Plasma 2
Lecture 23:
More Kink Mode Modeling in Strong Axial Field

APPH E6102y
Columbia University

Review

$\bar{B} = B_z \hat{z} + \bar{B}_\perp$

B_z = LARGE AND CONSTANT ($\beta_p = 1$)

$B_\perp = \hat{z} \times \nabla(\psi_0 + \tilde{\psi})$

↑ EQUILIBRIUM POLOIDAL FLUX

MHD $\quad \rho \frac{du_\perp}{dt} = -\nabla p + J \times B$

MAXWELL'S EQ $\quad \mu_0 \bar{J} = \nabla \times \bar{B}$

$$\frac{\partial \bar{B}}{\partial t} = -\nabla \times E$$

OHM'S LAW CONTAINS CRITICALLY IMPORTANT ELECTRON PHYSICS

PLASMA FLOW

OHM'S LAW

IDEAL: $\quad E + V \times B \equiv 0$ — KINKS, INTERCHANGE, BALLOONING

ELECTRON FLOW

TWO FLUID DRIFT $\quad E + u_e \times B = -\frac{1}{en} \nabla p_e$ — DRIFT WAVES

RESISTIVE $\quad E + U \times B = \eta J$ — TEARING MODES

STILL VERY VERY SMALL $\Rightarrow$ CAUSES SLOW MAGNETIC DIFFUSION

Review

WHEN $E + v \times B = 0$

$$\bar{E} \cdot \bar{B} = 0$$

FARADAY'S LAW BECOMES

$$\frac{\partial B_\perp}{\partial t} + (V_\perp \cdot \nabla) B_\perp = (B \cdot \nabla) v_\perp$$

OR

$$\boxed{\frac{d\psi}{dt} = (B \cdot \nabla) \chi}$$

$$\bar{V}_\perp = \hat{z} \times \nabla \chi$$

WHEN $E + V_e \times B = -\frac{1}{en} \nabla P_e$

$$\bar{E} = -\nabla \phi + \hat{z} \frac{\partial \psi}{\partial t}$$

THEN B · (OHM'S LAW) BECOMES

$$B \cdot \left(-\nabla \phi + \hat{z} \frac{\partial \psi}{\partial t} \right) = - \frac{B \cdot \nabla P_e}{en}$$

OR

WHEN $\frac{\partial \psi}{\partial t}$ IS SLOW ?

$$\frac{\partial \psi}{\partial t} = B \cdot \left(\frac{\nabla \phi}{B_z} - \frac{1}{en} \nabla P_e \right)$$

NOTE: $V_\perp \sim \frac{E \times B}{B^2} \sim \frac{1}{B} \hat{z} \times \nabla \phi$

SO FOR MHD PERPENDICULAR MOTION

$$\bar{V}_\perp = \hat{z} \times \nabla \chi$$

$$\bar{V}_\perp = \frac{1}{B_z} \hat{z} \times \nabla \phi$$

$$\boxed{\chi = \frac{\phi}{B_z}}$$

IN OUR REDUCED MHD

TWO EQUATIONS

(MOMENTUM)

(INDUCTION)

IN TWO-FLUID (REDUCED DYNAMICS)

THREE EQUATIONS

(MOMENTUM)

(INDUCTION) AND CONTINUITY $\boxed{\frac{\partial n}{\partial t} = -V \cdot \nabla n}$

Cylindrical Reduced MHD

the order of $\epsilon^2 B_0$. To lowest order in ϵ this unknown variation of the toroidal field can be eliminated from the problem by taking the curl of the momentum equation. The resulting equations are the standard low-β tokamak reduced equations that describe free-boundary kink modes[3]:

$$R_0^2 \frac{d\nabla^2 u}{dt} = \boxed{\mathbf{B} \cdot \nabla} (\nabla_1^2 \psi),$$

$$\frac{\partial \psi}{\partial t} = R_0^2 \boxed{\mathbf{B} \cdot \nabla} u,$$

important

$$\mathbf{B} = \nabla \psi \times \nabla \zeta + I_0 \nabla \zeta,$$

$$\mathbf{V} = R_0^2 \nabla u \times \nabla \zeta,$$

$$\nabla_1^2 = \frac{\partial^2}{\partial R^2} + \frac{\partial^2}{\partial z^2}.$$

Here $I_0 = B_0 R_0$ and $\nabla \zeta = \hat{\zeta} / R_0$.

$$A_\phi = I_0(r/2)$$

$$A_\| \approx A_z = \psi_0(r) + \tilde{\psi}(r, \phi)$$

Plasma Physics Series

Tokamak Plasma:

A Complex Physical System

B B Kadomtsev

I V Kurchatov Institute of Atomic Energy, Moscow, Russia

Translation Editor: **Professor E W Laing**

“In memory of Boris Kadomtsev,” by E P Velikhov et al 1998, *Phys.-Usp.* **41** 1155; [https://doi.org/10.1070/PU1998v041n11ABEH000508]

Institute of Physics Publishing

Bristol and Philadelphia

"Simplest" Kink Mode Theory

- Reduced MHD (plasma torus with a strong toroidal field)
- Kink modes

$$\frac{d}{dt}\psi = (\bar{B}\cdot\bar{\nabla})\chi$$

"POLOIDAL FLUX EVOLVES DYNAMICALLY DUE TO FIELD-ALIGNED CHANGES IN THE STREAM FUNCTION"

$$\rho\frac{d}{dt}\Omega_z = (\bar{B}\cdot\bar{\nabla})J_z$$

"AXIAL VORTICITY CHANGES ACCORDING TO FIELD-ALIGNED VARIATION OF AXIAL CURRENT"

Importance of B·∇

$$\rho\frac{d}{dt}\nabla_\perp^2\chi = \frac{1}{\mu_0}(B\cdot\bar{\nabla})\nabla_\perp^2\psi \qquad \text{(MHD)}$$

$$\frac{d}{dt}\psi = (\bar{B}\cdot\bar{\nabla})\chi \qquad \text{(INDUCTION)}$$

LINEAR

$$= i\bar{k}\cdot\bar{B}_0 \qquad \bar{k} \equiv -\frac{n}{R}\hat{z} + \frac{m}{r}\hat{\theta} \quad \text{(PLUS RADIAL TERMS)}$$

$$\bar{B}\cdot\bar{\nabla} = B_z\frac{\partial}{\partial z} + B_\perp\cdot\bar{\nabla}_\perp = -i\frac{n}{R}B_z + i\frac{m}{r}B_p = i\,\frac{B_p(r)}{r}\left(m - nq(r)\right)$$

$$\text{WITH } q(r) = \frac{r\,B_z}{R\,B_p(r)} = \text{SAFETY FACTOR}$$

$\bar{B}\cdot\bar{\nabla} \to 0$ WHEN $m/n = q(r)$ (RESONANCE)

WHEN $\bar{B}\cdot\bar{\nabla} \neq 0$, THEN IDEAL REDUCED MHD MAKE SENSE

$B\cdot\bar{\nabla} = 0$, THEN REDUCED MHD DOES NOT DESCRIBE DYNAMICS

(SIDE BAR! $\bar{B}\cdot\bar{\nabla} = 0$ DEFINES "INTERCHANGE" MODES. THESE ARE THE DOMINANT MODES IN MAGNETOSPHERES AND DIPOLES, ETC)

Linearized Reduced MHD

MHD: $\rho \frac{d}{dt} \nabla_\perp^2 \chi = \frac{1}{\mu_0} (\bar{B}\cdot\nabla) \nabla_\perp^2 \psi$

$\frac{d}{dt} = -i\omega$

INDUCTION: $\frac{d\psi}{dt} = (B\cdot\nabla)\chi$

$B_0\cdot\nabla \rightarrow i \frac{B_\theta(r)}{r} (m - n q(r))$

$$\boxed{(\bar{B}\cdot\bar{\nabla}) \nabla_\perp^2 \psi = (\tilde{B}\cdot\nabla) \underbrace{\nabla_\perp^2 \psi_0}_{\mu_0 J_z(r)} + (\bar{B}_0\cdot\bar{\nabla}) \bar{\nabla}_\perp^2 \tilde{\psi}}$$ + NONLINEAR TERMS

$\mu_0 J_z(r)$ ← WHEN EQUILIBRIUM CURRENT DENSITY VARIES WITHIN PLASMA

$$\tilde{B}\cdot\nabla J_z(r) = (\hat{z}\times\nabla\psi)\cdot \hat{r}\frac{\partial J_z}{\partial r} = -\hat{\theta}\cdot\nabla\psi \frac{\partial J_z^0}{\partial r}$$

LINEAR MHD:

$$\boxed{\underbrace{-\rho\omega \nabla_\perp^2 \tilde{\chi}}_{\text{PLASMA INERTIA}} = -\frac{m}{r}\frac{\partial J_z}{\partial r}\tilde{\psi} + \frac{B_\theta}{\mu_0 r}(m - nq(r)) \nabla_\perp^2 \tilde{\psi}}$$

LINEAR INDUCTION:

$$-\omega\tilde{\psi} = \frac{B_\theta}{r}(m - nq(r))\tilde{\chi}$$

Alfvén Waves in Shafranov's Equilibrium (q(r) = q_a = constant)

WITHIN PLASMA

$$-\rho\omega \nabla_\perp^2 \hat{\chi} = \frac{B_\theta}{\mu_0 r}(m - nq) \nabla_\perp^2 \hat{\psi}$$

$$-\omega\hat{\psi} = \frac{B_\theta}{\mu_0 r}(m - nq)\chi$$

NOTE: $\frac{B_\theta}{r}$ = CONSTANT = $\frac{B_\theta(a)}{a}$

$$\omega_A^2 \equiv \frac{B_\theta^2/\mu_0\rho}{a^2} = \frac{B_z^2/\mu_0\rho}{(q_a R)^2}$$

$= V_A^2/(qR)^2$ ALFVEN TRANSIT TIME!

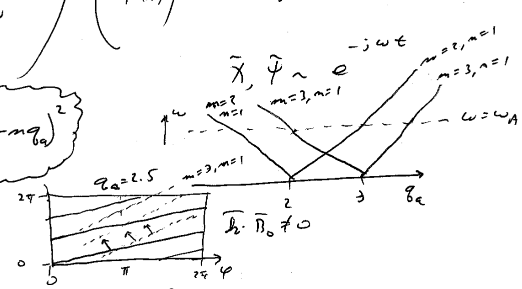
NORMAL MODES (ALFVEN WAVES)

$$\begin{pmatrix} \rho\omega & \frac{B_\theta}{\mu_0 a}(m - nq) \\ \frac{B_\theta}{\mu_0 a}(m - nq) & \omega \end{pmatrix} \cdot \begin{pmatrix} \tilde{\chi}(r) \\ \tilde{\psi}(r) \end{pmatrix}$$ WITH $\rho \simeq$ CONSTANT

$\tilde{\chi}, \tilde{\psi} \sim e^{-j\omega t}$

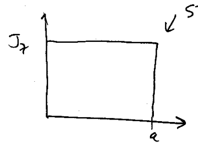
$$\omega^2 = \omega_A^2 (m - nq_a)^2$$

RADIAL STRUCTURE NOT SPECIFIED



Global Kink Eigenmodes

$$\nabla_\perp \cdot \vec{A} = \frac{1}{r}\frac{\partial}{\partial r}(r A_r) + \cdots$$



STEP DISCONTINUITY AT PLASMA EDGE

$$\frac{d}{dt}\nabla_\perp \cdot (\rho \nabla_\perp \chi) = (\hat{z} \times \nabla_\perp \tilde{\psi}) \cdot \hat{r}\, \frac{\partial J_z}{\partial r} + i \frac{B_\theta}{\mu_0 r}(m - nq)\nabla_\perp^2 \tilde{\psi}$$

VERY BIG AT EDGE

$$\int_{a^-}^{a^+} 2\pi r\, dr \left\{ \text{“ MHD ”} \right\} \Rightarrow$$

$$\omega \rho \frac{\partial \chi}{\partial r}\Big|_{a^-} = \frac{2m B_\theta(a)}{\mu_0 a^2}\tilde{\psi}_a + \frac{B_\theta(a)}{\mu_0 a}(m - nq)\tilde{\psi}_a \underbrace{\left[\frac{1}{\tilde{\psi}}\frac{\partial \tilde{\psi}}{\partial r}\Big|_{a^+} - \frac{1}{\tilde{\psi}}\frac{\partial \tilde{\psi}}{\partial r}\Big|_{a^-} \right]}$$

$$\boxed{\Delta'(a) \equiv \frac{1}{\psi}\left(\frac{\partial \tilde{\psi}}{\partial r}\Big|_{a^+} - \frac{\partial \psi}{\partial r}\Big|_{a^-} \right)}$$

THIS MEASURES PERTURBED SURFACE CURRENT ON PLASMA

IMPORTANT:

$$\omega \rho \frac{\partial \chi}{\partial r}\Big|_{a^-} = \frac{2m B_\theta(a)}{\mu_0 a^2}\tilde{\psi}_a \left[(m - nq)\left(\frac{-\Delta'(a)}{2m/a} \right) - 1 \right]$$

$$\Delta'(a)\,\tilde{\psi}_a = \mu_0 \tilde{K}_z(\theta, \phi)/a$$

Global Kink Eigenmodes

WHAT ARE $(\psi(r), \chi(r))$ INSIDE AND OUTSIDE PLASMA?

Boundary conditions

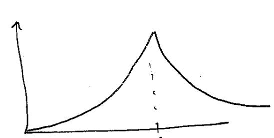
$\nabla^2 \psi = 0$ (NO CURRENT) OUTSIDE PLASMA

$\nabla^2 \chi = 0$ (NO FLOW, VORTICITY, NO SHEAR)

$\nabla^2 \psi = 0$ (NO CURRENTS INSIDE PLASMA TOO)

$\nabla^2 \chi = 0$ (NO VORTICITY WITHIN PLASMA)

PERTURBED FIELDS + PLASMA MOTION BUT NO CURRENTS OR VORTICITY



NO WALL		WITH WALL	
$\psi(r) \sim \left(\frac{r}{a}\right)^m$	$r < a$	$\sim \left(\frac{r}{a}\right)^m$	$r < a$
$\sim \left(\frac{a}{r}\right)^m$	$r > a$	$\sim \dfrac{\left(\frac{b}{r}\right)^m - \left(\frac{r}{b}\right)^m}{\left(\frac{b}{a}\right)^m - \left(\frac{a}{b}\right)^m}$	$(a < r < b)$

$$\vec{V}_\perp = \hat{z} \times \nabla\chi = \hat{\theta}\, \frac{m}{r}\left(\frac{r}{a}\right)^m - \hat{r}\, \frac{im}{r}\left(\frac{r}{a}\right)^m \quad \text{INSIDE}$$

$$= -\hat{\theta}\, \frac{m}{r}\left(\frac{a}{r}\right)^m - \hat{r}\, \frac{im}{r}\left(\frac{a}{r}\right)^m \quad \text{OUTSIDE}$$

Kink Mode

$$-\omega\,\tilde{\varphi}_a = \frac{B_\theta}{r}(m-nq)\,\tilde{\chi}_a$$

$$\omega\rho\left.\frac{\partial\chi}{\partial r}\right|_{a^-} = \frac{2m B_\theta}{\mu_0 a^2}\,\tilde{\varphi}_a\left[(m-nq)\left(\frac{-\Delta'(a)}{2m/a}\right)-1\right]$$

$$\Delta'(a) = -\frac{2m}{a}\,\frac{(b/a)^m}{(b/a)^m-(a/b)^m}$$ FOR A WALL AT $r=b$

$$\left.\frac{\partial\chi}{\partial r}\right|_{a^-} = -\frac{m}{a}\,\tilde{\chi}_a$$

$$\frac{\tilde{\chi}_a}{\tilde{\varphi}_a} = -\frac{\omega q_a R}{B_0(m-nq_a)}$$

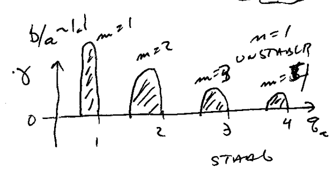
EIGENVECTOR

GLOBAL KINK MODES

EIGENVALUE

$$\omega^2 = 2\omega_A^2(m-nq)\times\left[(m-nq)\frac{(\Lambda+1)}{2}-1\right]$$

$$\begin{pmatrix} \omega & \frac{B_\theta}{a}(m-nq) \\ \frac{2mB_\theta}{\mu_0 a}\left[(m-nq)\frac{(\Lambda+1)}{2}-1\right] & \omega\rho \end{pmatrix}\begin{pmatrix}\tilde{\varphi}_a \\ \tilde{\chi}_a\end{pmatrix} = 0$$

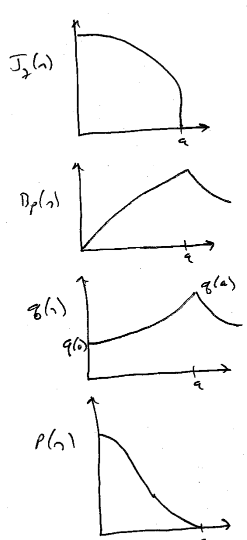


$$\boxed{\Delta'(a) = -\frac{m}{a}(\Lambda+1)}$$

SHAFRANOV'S FORMULA

Wesson's Cylindrical Equilibrium

REF: WESSON, NUC FUSION 18 (1978) P. 87



$$J_z(r) = J_0\left(1-r^2/a^2\right)^\nu$$

J_0 = CENTRAL CURRENT DENSITY

$$q(0) = \frac{2B_z}{\mu_0 R J_0} \qquad \mu_0 I_p = \frac{2\pi^2 a^2 B_z}{\mu_0 q(0) R(1+\nu_s)}$$

$$\nu = \frac{q(a)}{q(0)} - 1$$

$$\ell_i \neq \tfrac{1}{2}$$

$$\langle\beta\rangle \sim \epsilon^2/q_a^2 \qquad \beta_p \sim 1 \qquad \langle\beta_N\rangle \sim 20\,\epsilon/q_a$$

EQUILIBRIUM SET BY $q(0)$, $q(a)$

TWO PARAMETERS

$$\frac{r}{q}\frac{dq}{dr} = s$$ = MAGNETIC SHEAR ** $\neq 0$

$B\cdot\nabla \to 0$ AT INTERNAL RESONANT LAYER

$$(m - n/q(r)) \hat{=} 0$$

PRESSURE PROFILE CAN BE MORE PEAKED

Wesson's Kink Modes

Linearized equations for perturbed stream function (χ) and perturbed poloidal flux (ψ)

$$-\rho\,\omega\,\nabla_\perp^2\tilde{\chi} = -\frac{m}{r}\frac{\partial J_z}{\partial r}\tilde{\psi} + \frac{B_\theta}{\mu_0 r}(m-nq)\nabla_\perp^2\tilde{\psi} \qquad \text{(MHD)}$$

$$-\omega\,\tilde{\psi} = \frac{B_\theta}{r}(m-nq)\tilde{\chi} \qquad \text{(induction)}$$

Let's take $\rho \simeq$ uniform, with a sharp jump at the plasma's edge:

$$\omega\,\rho\,\frac{\partial\chi}{\partial r}\Big|_{a^-} = \frac{B_\theta(a)}{\mu_0 a}(m-nq_a)\,\tilde{\psi}_a\,\Delta'(a)$$

$\Delta'(a)$ ← perturbed surface current at plasma's edge

With induction equation:

$$\omega^2 = -\omega_A^2(m-nq_a)^2\,\Delta'(a)\,\frac{\chi_a}{(\partial\chi/\partial r)_{a^-}}$$

But, how to figure out $\tilde{\psi}(r)$?

Instability requires $\Delta'(a) > 0$

Wesson's Kink Modes

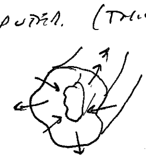
Since $|\omega| < \omega_A$, the kink mode causes the "internal" plasma to respond "quickly", so quickly that we can ignore the time it takes to form a distorted, 3D, quasi-equilibrium.

Inside, the plasma is a "forced" equilibrium

$$0 \approx -\frac{m}{r}\frac{\partial J_z}{\partial r}\tilde{\psi} + \frac{B_\theta}{\mu_0 r}(m-nq)\nabla_\perp^2\tilde{\psi}$$

Outside, the response is the "vacuum" response.

With $J_z(r)$, we have to solve for $\tilde{\psi}$ using a computer. (This is very easy for the cylindrical "tokamak")



← The surface current "pushes"/"pulls" plasma, and the "distorted" plasma is measured by $\tilde{\psi}(a, \theta, z)$

Solving the Eigenvalue Problem

Examining the properties of kink modes in the (straight) reduced MHD formalism.

See Kink_Mode_Plasma-II.nb

Cylindrical Kink

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For *Plasma II*