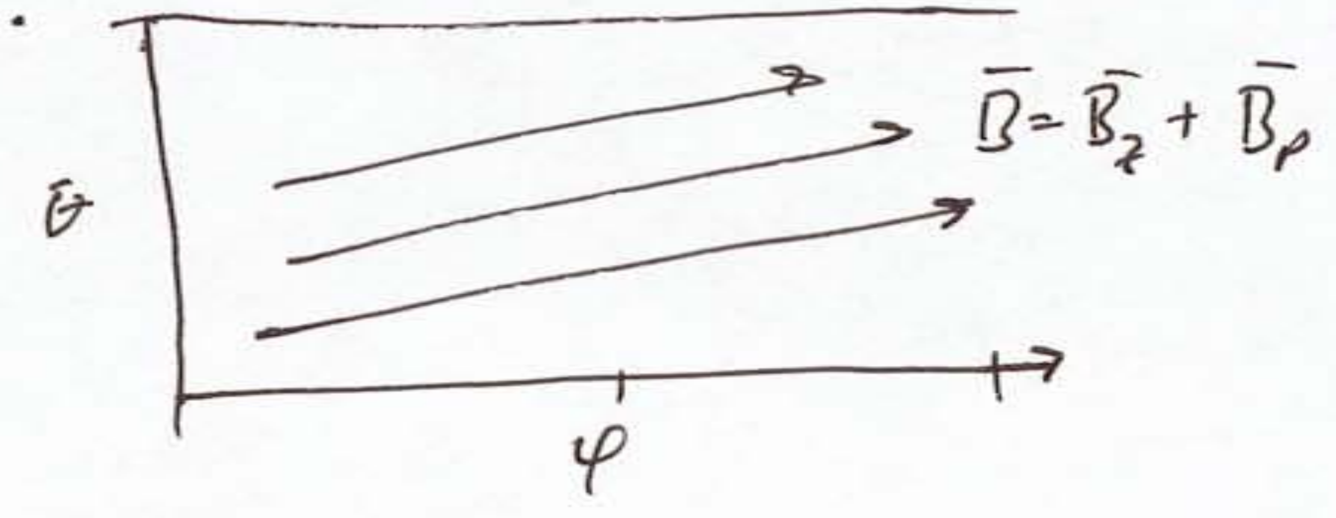
$$B_\theta^* \cong \underbrace{B_\theta^*(r=r_s)}_{=0} + (r-r_s) \frac{B_z}{Rq} s + \dots \quad \left(s \equiv \frac{1}{q} \frac{dq}{dr} \right)$$

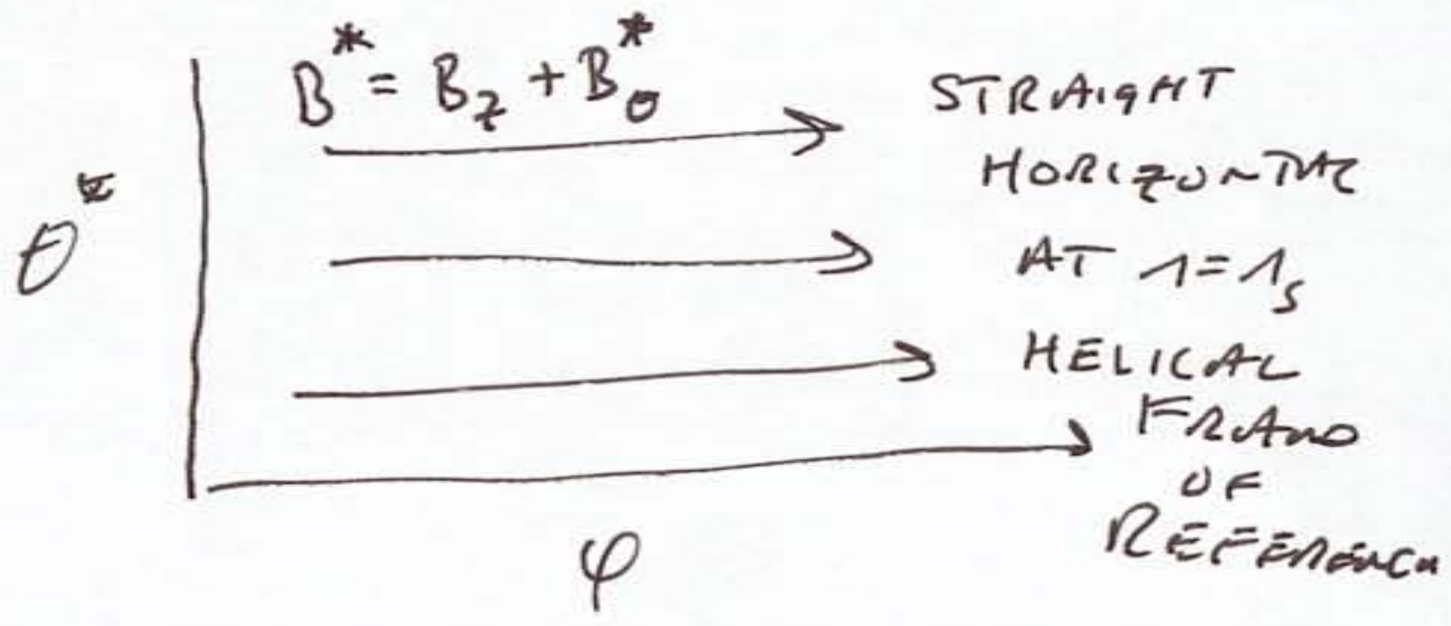
THEREFORE

$$\psi^*(r) \cong \psi^*(r_s) - \frac{(r-r_s)^2}{2} \frac{B_z}{Rq} s + \dots \quad (\text{NEAR } r \simeq r_s)$$

EQUILIBRIUM POLOIDAL FLUX







How Big are Magnetic Islands?

LET $\psi^* = \psi_0^*(r) + \tilde{\psi}^*(r, \theta^*)$

ψ_0^*: EQUILIBRIUM

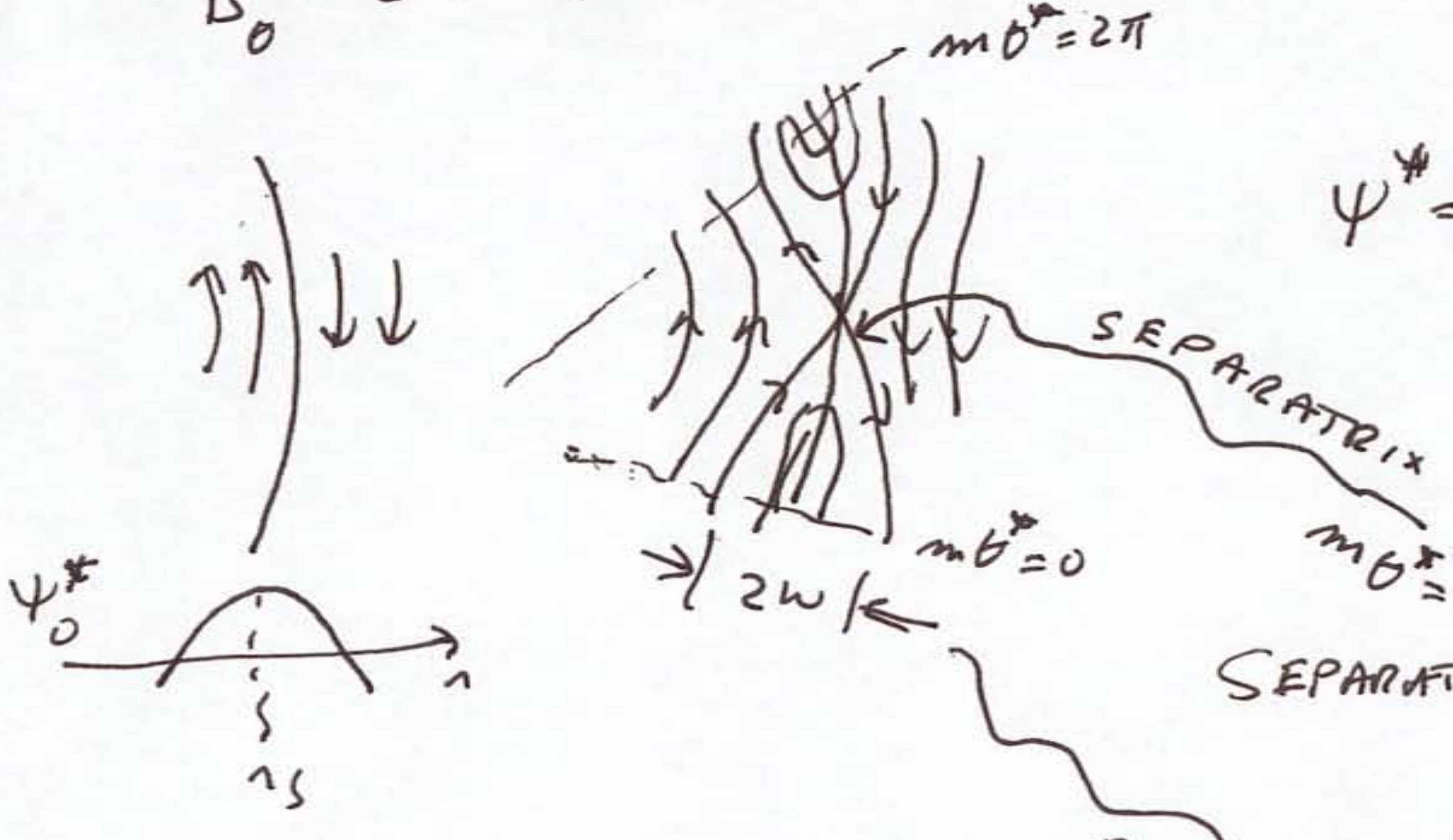
$\tilde{\psi}^*$: PERTURBED MAGNETIC FLUX

IF $\tilde{\psi}^* \sim A\cos(m\theta^*)$, THEN

WHAT ARE THE MAGNETIC FLUX SURFACES?

$B_0^\theta = \hat{z} \times \nabla\psi^*$

$\psi^* =$ CONSTANT DEFINE MAGNETIC FLUX SURFACES



$\underline{\delta r \equiv r - r_s}$

$$\psi^* = \text{CONSTANT} = -\frac{\delta r^2}{2}\frac{B_z}{Rq}s + A\cos(m\theta^*)$$

$$\delta r^2 \frac{B_z s}{Rq} = \text{CONSTANT} + 2A\cos(m\theta^*)$$

SEPARATRIX $\Rightarrow$ $$\delta r^2 \frac{B_z s}{Rq} = 2A(1+\cos(m\theta^*))$$

MAX ISLAND WIDTH

$$w^2 \frac{B_z s}{Rq} = 4A$$

VERY IMPORTANT

$$w = \pm\sqrt{\frac{4\hat{\psi}\, Rq}{B_z s}}$$

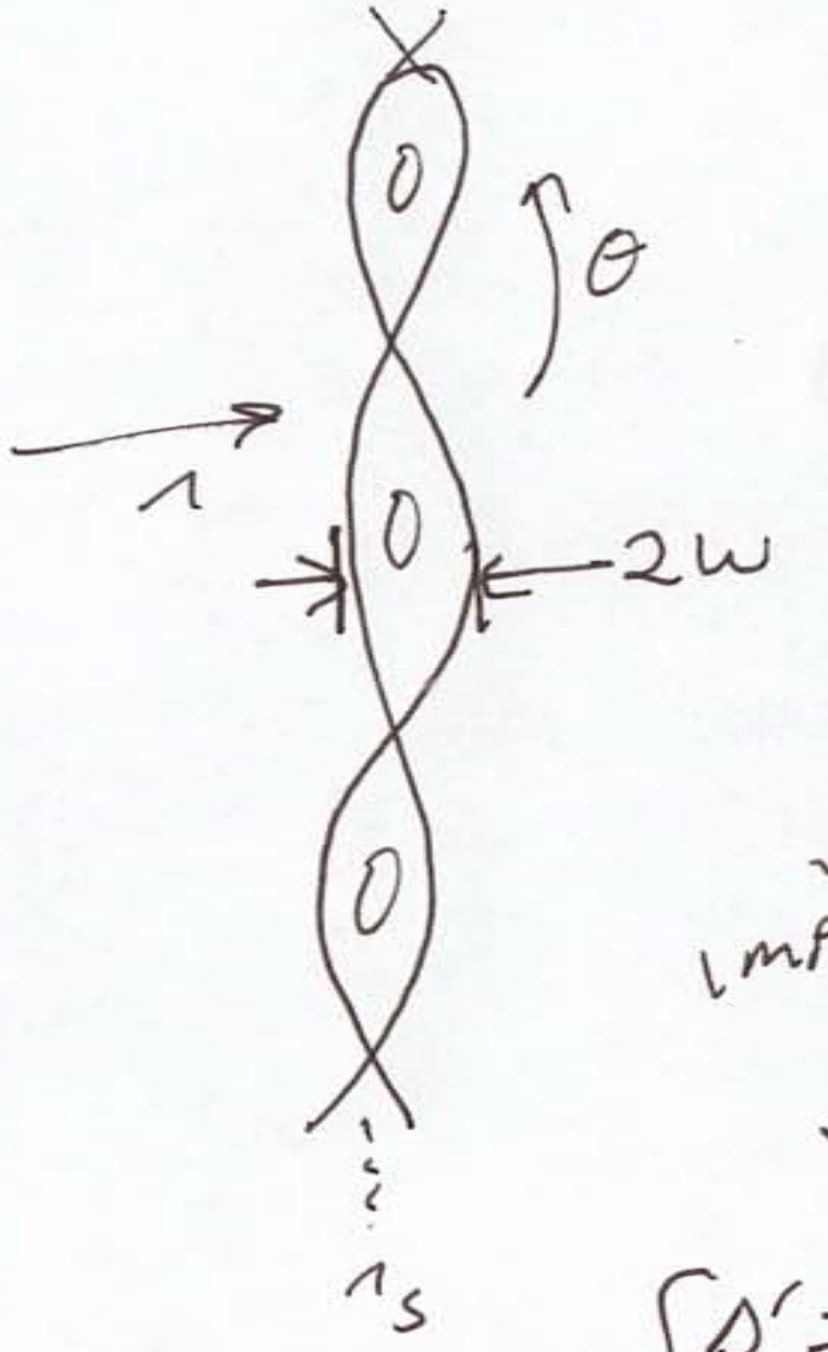
Do Magnetic Islands Grow?

IF WE ASSUME THAT THE PLASMA MOTION IS QUASI-STATIC AS THE ISLAND EVOLVES, THEN INDUCTION EQUATION IS

$$\frac{\partial \bar{B}}{\partial t} = \nabla\times(\underbrace{v\times B}_{\simeq 0}) - \nabla\times\left(\frac{\eta}{\mu_0}\nabla\times B\right)$$

OR

$$\frac{d\psi}{dt} \simeq \frac{\eta}{\mu_0}\nabla^2\psi$$ ← MAGNETIC DIFFUSION



INTEGRATE INDUCTION ACROSS ISLAND

$$\int_{r_s-w}^{r_s+w} 2\pi r\,dr\,\frac{d\psi}{dt} \simeq \frac{\eta}{\mu_0}\int_{r_s-w}^{r_s+w} 2\pi r\,dr\,\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right)$$

$$2\pi r_s\, 2w\,\frac{d\bar\psi}{dt} = \frac{\eta}{\mu_0}\, 2\pi r_s \left(\left.\frac{\partial\psi}{\partial r}\right|_{r_s+w} - \left.\frac{\partial\psi}{\partial r}\right|_{r-w} \right)$$

CALLED "RUTHERFORD REGIME"

$$\boxed{\frac{d\bar\psi}{dt} = \frac{\eta}{\mu_0}\frac{\bar\psi\,\Delta'}{2w}}$$

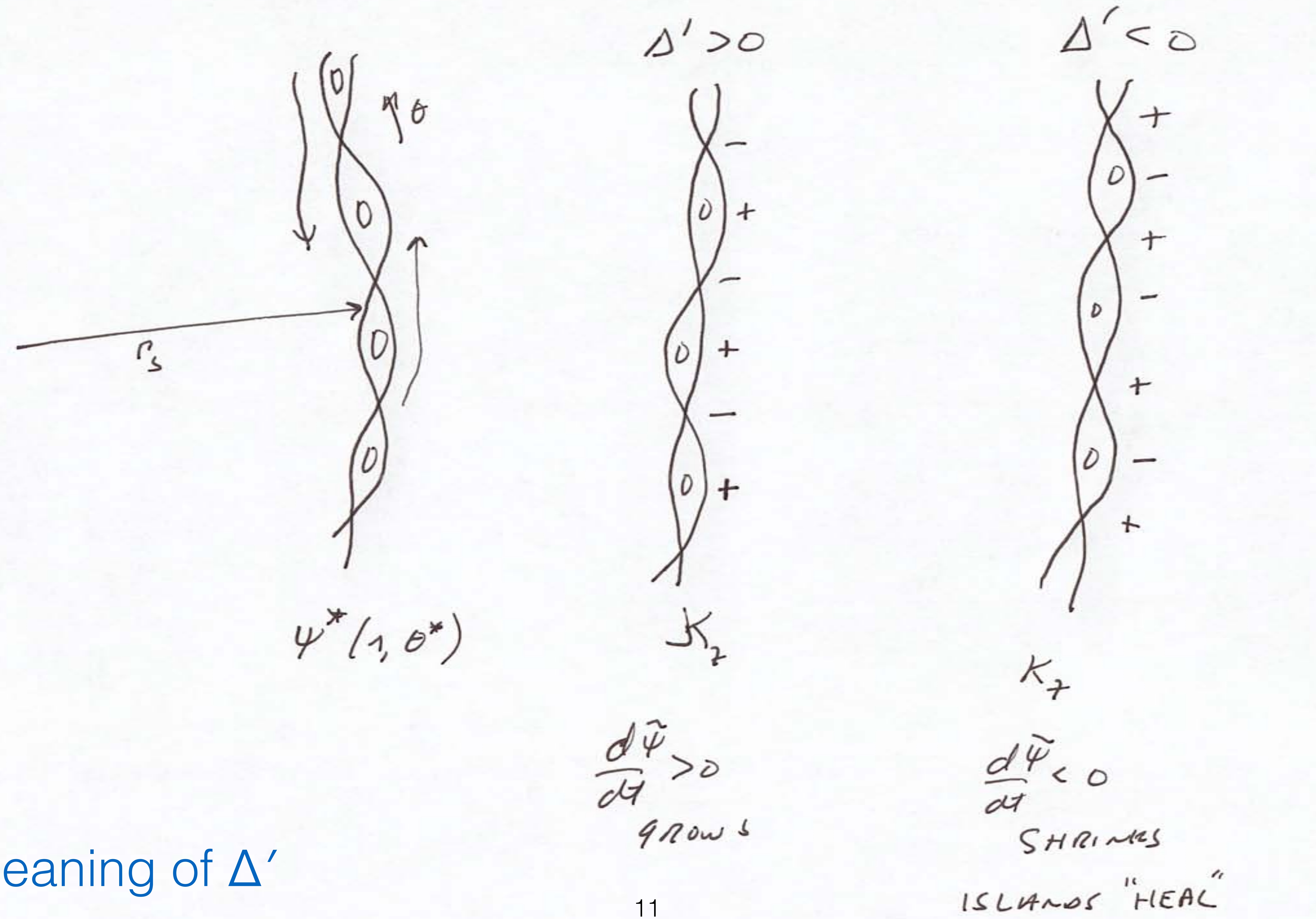
VERY IMPORTANT
↓

IF $\begin{cases}\Delta' > 0 & \text{GROWS}\\ \Delta' < 0 & \text{SHRINKS}\end{cases}$

RECALL: $\mu_0 \tilde{K}_z(\theta,\varphi) = r_s\Delta'\tilde{\psi}_s$

$w \propto \sqrt{\psi}$

Helical Current at Resonant Surface



The meaning of Δ′

What is Δ'?

INSIDE ISLAND REGION, FLUX EVOLVES AS

$$\frac{d\tilde{\Psi}}{dt} \simeq \frac{\eta}{\mu_0}\frac{\tilde{\Psi}\Delta'}{2w} \; ; \quad \mu_0 \tilde{K}_z(\theta,\varphi) \sim r_s \Delta' \tilde{\Psi}$$

OUTSIDE ISLAND REGION, PLASMA IS A QUASI-EQUILIBRIUM (JUST AS FOR THE PLASMA RESPONSE TO KINK MODES)

ALFVENIC RESPONSE (PLASMA INERTIA) IS IGNORED

$$\rho \frac{d}{dt}\nabla_\perp^2 \chi \to 0 \simeq (\bar{B}\cdot\bar{\nabla})\nabla_\perp^2 \Psi \quad (\text{i.e. } \nabla\times(J\times B) \simeq 0)$$

A FORCE FREE RESPOND

LINEAR RESPONSE: $(\bar{B}\cdot\bar{\nabla})\nabla^2\tilde{\Psi} = 0$

THIS IS NON LINEAR

THIS IS LINEARIZED

$$\frac{B_p}{r}(m - nq)\nabla^2\tilde{\Psi} + (\hat{z}\times\nabla\tilde{\Psi})\cdot\bar{\nabla}_{\!/\!B} J_z(\text{EQUIL}) \simeq 0$$

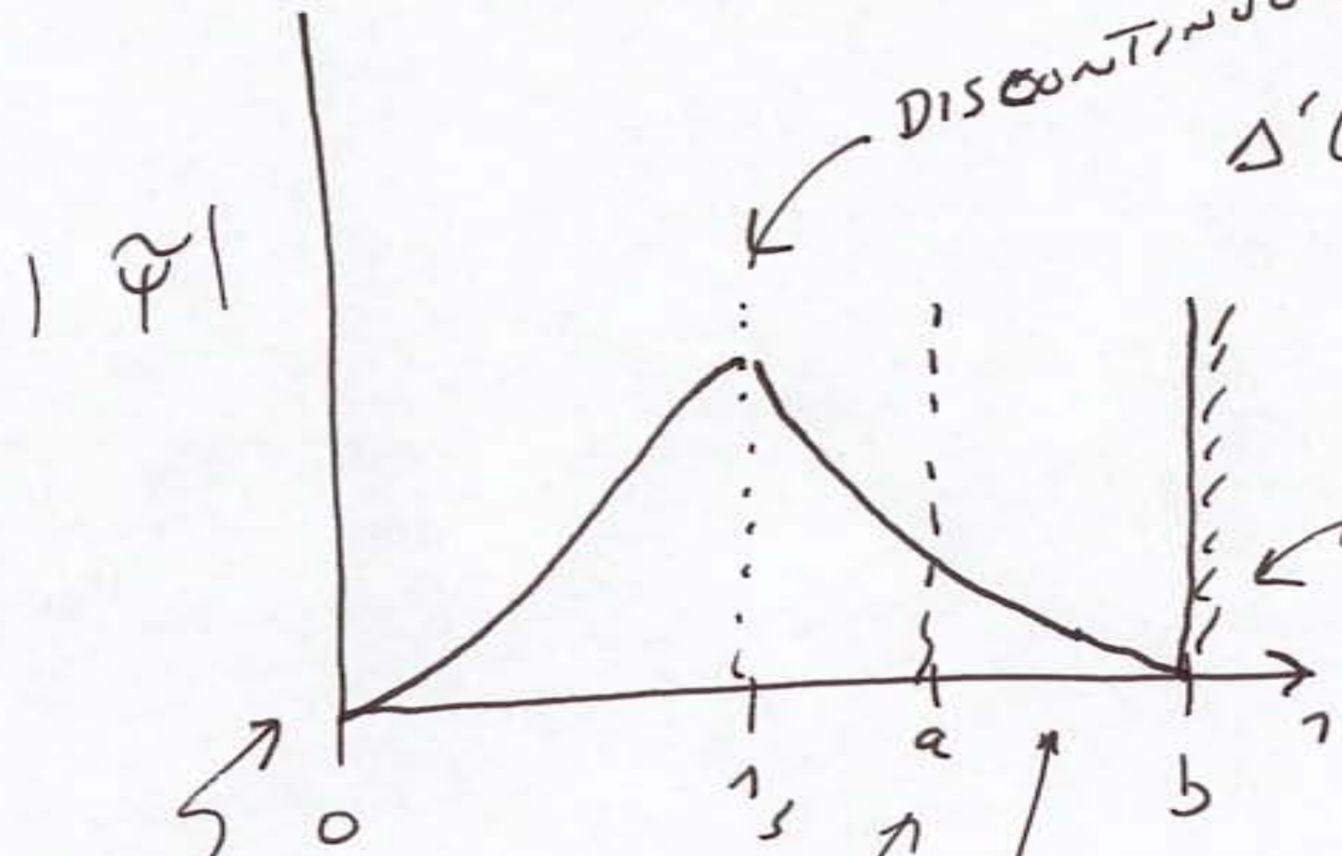
THIS IS THE SAME EQUATION WE SOLVED FOR KINK MODES!

OR $$\frac{B_p}{\mu_0 r}(m - nq(r))\nabla_\perp^2\tilde{\Psi} - \frac{m}{r}\frac{\partial J_z}{\partial r}\tilde{\Psi} = 0$$

Boundary Conditions on ψ(r,θ,φ)

DISCONTINUOUS GRADIENT

$$\Delta'(r_s) = \frac{1}{\psi}\left(\left.\frac{\partial \psi}{\partial r}\right|_{r_s+w} - \left.\frac{\partial \psi}{\partial r}\right|_{r_s-w} \right)$$



WALL CAN STABILIZE TEARING MODES

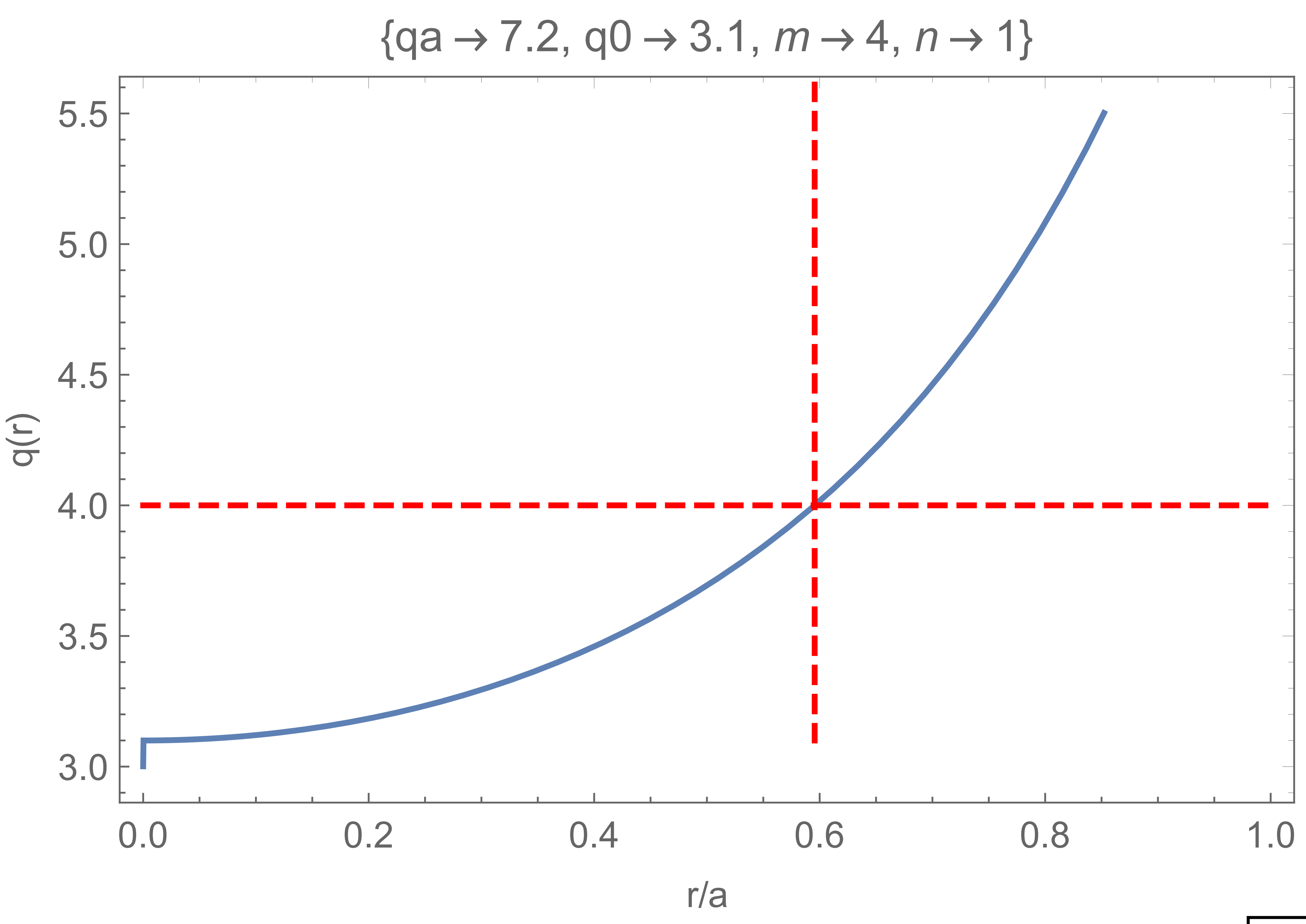
$m>0$ IMPLIES NO PERTURBED FLUX AT AXIS

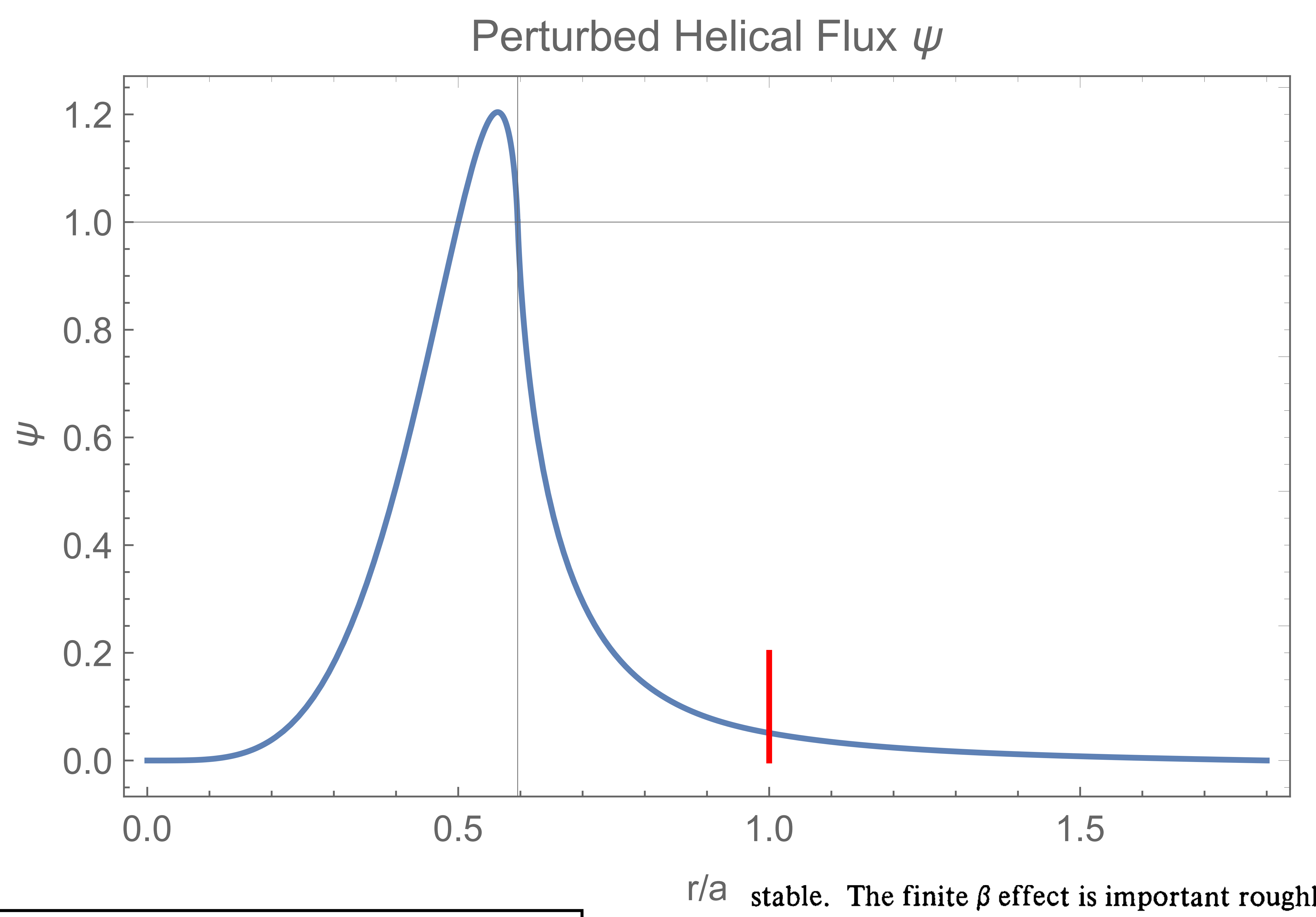
NO PERTURBED SURFACE CURRENT AT PLASMA BOUNDARY $\Delta'(a)=0$

VACUUM $\tilde{\psi}$

PLASMA INNER CURRENT GRADIENT DRIVES TEARING MODES

THIS IS EASILY SOLVED JUST AS WE SOLVED FOR THE KINK MODE RESPONSE...





See Kink_Mode_Plasma-II.nb

HYDROMAGNETIC TOKAMAK STABILITY

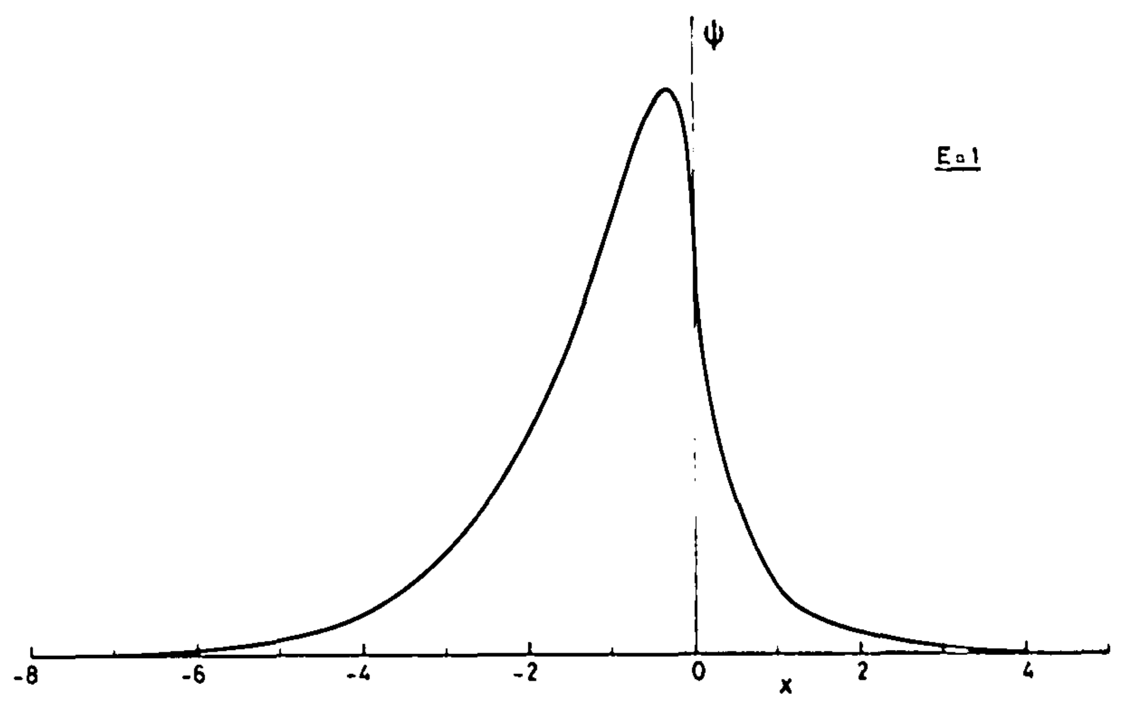


FIG.33. The fundamental eigenfunction for the radial perturbed magnetic field, ψ, for large-m tearing modes at marginal stability (E=1).

stable. The finite β effect is important roughly when $\beta_p \gtrsim (R/a)^{12/5}\, S^{-2/5}$ where S is the ratio of the magnetic field diffusion time σa^2 to the Alfvén transit time $a\sqrt{\rho}/B_\varphi$. It is, therefore, possible to identify an intermediate range of aspect ratios $1 \ll R/a \lesssim \beta_p^{5/12}\, S^{1/6}$ where the finite β effects play a role and a large-aspect-ratio theory is applicable.

In the very-large-aspect-ratio case the stability criterion for tearing modes is

$$\Delta < 0$$

where

$$\Delta = r_s \lim_{\epsilon \to 0} \left[\left(\frac{\psi_+'}{\psi_+}\right)_{r=r_s+\epsilon} - \left(\frac{\psi_-'}{\psi_-}\right)_{r=r_s-\epsilon} \right]$$

(Summary) Tearing Modes are Critically Important for Tokamaks

Locking of TMs…

Neoclassical Tearing Modes…

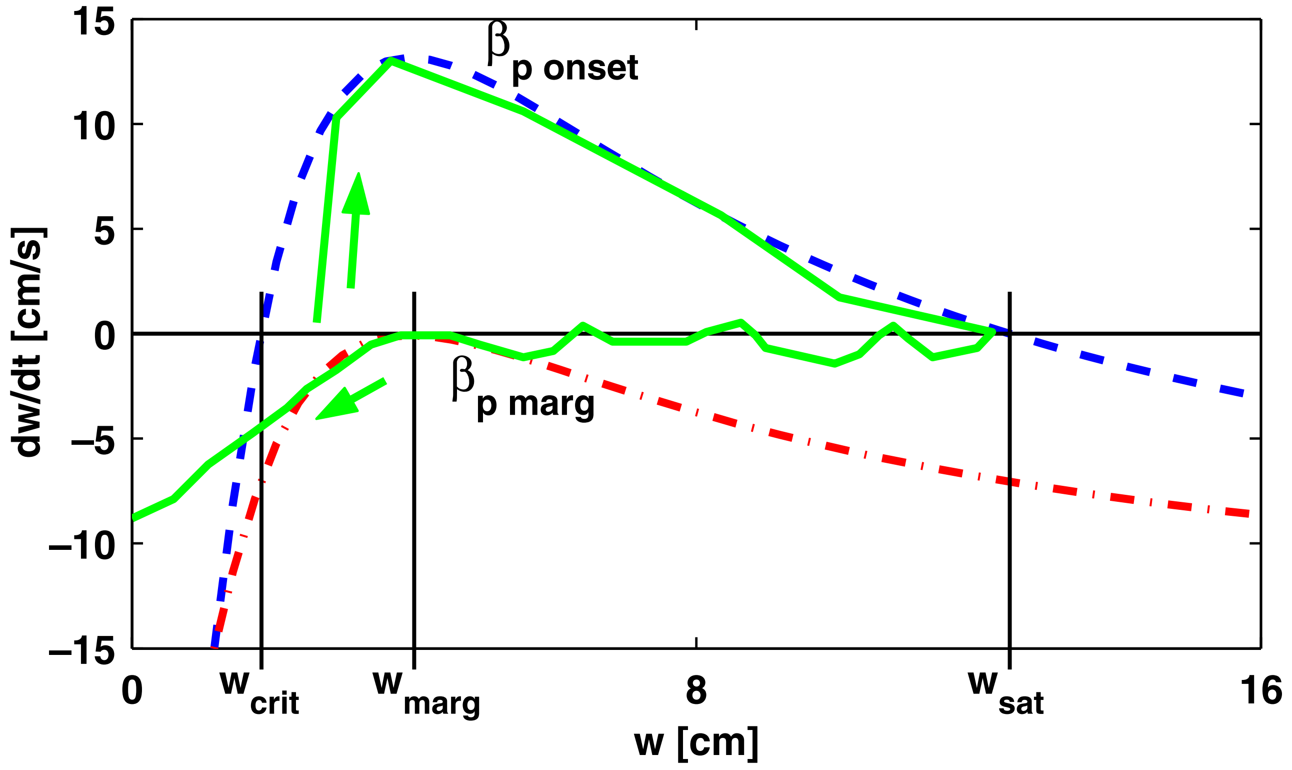


Figure 3. Sketch of the time evolution of the island growth rate as given by equation (6) at the onset of the NTM when the critical seed island size ($W_{\rm crit}$) is exceeded and an NTM forms at $\beta_{\rm p,onset}$. A slow decrease in beta from $\beta_{\rm p,onset}$ to $\beta_{\rm p,marg}$ (when $\max(\mathrm{d}W/\mathrm{d}t) = 0$) is assumed, as in power ramp-down experiments, such that $\mathrm{d}W/\mathrm{d}t \approx 0$ (reproduced from [54] 'Marginal β-limit for neoclassical tearing modes in JET H-mode discharges').

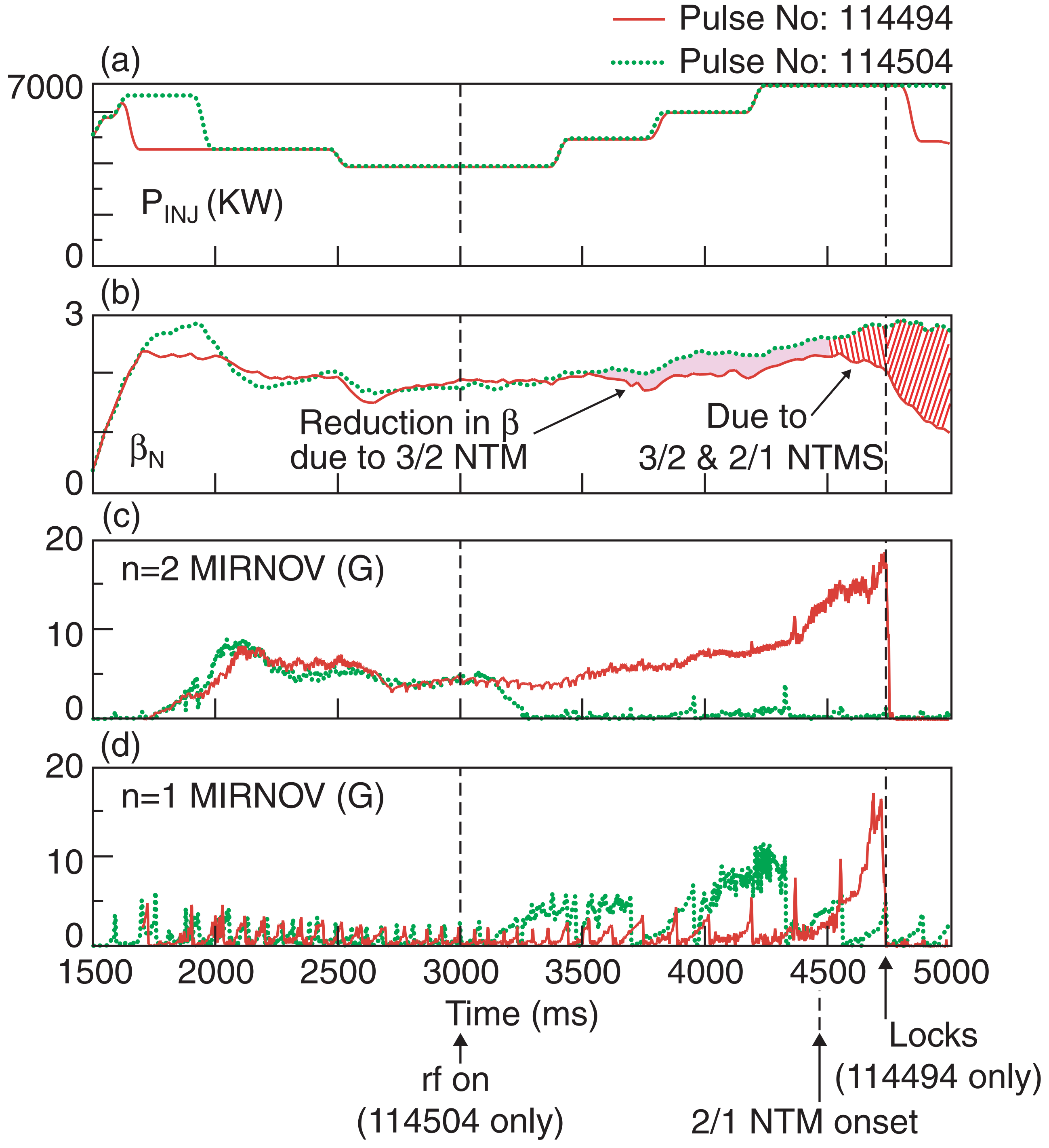


Figure 2. DIII–D discharges with (114504, dotted lines) and without (114494, solid lines) ECCD suppression of an $m/n = 3/2$ neoclassical tearing mode. (a) Neutral beam power, (b) β_N, (c) $n = 2$ Mirnov $|\tilde{B}_\theta|$, (d) $n = 1$ Mirnov $|\tilde{B}_\theta|$. The degradation in energy confinement due to the NTM from 3/2 and 2/1 NTMs can be seen in the effect on β_N.

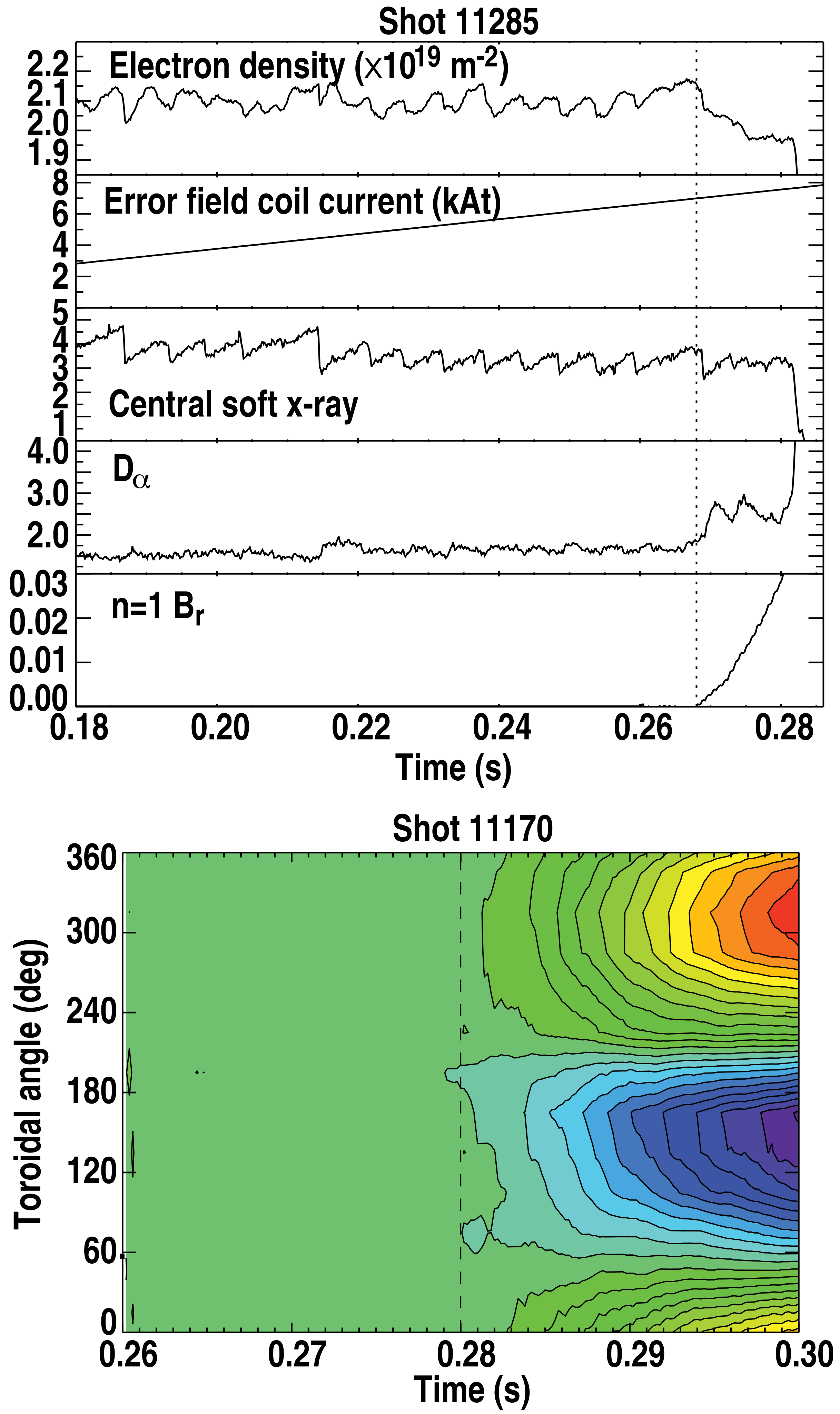


FIG. 2. Error field penetration and locked mode onset in MAST. A slowly increasing $n = 1$ field [second panel of (a)] leads to the sudden onset of an $n = 1$ instability [last panel of (a) and contour plot of δB_r in (b)]. Note that (a) and (b) come from different discharges with slightly different timing. Reprinted with permission from Howell *et al.*, Nucl. Fusion **47**, 1336 (2007). Copyright 2007 International Atomic Energy Agency, Institute of Publishing.[14]