Plasma 2
Lecture 22:
More Reduced MHD (and more equilibrium)

APPH E6102y

Columbia University

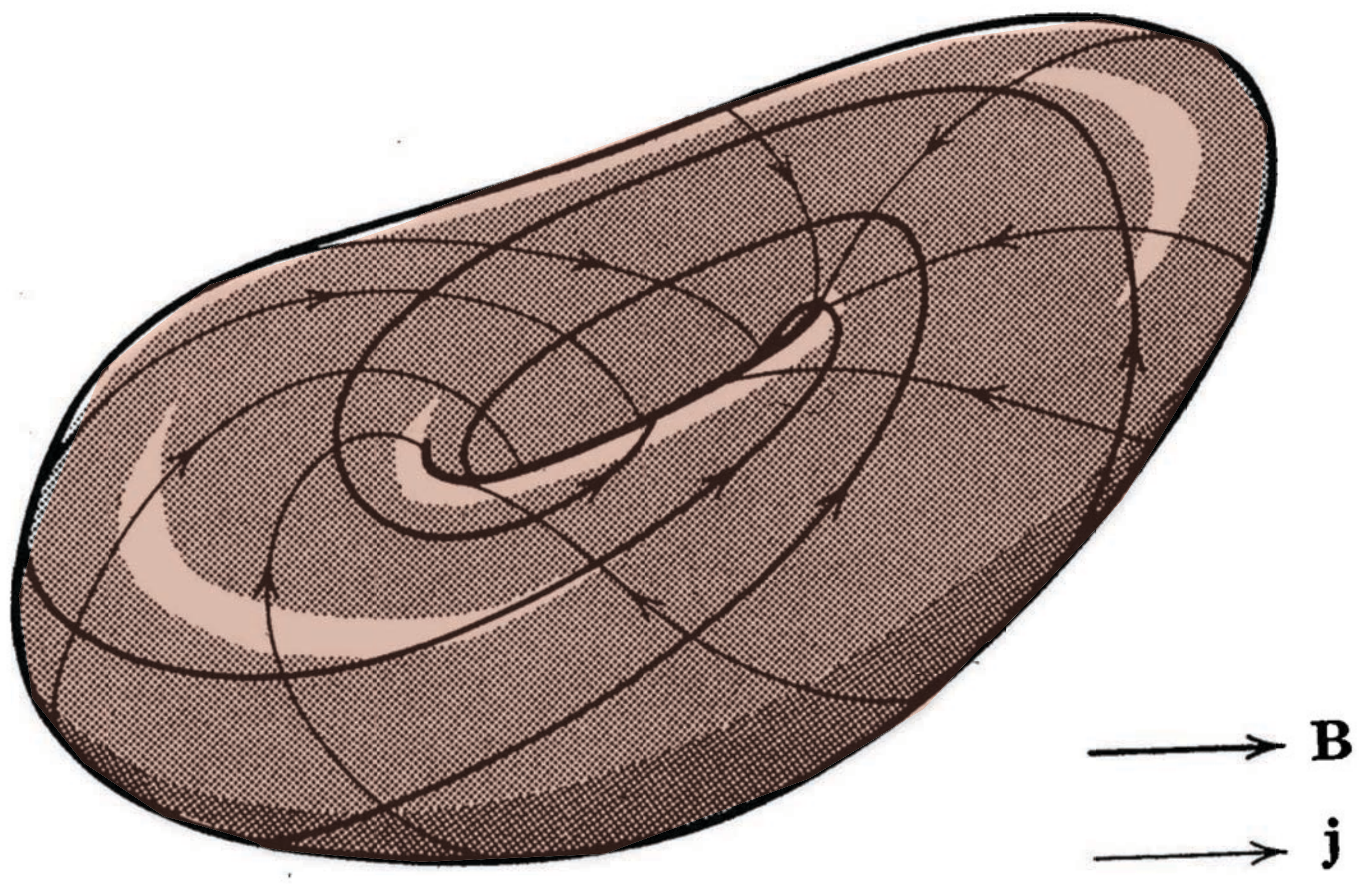
Toroidal Magnetic Confinement (and Instabilities)

Plasma Pressure

Plasma Current

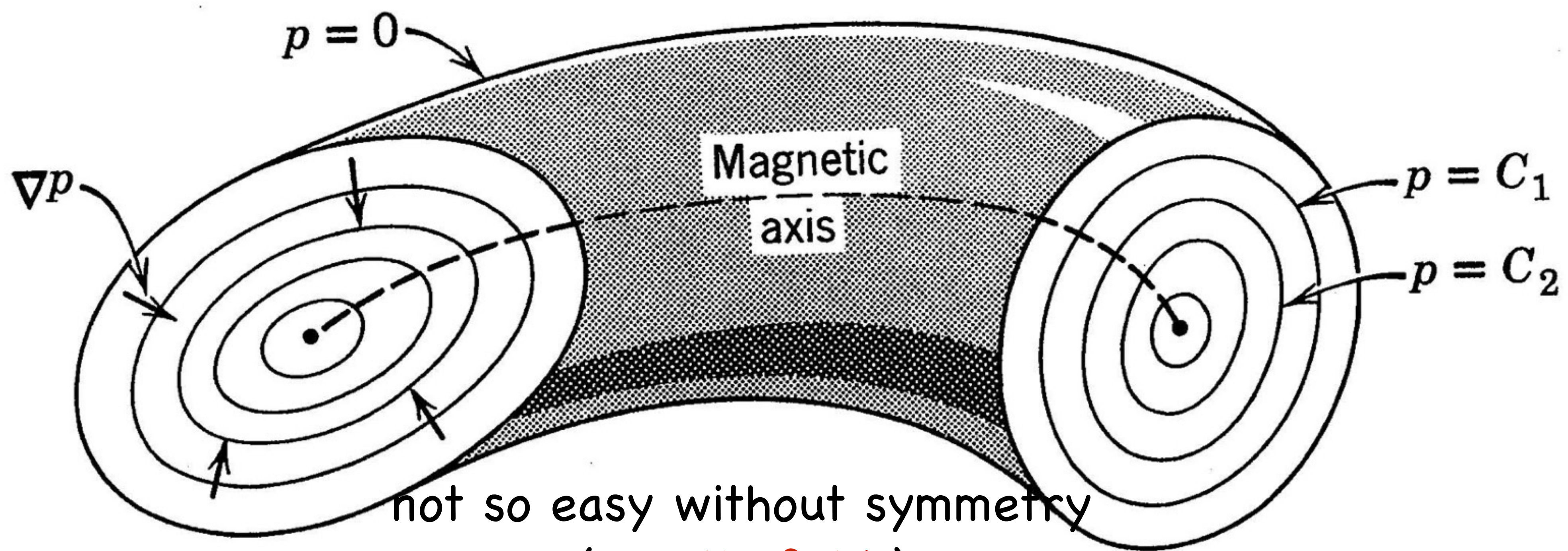
$$
\begin{aligned}
\text{(No monopoles)} \quad \nabla \cdot \mathbf{B} &= 0 \\
\text{(No charge accumulation)} \quad \nabla \cdot \mathbf{J} &= 0 \\
\text{(No unbalanced forces)} \quad 0 &= -\nabla P + \mathbf{J} \times \mathbf{B} \\
\text{(Magnetostatics)} \quad \nabla \times \mathbf{B} &= \mu_0 \mathbf{J}
\end{aligned}
$$

Magnetic Torus



$$
\begin{aligned}
\mathbf{J} \times \mathbf{B} &= \nabla P \\
\mathbf{B} \cdot \nabla P &= 0 \\
\mathbf{J} \cdot \nabla P &= 0
\end{aligned}
$$

Surfaces of constant plasma pressure form nested tori



not so easy without symmetry

(chaotic fields)

Tokamak Equilibrium

Review paper NUCLEAR FUSION 11 (1971)

PLASMA EQUILIBRIUM IN A TOKAMAK

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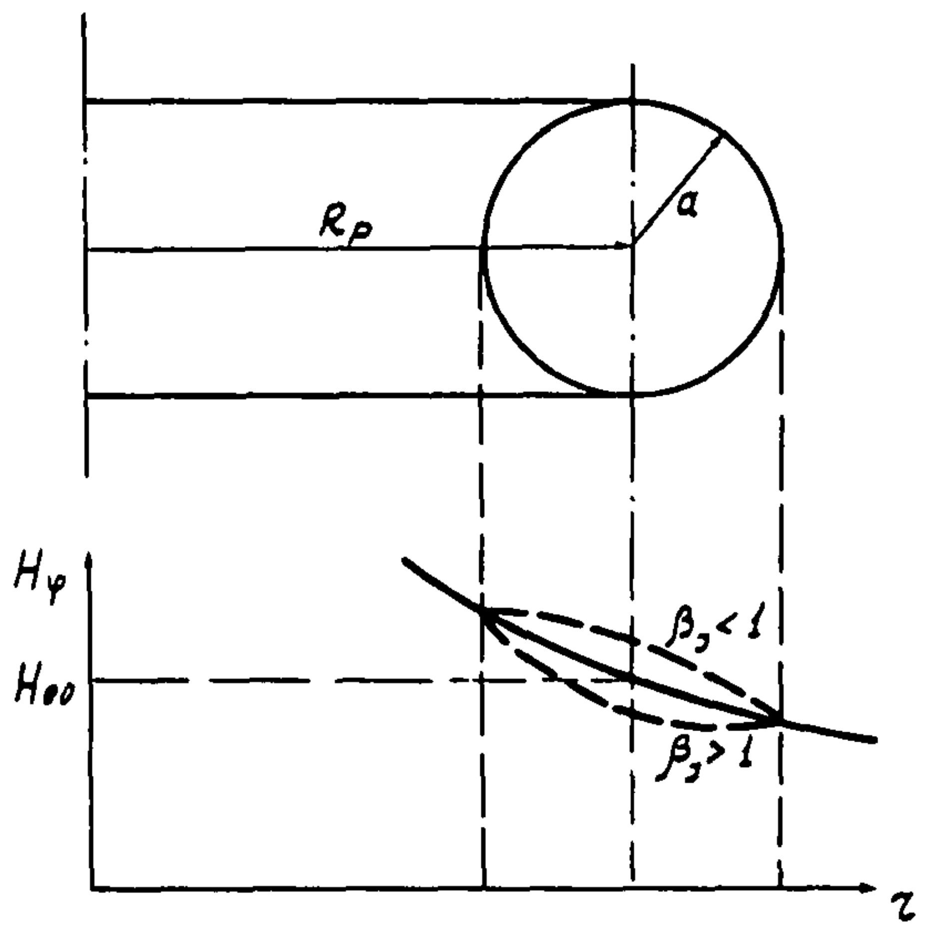


FIG.1. Distribution of a toroidal magnetic field.

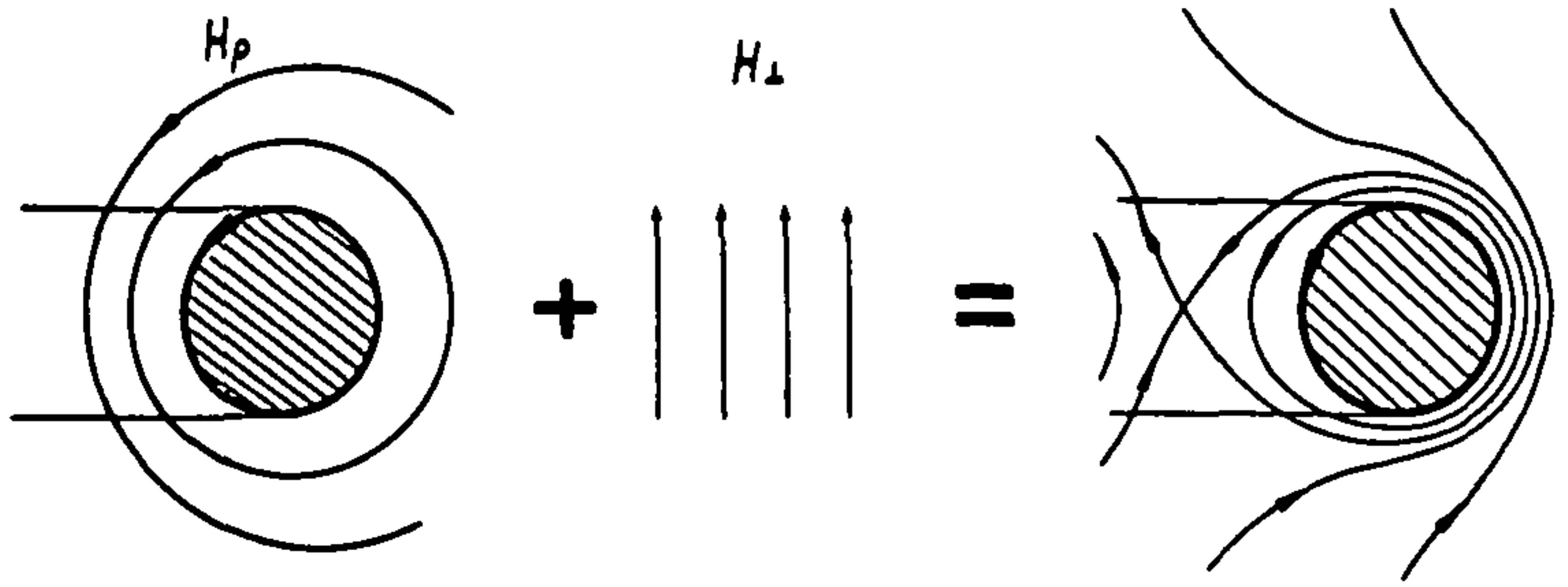


FIG.4. Diagram of the combination of the proper magnetic field of ring current with transverse balancing magnetic field.

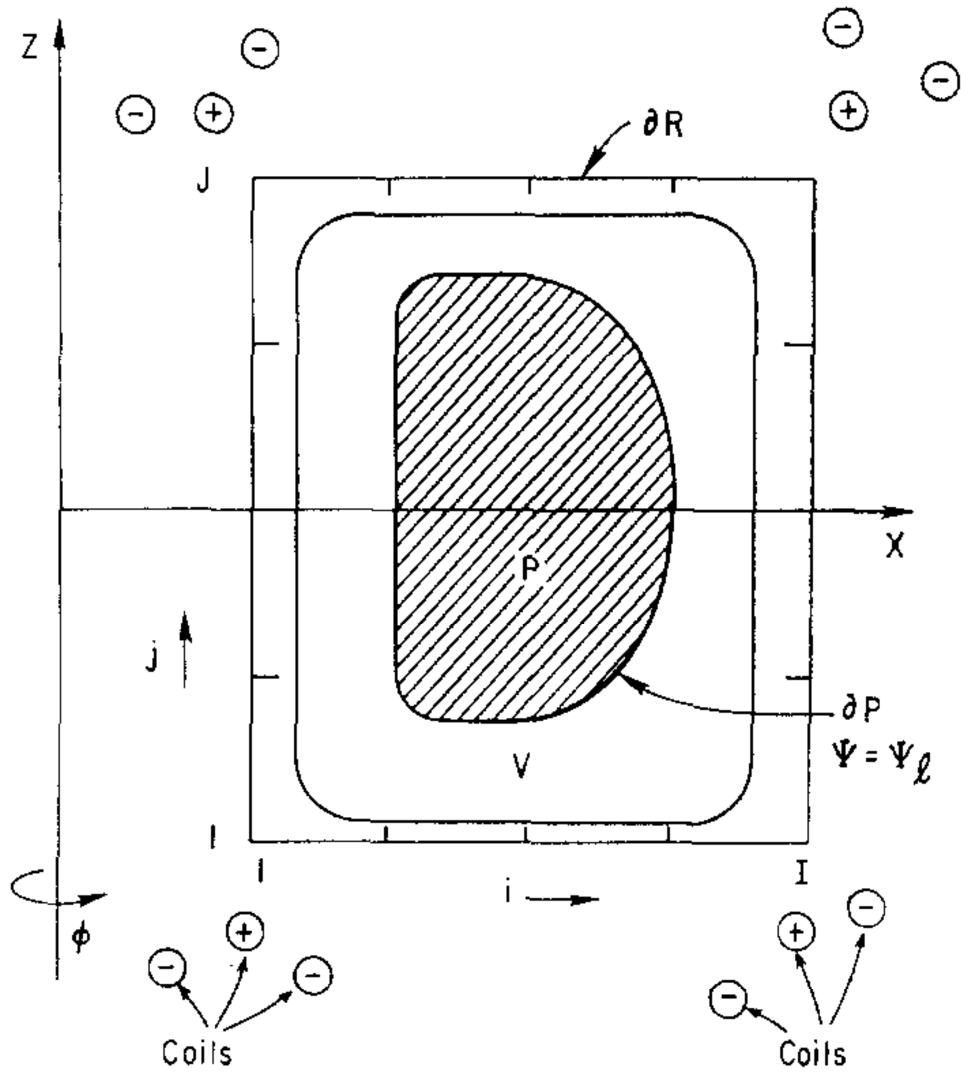


FIG. 1. Computational domain $\mathscr{R}$.

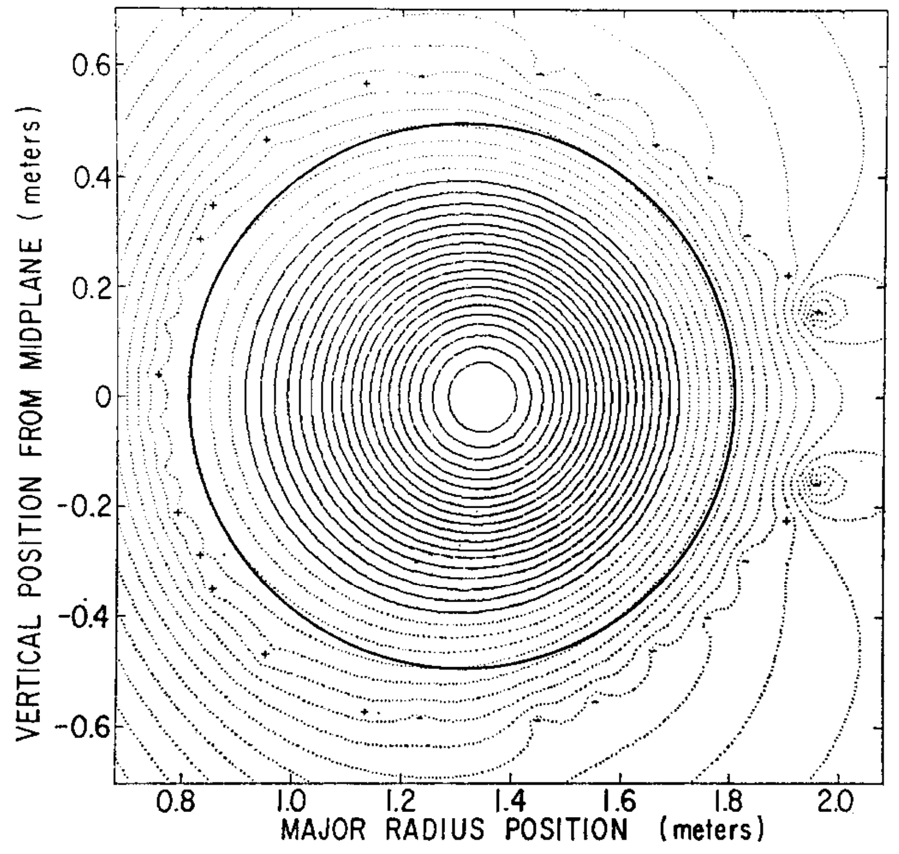


FIG. 4. A typical PLT equilibrium with $\beta_p = 0.23$ and $1.05 < q < 5.3$. The solid curve marks the position of the vacuum vessel. The pluses and minuses denote the poloidal field coils.

JOURNAL OF COMPUTATIONAL PHYSICS 32, 212–234 (1979)

Numerical Determination of Axisymmetric Toroidal Magnetohydrodynamic Equilibria

J. L. JOHNSON,* H. E. DALHED, J. M. GREENE, R. C. GRIMM, Y. Y. HSIEH, S. C. JARDIN, J. MANICKAM, M. OKABAYASHI, R. G. STORER,† A. M. M. TODD, D. E. VOSS, AND K. E. WEIMER

Plasma Physics Laboratory, Princeton University, Princeton, New Jersey 08544

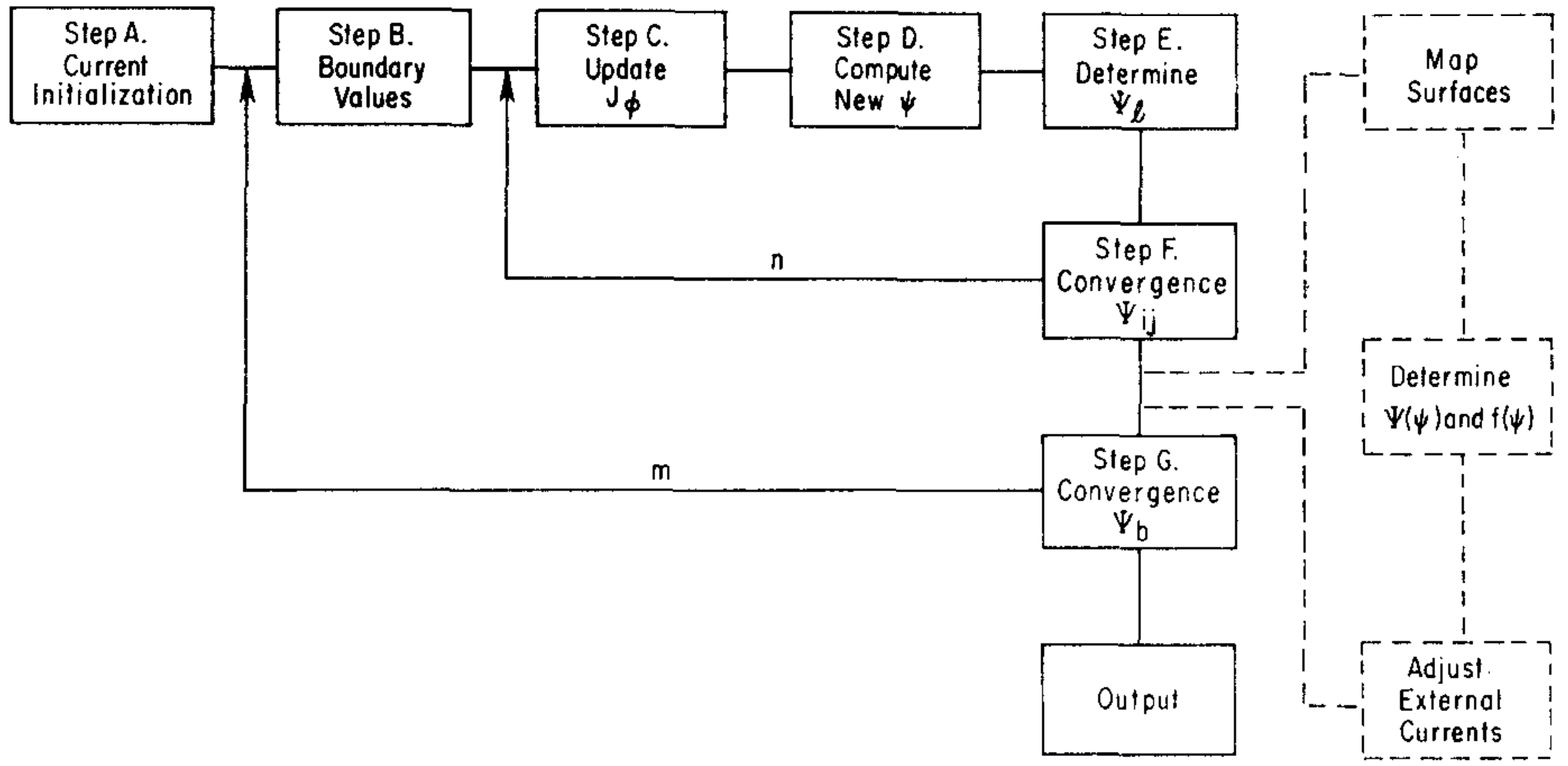


FIG. 2. Code flow chart. The dashed section is inserted when $p(\Psi)$ and $q(\Psi)$, rather than $p(\Psi)$ and $g(\Psi)$ are specified.

Grad-Shafranov Equation

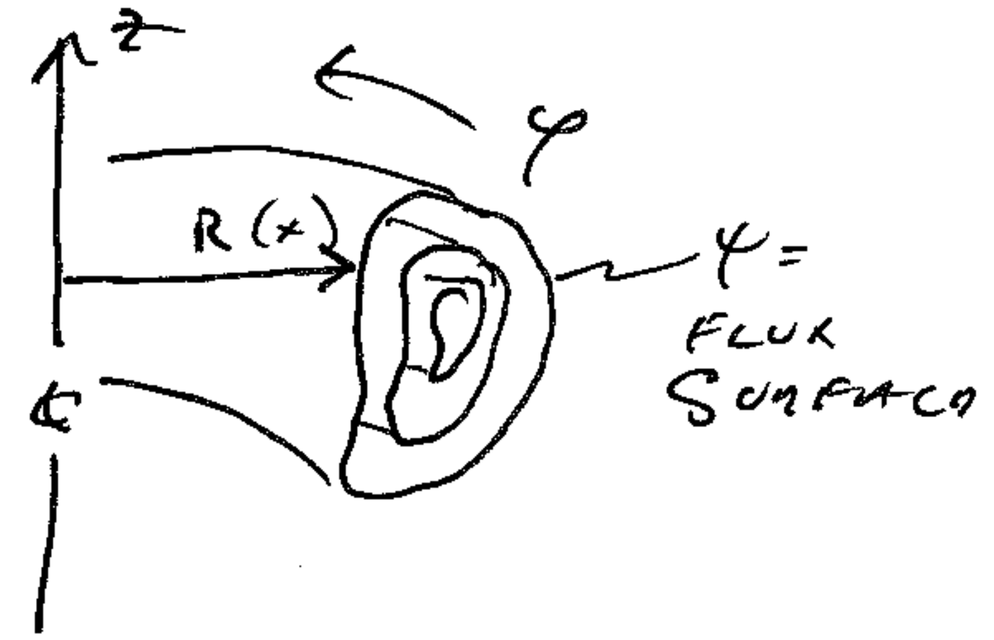
2D EQUILIBRIUM 1958 HAROLD GRAD (NYU)
VITALI SHAFRANOV (KURCHATOV, MOSCOW)

$\nabla p = J \times B$ $\mu_0 \bar{J} = \nabla \times B$

$\bar{B} = \nabla\varphi \times \nabla\psi + \nabla\varphi\, G(\psi)$ WHERE $\nabla\varphi = \frac{\hat{\varphi}}{R}$

IN CYLINDRICAL COORDINATES (R, φ, z)

NOTE: $B \cdot \nabla p = 0$
$J \cdot \nabla p = 0$
SO PLASMA FORMS NESTED FLUX SURFACES



THIS IS A 2D EQUILIBRIUM

$\bar{\nabla} p = \bar{\nabla}\psi \frac{\partial p}{\partial \psi}$ $\nabla G = \nabla\psi \frac{\partial G}{\partial \psi}$

PROBLEM: GIVEN $P(\psi)$, $G(\psi)$, WHAT IS $\psi(R, z)$
R, z OR (x, z)

FORCE BALANCE: $\mu_0 |\nabla\psi|^2 p' = -\bar{\nabla}\psi \cdot \left[\bar{B} \times (\nabla \times B)\right]$

I USE x, R INTERCHANGEABLY

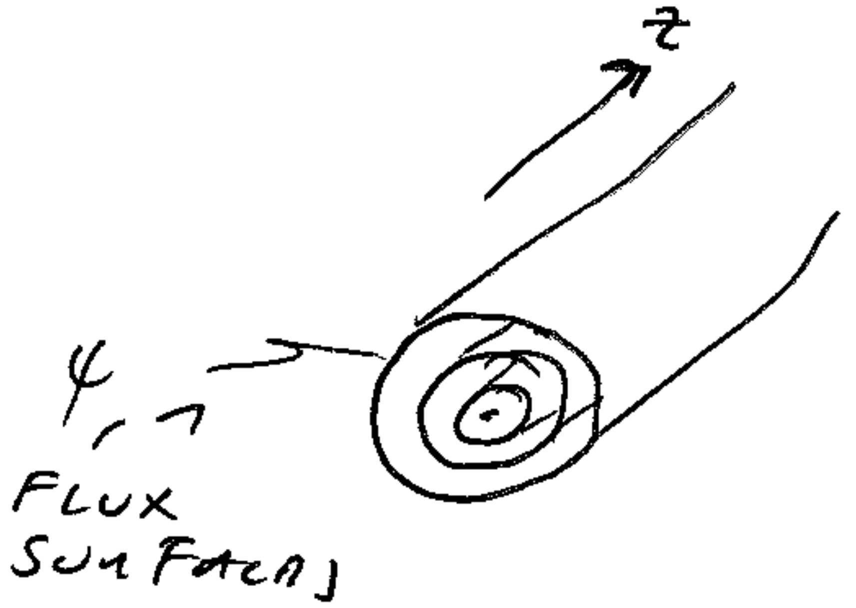
Grad-Shafranov Equation (1D example)

For comparision let $\bar{B} = \hat{z}\times\nabla\psi + \hat{z}\,G(\psi)$

Force balance: $\mu_0 |\nabla\psi|^2 p' = -\nabla\psi\cdot\left[B\times(\nabla\times B)\right]$

where $p(\psi) = p(r)$

$$\nabla\psi \sim \hat{r}\frac{\partial\psi}{\partial r}$$



But $\nabla\times\bar{B} = \nabla\times(\hat{z}\times\nabla\psi) + \nabla\times(\hat{z}\,G)$

$$= \hat{z}\,\nabla^2\psi - \nabla\psi\underbrace{(\nabla\cdot\hat{z})}_{0} + \underbrace{(\nabla\psi\cdot\bar{\nabla})\hat{z}}_{0} - \underbrace{(\hat{z}\cdot\nabla)\nabla\psi}_{0} + G\underbrace{\nabla\times\hat{z}}_{0} + G'\nabla\psi\times\hat{z}$$

$$= \hat{z}\,\nabla^2\psi - (\hat{z}\times\nabla\psi)\,G'$$

$$\bar{B}\times(\nabla\times B) = (\hat{z}\times\nabla\psi)\times(\nabla\times B) + G\,\hat{z}\times(\nabla\times B)$$

$$= (\hat{z}\times\nabla\psi)\times\hat{z}\,(\nabla^2\psi) - GG'\,\hat{z}\times(\hat{z}\times\nabla\psi)$$

$$= \bar{\nabla}\psi\,(\nabla^2\psi) + GG'\,\bar{\nabla}\psi$$

$\therefore$ $\boxed{\mu_0\, p' = -\nabla^2\psi + GG'}$ ← Radial force balance in a cylindrical screw pinch!

Grad-Shafranov Equation (2D example)

$$\nabla \times B = \nabla \times (\nabla\varphi \times \nabla\psi) + \nabla \times (G\,\bar{\nabla}\varphi)$$

$$= \bar{\nabla}\varphi\, \nabla^2\psi - \nabla\psi \underbrace{(\nabla\cdot\nabla\varphi)}_{=0} + (\nabla\psi\cdot\nabla)\nabla\varphi - (\nabla\varphi\cdot\nabla)\nabla\psi$$

$$+ G\, \nabla\times\nabla\varphi + G'\,\nabla\psi\times\nabla\varphi$$

$$= \bar{\nabla}\varphi\,(\nabla^2\psi) + G'(\nabla\psi\times\nabla\varphi) + (\nabla\psi\cdot\bar{\nabla})\bar{\nabla}\varphi - (\bar{\nabla}\varphi\cdot\bar{\nabla})\nabla\psi$$

BOTH IN $\hat{\varphi}$ DIRECTION!!

<u>$B\times(\nabla\times B)$ IN A 2D TORUS</u>

$$B\times(\nabla\times B) = (\nabla\varphi\times\nabla\psi)\times(\nabla\times B) + G\,\bar{\nabla}\varphi\times(\nabla\times B)$$

$$= (\nabla\times \bar{B})\times(\nabla\psi\times\nabla\varphi) + G\,\bar{\nabla}\varphi\times\left[G'(\nabla\psi\times\nabla\varphi) + \bar{\nabla}\varphi(\cdots)\right]$$

$$= \bar{\nabla}\psi\,(\nabla\varphi\cdot\nabla\times B) - \bar{\nabla}\varphi\,(\nabla\psi\cdot\nabla\times B) + GG'\underbrace{\nabla\varphi\times(\nabla\psi\times\nabla\varphi)}_{|\nabla\varphi|^2\,\bar{\nabla}\psi}$$

$$= \bar{\nabla}\psi\left[|\nabla\varphi|^2\,\nabla^2\psi + \bar{\nabla}\varphi\cdot(\nabla\psi\cdot\bar{\nabla})\bar{\nabla}\varphi - \bar{\nabla}\varphi\cdot(\nabla\varphi\cdot\bar{\nabla})\bar{\nabla}\psi\right]$$

$$- \bar{\nabla}\varphi\left[\underbrace{\nabla\bar{\psi}\cdot(\cdots)}_{=0}\right] + \bar{\nabla}\psi\,|\nabla\varphi|^2\,GG'$$

Grad-Shafranov Equation (2D example)

$$\therefore \quad \bar{\nabla}\psi \cdot \left(\bar{B} \times (\nabla \times B)\right) = |\nabla\psi|^2 \Big[|\nabla\varphi|^2 \, \nabla^2\psi + \bar{\nabla}\varphi \cdot (\nabla\psi \cdot \bar{\nabla})\nabla\varphi - \bar{\nabla}\varphi \cdot (\bar{\nabla}\varphi \cdot \bar{\nabla})\nabla\psi \Big]$$

$$+ |\nabla\psi|^2 |\nabla\varphi|^2 \, GG'$$

BUT $$(\nabla\psi \cdot \nabla)\nabla\varphi = \hat{\varphi} \frac{\partial\psi}{\partial r} \frac{\partial}{\partial r}\left(\frac{1}{r}\right) = \bar{\nabla}\varphi \, R \frac{\partial\psi}{\partial R} \frac{\partial}{\partial R}\left(\frac{1}{R}\right)$$

$$(\nabla\varphi \cdot \bar{\nabla})\nabla\psi = \hat{\varphi} \frac{1}{R^2} \frac{\partial\psi}{\partial r} = \bar{\nabla}\varphi \frac{1}{R} \frac{\partial\psi}{\partial R}$$

THEREFORE

$$\mu_0 \frac{\partial p}{\partial \psi} = -\frac{1}{R^2} \Big[\nabla^2\psi + GG' + \underbrace{R \frac{\partial\psi}{\partial R} \frac{\partial}{\partial R}\left(\frac{1}{R}\right) - \frac{1}{R} \frac{\partial\psi}{\partial R}}_{-\frac{2}{R}\frac{\partial\psi}{\partial R}} \Big]$$

OR

$$\boxed{\mu_0 R^2 \frac{\partial p}{\partial \psi} - GG' = \nabla^2\psi - \frac{2}{R} \frac{\partial\psi}{\partial R}} \leftarrow \text{GRAD-SHAFRANOV EQ.}$$

Cylindrical Reduced MHD

the order of $\epsilon^2 B_0$. To lowest order in ϵ this unknown variation of the toroidal field can be eliminated from the problem by taking the curl of the momentum equation. The resulting equations are the standard low-β tokamak reduced equations that describe free-boundary kink modes[3]:

$$R_0^2 \frac{d\nabla^2 u}{dt} = \boxed{\mathbf{B}\cdot\nabla}(\nabla_\perp^2 \psi),$$

$$\frac{\partial\psi}{\partial t} = R_0^2 \boxed{\mathbf{B}\cdot\nabla} u,$$

important

$$\mathbf{B} = \nabla\psi\times\nabla\zeta + I_0\nabla\zeta,$$

$$\mathbf{V} = R_0^2 \nabla u\times\nabla\zeta,$$

$$\nabla_\perp^2 = \frac{\partial^2}{\partial R^2} + \frac{\partial^2}{\partial z^2}.$$

Here $I_0 = B_0 R_0$ and $\nabla\zeta = \hat{\zeta}/R_0$.

$$A_\phi = I_0(r/2)$$

$$A_\| \approx A_z = \psi_0(r) + \tilde{\psi}(r,\phi)$$

Plasma Physics Series

Tokamak Plasma:

A Complex Physical System

B B Kadomtsev

I V Kurchatov Institute of Atomic Energy, Moscow, Russia

Translation Editor: **Professor E W Laing**

5 Plasma Stability 64

5.1 Kink Instability 65
5.2 Tearing Instability 73
5.3 Flute Instability 77
5.4 The Ballooning Instability 81
5.5 Internal Kink Mode 85
5.6 Drift Instabilities 85

Institute of Physics Publishing
Bristol and Philadelphia

“In memory of Boris Kadomtsev,” by E P Velikhov et al 1998, *Phys.-Usp.* **41** 1155; [https://doi.org/10.1070/PU1998v041n11ABEH000508]

Basic Derivation

REF: KADOMTSEV, "TOKAMAK PLASMA: A COMPLEX PHYSICAL SYSTEM"
IOP (1992)

$$\bar{B} = \bar{B}_\perp + \hat{z} B_z$$

WITH $B_z \approx$ CONSTANT

CYLINDRICAL COORDINATES

MHD:

$$\rho \frac{d\bar{v}_\perp}{dt} = -\nabla p + \bar{J} \times \bar{B}$$

$$\bar{E} = -\bar{v} \times \bar{B} \quad \text{(IDEAL)}$$

$$\bar{v} \cdot \hat{z} = 0$$

$$\nabla \cdot \bar{v}_\perp = 0$$

INCOMPRESSIBLE

MAXWELL'S EM

$$\bar{J} = \frac{1}{\mu_0} \nabla \times \bar{B}$$

(NO DISPLACEMENT CURRENT)

$$\nabla \cdot \bar{B} = 0 = \nabla \cdot \bar{B}_\perp$$

$$\nabla \cdot J = 0$$

$$\frac{\partial \bar{B}}{\partial t} = -\nabla \times \bar{E}$$

IDEAL MHD DESCRIBE PLASMA DYNAMICS AT ALFVEN TIME SCALE: $V \sim V_A \sim \sqrt{B^2/\mu_0 \rho}$ (FAST!)

Stream Function and Poloidal Flux

WITH B_z = CONSTANT, $V_z = 0$ THE REDUCED MHD DYNAMICS IS DESCRIBED BY FOUR UNKNOWN FUNCTIONS OF (r, θ, z, t):

$$\bar{B}_\perp(r, \theta, z, t) \quad \text{AND} \quad \bar{V}_\perp(r, \theta, z, t)$$

TWO POTENTIALS INSTEAD OF TWO VECTOR FIELDS

WE "GREATLY SIMPLIFY" THE MATH BY INTRODUCING THE STREAM FUNCTION, χ, AND THE POLOIDAL FLUX FUNCTION, ψ

$$\bar{B}_\perp(r, \theta, z, t) = \hat{z} \times \nabla \psi$$

$$\nabla \cdot \bar{B} = 0$$

$$\bar{V}_\perp(r, \theta, z, t) = \hat{z} \times \nabla \chi$$

$$\nabla \cdot \bar{V} = 0$$

AMPERE'S LAW

$$\bar{J} = \frac{1}{\mu_0} \nabla \times (\hat{z} \times \psi)$$

$$\mu_0 \bar{J} = \hat{z}\, \nabla^2 \psi - (\hat{z} \cdot \bar{\nabla})\, \bar{\nabla} \psi$$

AXIAL VORTICITY

$$\Omega_z \equiv \hat{z} \cdot \nabla \times \bar{V}_\perp = \nabla^2 \chi$$

TWO UNKNOWN POTENTIALS:

$$\psi(r, \theta, z, t) \quad \chi(r, \theta, z, t)$$

Simplifying the MHD Equations

$$\rho \frac{d\bar{V}_\perp}{dt} = -\nabla p + \frac{1}{\mu_0}(\nabla \times B) \times B$$

$$= -\nabla\left(p + B^2/2\mu_0\right) + \frac{1}{\mu_0}(\bar{B}\cdot\bar{\nabla})\bar{B}$$

$$\hat{z}\cdot\nabla\times\left[\;'' \quad = \quad ''\;\right]$$

ASSUME $\rho \simeq$ UNIFORM

$$\rho \frac{d}{dt}\left(\hat{z}\cdot\nabla\times V_\perp\right) = \frac{1}{\mu_0}(\bar{B}\cdot\bar{\nabla})\left(\hat{z}\cdot\nabla\times\bar{B}\right)$$

$$\rho \frac{d}{dt}\left(\nabla_\perp^2 \chi\right) = \frac{1}{\mu_0}(\bar{B}\cdot\bar{\nabla})\,\nabla_\perp^2\psi = (\bar{B}\cdot\bar{\nabla})\,J_z$$

$$\boxed{\rho \frac{d}{dt}\Omega_z = (\bar{B}\cdot\bar{\nabla})\,J_z}$$

"AXIAL VORTICITY CHANGES ACCORDING TO FIELD-ALIGNED VARIATION OF AXIAL CURRENT"

Simplifying the Induction Equation

$$\frac{\partial \bar{B}}{\partial t} = \nabla \times (\bar{V}_\perp \times \bar{B}) = \bar{V}_\perp \bar{\nabla} \cdot \bar{B} - \bar{B} \bar{\nabla} \cdot \bar{V} + (\bar{B} \cdot \bar{\nabla}) \bar{V} - (\bar{V} \cdot \bar{\nabla}) \bar{B}$$

$$= \nabla \times (\bar{V}_\perp \times \bar{B}_\perp) + B_z \frac{\partial \bar{V}_\perp}{\partial z}$$

$$= (\bar{B}_\perp \cdot \bar{\nabla}) \bar{V}_\perp - (\bar{V}_\perp \cdot \bar{\nabla}) \bar{B}_\perp + B_z \frac{\partial \bar{V}_\perp}{\partial z}$$

$$\frac{\partial \bar{B}_\perp}{\partial t} + (\bar{V}_\perp \cdot \bar{\nabla}) B_\perp = \frac{d \bar{B}_\perp}{dt} = (\bar{B} \cdot \bar{\nabla}) \bar{V}_\perp$$

SUBSTITUTING FLUX FUNCTIONS

$$\boxed{\frac{d}{dt} \psi = (\bar{B} \cdot \bar{\nabla}) \chi}$$

"POLOIDAL FLUX EVOLVES DYNAMICALLY DUE TO FIELD-ALIGNED CHANGES IN THE STREAM FUNCTION"

"Simplest" Kink Mode Theory

- Reduced MHD (plasma torus with a strong toroidal field)
- Kink modes

$$\frac{d}{dt}\psi = (\mathbf{B}\cdot\nabla)\chi$$

"POLOID FLUX EVOLVES DYNAMICALLY DUE TO FIELD-ALIGNED CHANGES IN THE STREAM FUNCTION"

$$\rho\frac{d}{dt}\Omega_z = (\mathbf{B}\cdot\nabla)J_z$$

"AXIAL VORTICITY CHANGES ACCORDING TO FIELD-ALIGNED VARIATION OF AXIAL CURRENT"

Importance of B·∇

$$\rho \frac{d}{dt} \nabla_\perp^2 \chi = \frac{1}{\mu_0}(B\cdot\bar\nabla)\nabla_\perp^2 \psi \qquad \text{(MHD)}$$

$$\frac{d}{dt}\psi = (\bar B\cdot\bar\nabla)\chi \qquad \text{(INDUCTION)}$$

$$= i\bar k\cdot\bar B_0 \qquad \bar k \doteq -\frac{n}{R}\hat z + \frac{m}{r}\hat\theta \quad \text{(PLUS RADIAL TERMS)}$$

LINEAR

$$\bar B\cdot\bar\nabla = B_z\frac{\partial}{\partial z} + B_\perp\cdot\bar\nabla_\perp = -i\frac{n}{R}B_z + i\frac{m}{r}B_\rho = i\frac{B_\rho(r)}{r}(m - nq(r))$$

$$\text{WITH } q(r) = \frac{r B_z}{R B_\rho(r)} = \text{SAFETY FACTOR}$$

$\bar B\cdot\bar\nabla \to 0$ WHEN $m/n = q(r)$ {RESONANCE}

WHEN $\bar B\cdot\bar\nabla \neq 0$, THEN IDEAL REDUCED MHD MAKE SENSE

$B\cdot\bar\nabla = 0$, THEN REDUCED MHD DOES NOT DESCRIBE DYNAMICS

(SIDE BAR! $\bar B\cdot\bar\nabla = 0$ DEFINES "INTERCHANGE" MODES.
THESE ARE THE DOMINANT MODES IN
MAGNETOSPHERES AND DIPOLES, ETC)

First: Equilibrium

$V_\perp = 0$, $\frac{\partial}{\partial t} = 0$,

$$0 = -\nabla P + J \times B$$
$$= -\nabla P + \bar{J} \times (\hat{z} \times \nabla\psi)$$
$$= -\nabla P - J_z \nabla\psi$$

ALL EQUILIBRIUM VARIATION IS RADIAL, IN $\bar{\nabla}\psi$ DIRECTION

SO

$$\frac{\nabla\psi \cdot \nabla P}{(\nabla\psi)^2} = -J_z \Rightarrow \frac{\partial P}{\partial\psi} = -J_z$$

(= CONSTANT) WHEN J_z = CONSTANT SHAFRANOV'S

$$q(r) = \frac{r B_z}{R B_p(r)}$$

$$\mu_0 J_z = \nabla \times B_p = \nabla_\perp^2 \psi = -\mu_0 \frac{\partial P}{\partial\psi}$$

EQUILIBRIUM EQUATION FOR $\psi(r)$

EQUILIBRIUM FUNCTIONS

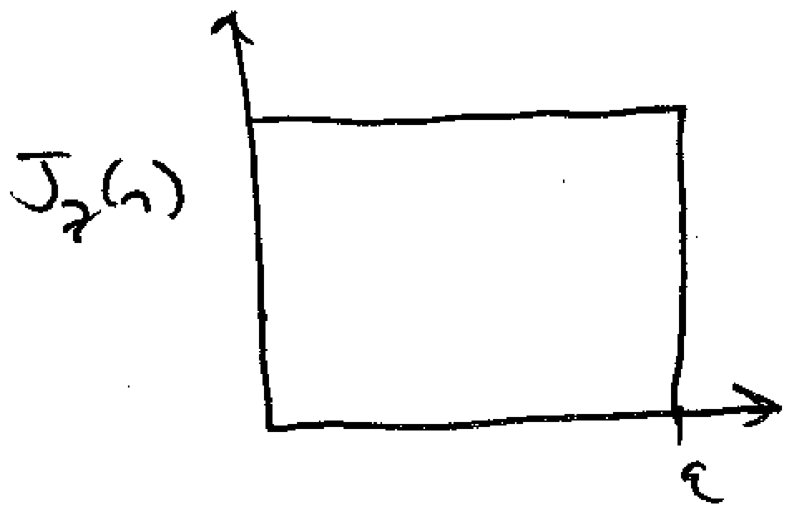
$$= \frac{1}{r}\frac{\partial}{\partial r}(r B_p)$$

$P(\psi)$, $J_z(\psi)$, $q(\psi)$

$$= \frac{B_z}{rR}\frac{\partial}{\partial r}\left(r^2/q(r)\right)$$

$P(r)$, $J_z(r)$, $q(r)$

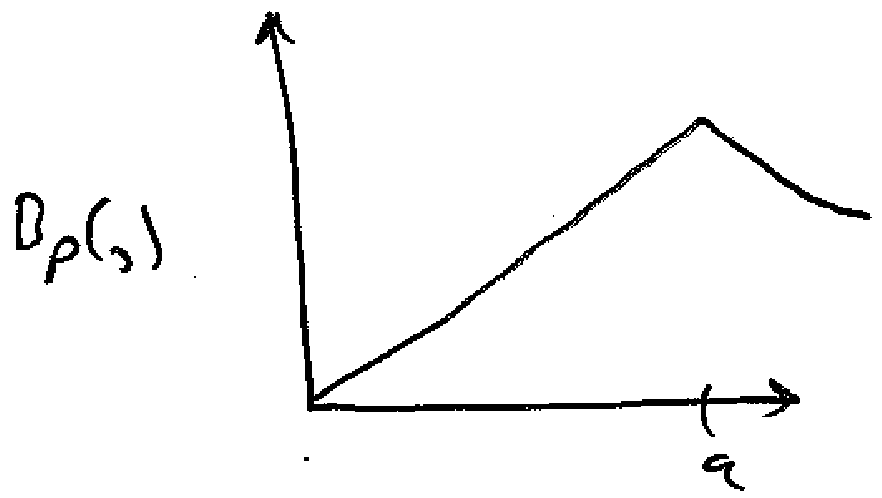
Step 1: Equilibrium (Shafranov's Simplest Case)



Constant J_z

$\ell_i = \frac{1}{2}$

$$B_p(a) = \frac{a\,\mu_0 J_z}{2} \qquad B_p(r) = \frac{r\,\mu_0 J_z}{2}$$

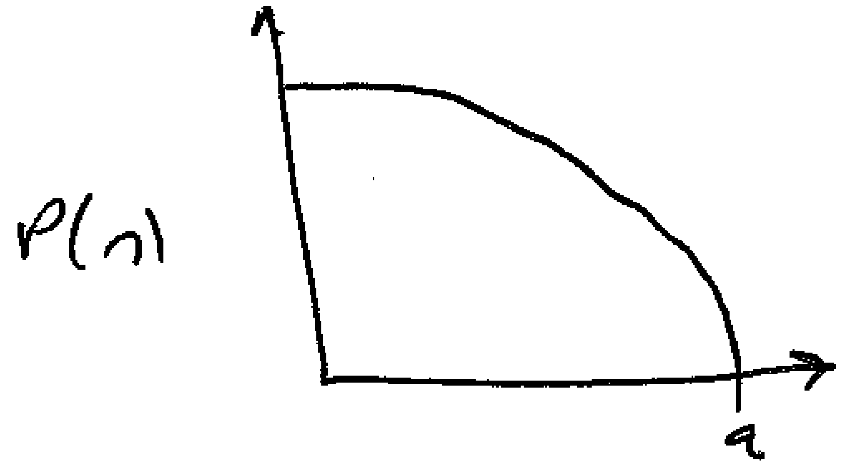


$$q(r) = q(a) = \text{constant} \ ** \;= \frac{2B_z}{\mu_0 R J_z}$$

$$B_p(r) = \hat{z} \times \nabla\psi$$

or

$$\psi(r) = \frac{r^2 B_p(a)}{2a} = \frac{r}{2} B_p(r) = \frac{r^2}{a^2}\left(\frac{aB_p}{2}\right)$$



$$\frac{\partial P}{\partial \psi} = -J_z = -\frac{2B_p(a)}{a\mu_0}$$

or

$$P(\psi) = \left(\psi(a) - \psi(r)\right)\frac{2B_p(a)}{a\mu_0} = \left(1 - r^2/a^2\right)\frac{B_p^2(a)}{\mu_0}$$

$$\langle\beta\rangle = \frac{2\pi R\int_0^a 2\pi r\,dr\,P(r)}{2\pi R\,\pi a^2\,B_z^2/2\mu_0} = \frac{B_p^2(a)}{B_z^2} = \epsilon^2/q^2(a) \ll 1$$

$$\ell_i = \frac{\text{Poloidal field energy}}{\frac{1}{4}\mu_0 R I_p^2} = \frac{2\pi R\int_0^a 2\pi r\,dr\,\frac{B_p^2(r)}{2\mu_0}}{\frac{1}{4}\mu_0 R\left(\pi^2 a^4 J_z^2\right)} = \frac{1}{2}$$

$$\beta_p = \langle\beta\rangle\frac{B_z^2}{B_p^2} = 1 \ (**) \qquad \langle\beta_N\rangle \equiv \frac{\langle\beta\rangle\,\%}{I/aB\ (\text{MKS})\ (\text{MA})} = \left(\frac{20}{q(a)}\right)\left(\frac{a}{R}\right)$$

Not tiny (of order 1)

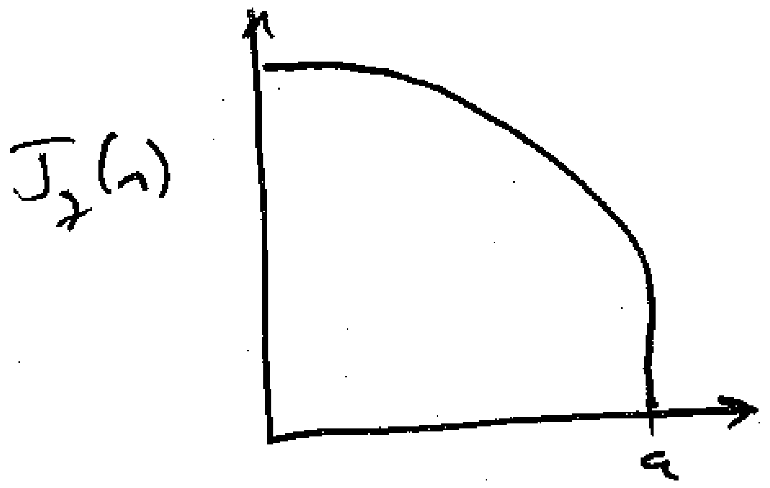
Plasma is not diamagnetic

is not paramagnetic

no change in B_z

Wesson's Cylindrical Equilibrium

REF: WESSON, NUC FUSION 18 (1978) P. 87

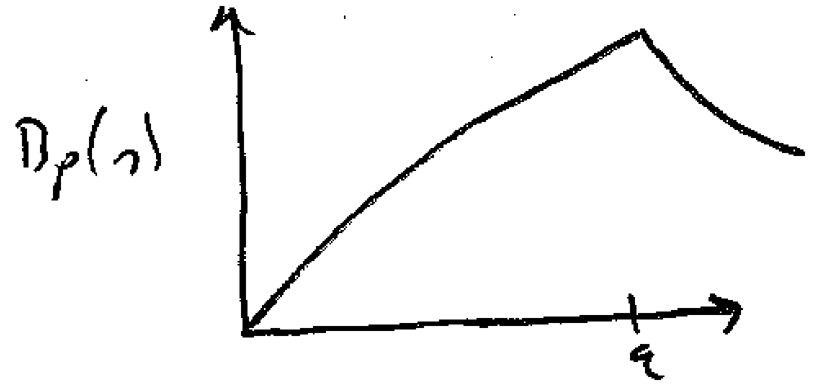


$$J_z(r) = J_0\left(1 - r^2/a^2\right)^{\nu}$$

J_0 = CENTRAL CURRENT DENSITY

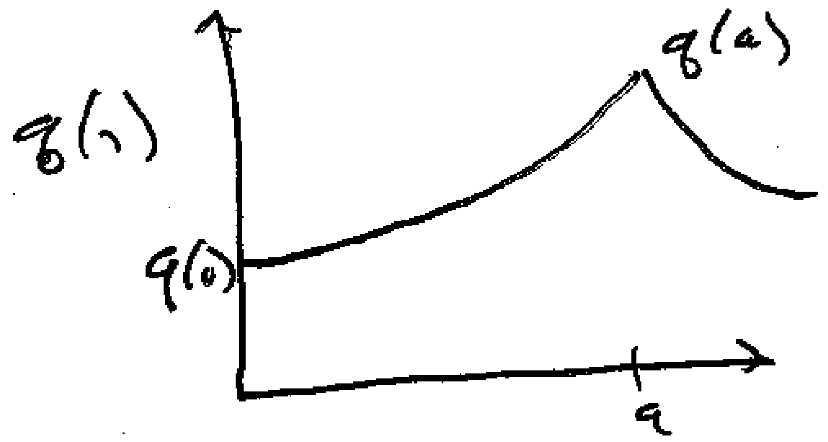
$$q(0) = \frac{2B_z}{\mu_0 R J_0} \qquad \mu_0 I_p = \frac{2\pi^2 a^2 B_z}{\mu_0\, q(0) R (1+\nu)}$$

$$\nu = \frac{q(a)}{q(0)} - 1$$



$\ell_i \neq \frac{1}{2}$

$\langle\beta\rangle \sim \epsilon^2/q_a^2 \qquad \beta_p \sim 1 \qquad \langle\beta_N\rangle \sim 20\,\epsilon/q_a$



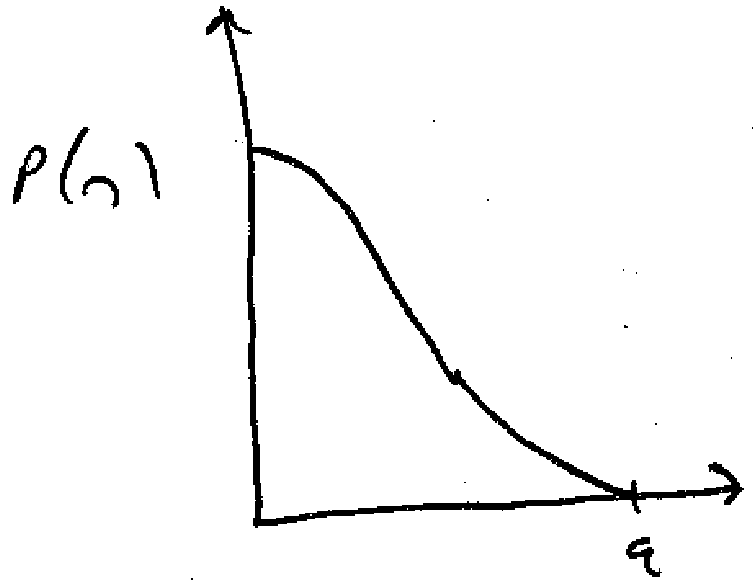
EQUILIBRIUM SET BY $q(0), q(a)$

TWO PARAMETERS

$$\frac{r}{q}\frac{dq}{dr} = s = \text{MAGNETIC SHEAR}^{**} \neq 0$$

$B\cdot\nabla \to 0$ AT INTERNAL RESONANT LAYER

$$(m - n/q(r)) \doteq 0$$



PRESSURE PROFILE CAN BE MORE PEAKED

Linearized Reduced MHD

MHD: $\rho \frac{d}{dt} \nabla_\perp^2 \chi = \frac{1}{\mu_0} (\vec{B} \cdot \nabla) \nabla_\perp^2 \psi$ $\qquad \frac{d}{dt} = -i\omega$

INDUCTION: $\frac{d\psi}{dt} = (B \cdot \nabla) \chi$ $\qquad B_0 \cdot \nabla \to i \frac{B_p(r)}{r} (m - n q(r))$

$$\boxed{(\bar{B} \cdot \bar{\nabla}) \nabla_\perp^2 \psi = (\tilde{B} \cdot \nabla) \underbrace{\nabla_\perp^2 \psi_0}_{\frac{\mu_0 J_z(r)}{\ldots}} + (\bar{B}_0 \cdot \bar{\nabla}) \bar{\nabla}_\perp^2 \tilde{\psi}} + \text{NONLINEAR TERMS}$$

$\leftarrow$ WHEN EQUILIBRIUM CURRENT DENSITY VARIES WITHIN PLASMA

$$\tilde{B} \cdot \nabla J_z(r) = (\hat{z} \times \nabla \psi) \cdot \hat{r} \frac{\partial J_z}{\partial r} = -\hat{\theta} \cdot \nabla \psi \frac{\partial J_z^0}{\partial r}$$

LINEAR MHD:

$$\boxed{-\underbrace{\rho \omega}_{\text{PLASMA INERTIA}} \nabla_\perp^2 \tilde{\chi} = -\frac{m}{r} \frac{\partial J_z}{\partial r} \tilde{\psi} + \frac{B_p}{\mu_0 r} (m - n q(r)) \nabla_\perp^2 \tilde{\psi}}$$

LINEAR INDUCTION:

$$-\omega \tilde{\psi} = \frac{B_p}{r} (m - n q(r)) \tilde{\chi}$$

Alfvén Waves in Shafranov's Equilibrium

WITHIN PLASMA

$$-\rho\,\omega\,\nabla_\perp^2\hat{X} = \frac{B_p}{\mu_0 r}(m-nq)\,\nabla_\perp^2\hat{\varphi}$$

$$-\omega\,\hat{\varphi} = \frac{B_p}{\mu_0 r}(m-nq)\,X$$

NOTE! $\frac{B_p}{r}$ = CONSTANT = $\frac{B_p(a)}{a}$

$$\omega_A^2 \equiv \frac{B_p^2/\mu_0\rho}{a^2} = \frac{B_z^2/\mu_0\rho}{(q_a R)^2}$$

$$= V_A^2/(qR)^2$$ ALFVEN TRANSIT TIME!

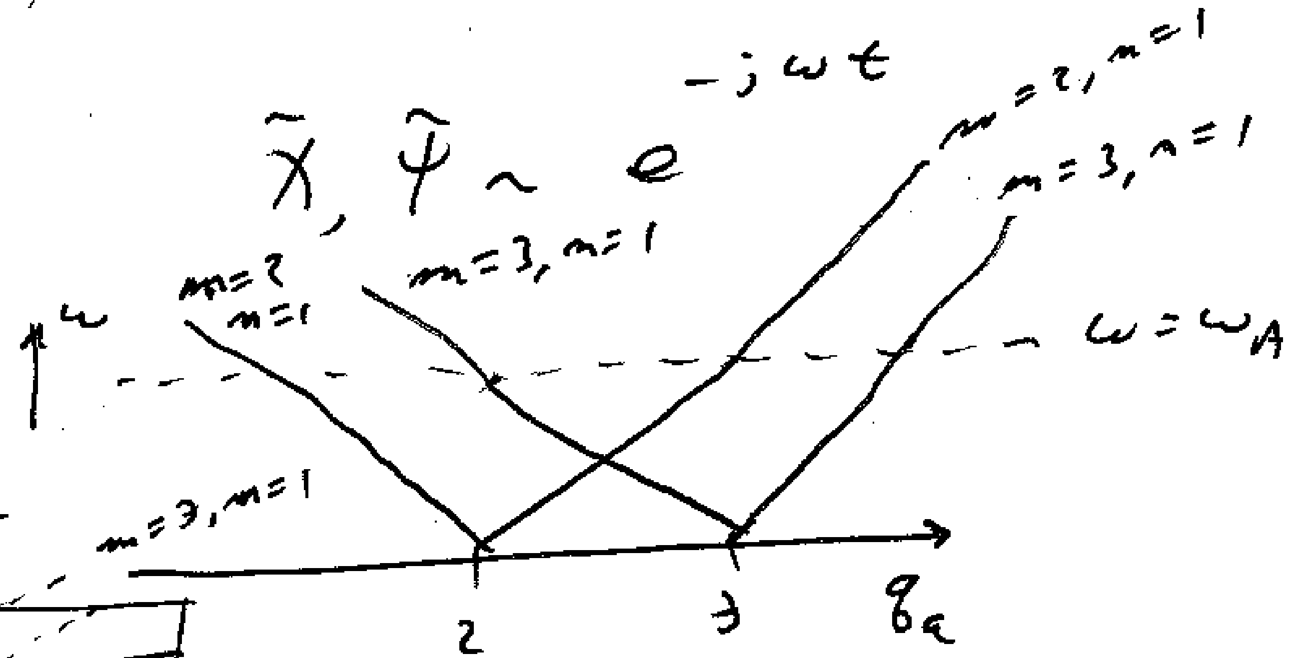
NORMAL MODES (ALFVEN WAVES)

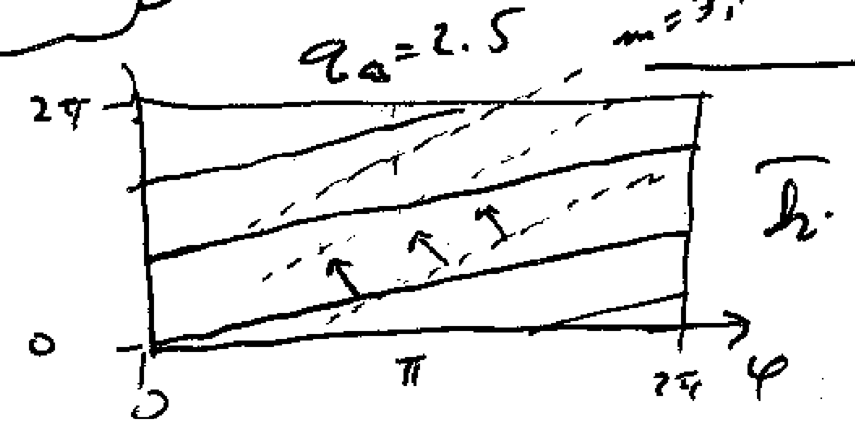
$$\begin{pmatrix} \rho\,\omega & \frac{B_p}{\mu_0 a}(m-nq) \\ \frac{B_p}{\mu_0 a}(m-nq) & \omega \end{pmatrix} \cdot \begin{pmatrix} \hat{X}(r) \\ \tilde{\varphi}(r) \end{pmatrix}$$

WITH $\rho \approx$ CONSTANT

$$\tilde{X},\ \tilde{\varphi} \sim e^{-j\omega t}$$

$$\boxed{\omega^2 = \omega_A^2\,(m-nq_a)^2}$$



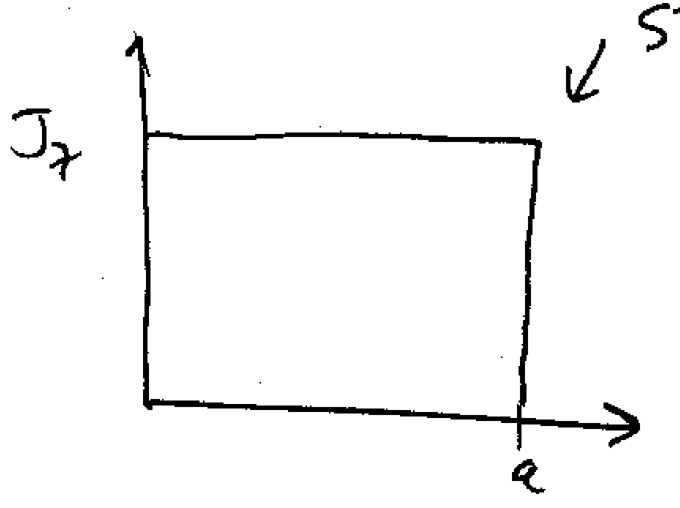


$\bar{k}\cdot\bar{B}_0 \neq 0$

RADIAL STRUCTURE NOT SPECIFIED

Global Kink Eigenmodes

$$\nabla_\perp \cdot \vec{A} = \frac{1}{r}\frac{\partial}{\partial r}(r A_r) + \dots$$



STEP DISCONTINUITY AT PLASMA EDGE

$$\frac{d}{dt}\nabla_\perp\cdot(\rho \nabla_\perp \chi) = (\hat{z}\times\nabla_\perp\tilde{\psi})\cdot\hat{r}\,\frac{\partial J_z}{\partial r} + i\frac{B_\theta}{\mu_0 r}(m-nq)\nabla_\perp^2\tilde{\psi}$$

VERY BIG AT EDGE

$$\int_{a^-}^{a^+} 2\pi r\,dr \left\{ \text{" MHD "} \right\} \Rightarrow$$

$$\omega\rho \frac{\partial\chi}{\partial r}\bigg|_{a^-} = \frac{2mB_\theta(a)}{\mu_0 a^2}\tilde{\psi}_a + \frac{B_\theta(a)}{\mu_0 a}(m-nq)\tilde{\psi}_a \underbrace{\left[\frac{1}{\tilde{\psi}}\frac{\partial\tilde{\psi}}{\partial r}\bigg|_{a^+} - \frac{1}{\tilde{\psi}}\frac{\partial\tilde{\psi}}{\partial r}\bigg|_{a^-}\right]}$$

$$\boxed{\Delta'(a) \equiv \frac{1}{\psi}\left(\frac{\partial\tilde{\psi}}{\partial r}\bigg|_{a^+} - \frac{\partial\psi}{\partial r}\bigg|_{a^-}\right)}$$

THIS MEASURES PERTURBED SURFACE CURRENT ON PLASMA

$$\Delta'(a)\,\tilde{\psi}_a = \mu_0 \tilde{K}_z(\theta,\phi)/a$$

IMPORTANT:

$$\omega\rho\frac{\partial\chi}{\partial r}\bigg|_{a^-} = \frac{2mB_\theta(a)}{\mu_0 a^2}\tilde{\psi}_a\left[(m-nq)\left(\frac{-\Delta'(a)}{2m/a}\right) - 1\right]$$

Global Kink Eigenmodes

WHAT ARE $(\psi(r), \chi(r))$ INSIDE AND OUTSIDE PLASMA?

Boundary conditions

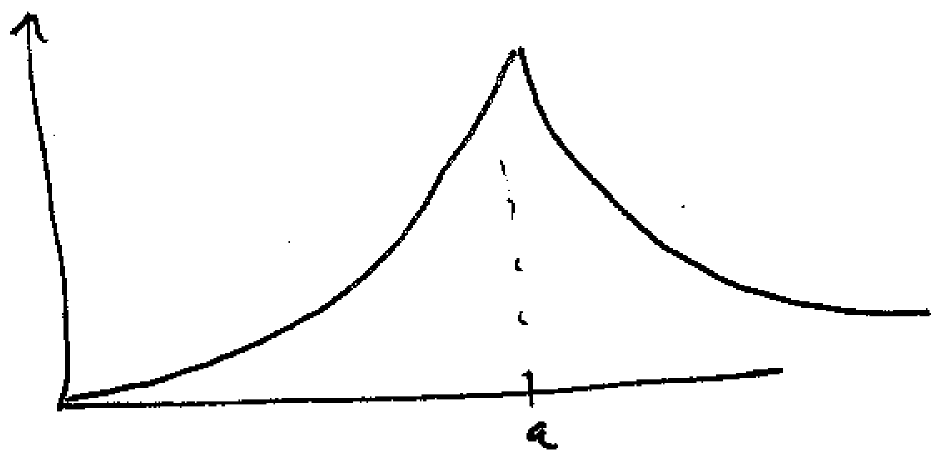
$\nabla^2 \psi = 0$ (NO CURRENT) OUTSIDE PLASMA

$\nabla^2 \chi = 0$ (NO FLOW, VORTICITY, NO SLIP)

$\nabla^2 \psi = 0$ (NO CURRENTS INSIDE PLASMA TOO)

$\nabla^2 \chi = 0$ (NO VORTICITY WITHIN PLASMA)

PERTURBED FIELDS + PLASMA MOTION
BUT NO CURRENTS OR VORTICITY



NO WALL

$\psi(r) \sim \left(\frac{r}{a}\right)^m \quad r<a$

$\sim \left(\frac{a}{r}\right)^m \quad r>a$

WITH WALL

$\sim \left(\frac{r}{a}\right)^m \quad r<a$

$\sim \dfrac{\left(\frac{b}{r}\right)^m - \left(\frac{r}{b}\right)^m}{\left(\frac{b}{a}\right)^m - \left(\frac{a}{b}\right)^m} \quad (a<r<b)$

$V_\perp = \hat{z} \times \nabla\chi = \hat{\theta}\,\frac{m}{r}\left(\frac{r}{a}\right)^m - \hat{r}\,\frac{im}{r}\left(\frac{r}{a}\right)^m$ INSIDE

$= -\hat{\theta}\,\frac{m}{r}\left(\frac{a}{r}\right)^m - \hat{r}\,\frac{im}{r}\left(\frac{a}{r}\right)^m$ OUTSIDE

Kink Mode

$$-\omega \tilde{\varphi}_a = \frac{B_\theta}{r}(m-nq)\tilde{\chi}_a$$

$$\omega\rho \left.\frac{\partial\chi}{\partial r}\right|_{a^-} = \frac{2mB_\theta}{\mu_s a^2}\tilde{\varphi}_a\left[(m-nq)\left(\frac{-\Delta'(a)}{2m/a}\right)-1\right]$$

$$\Delta'(a) = -\frac{2m}{a}\frac{(b/a)^m}{(b/a)^m-(a/b)^m}$$ FOR A WALL AT $r=b$

$$\left.\frac{\partial\chi}{\partial r}\right|_{a^-} = -\frac{m}{a}\tilde{\chi}_a$$

$$\frac{\tilde{\chi}_a}{\tilde{\varphi}_a} = -\frac{\omega q_a R}{B_0(m-nq_a)}$$

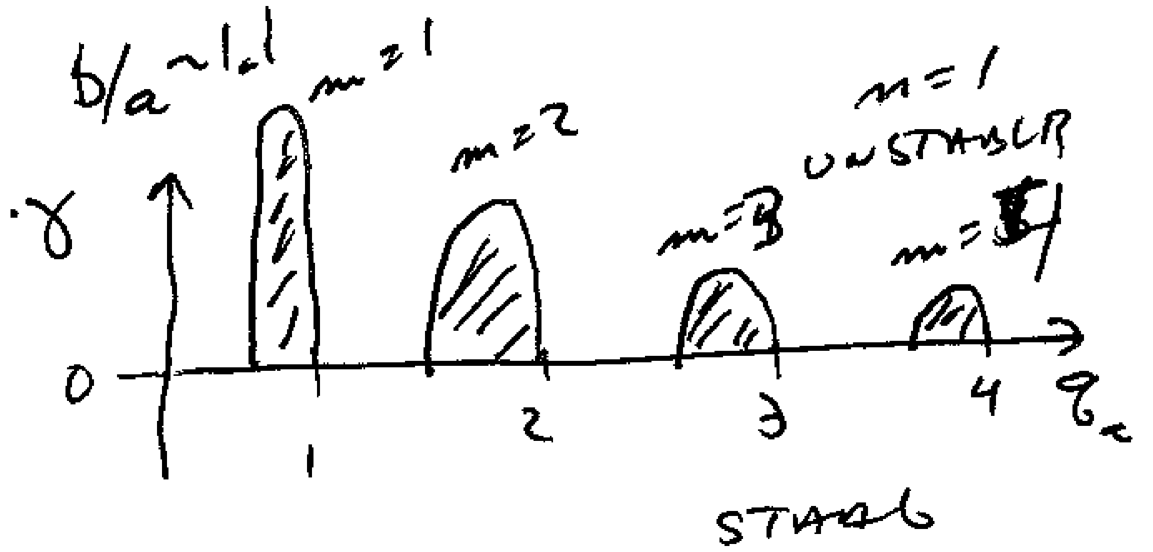
EIGENVECTOR

EIGENVALUE

$$\omega^2 = 2\omega_A^2(m-nq) \times \left[(m-nq)\frac{(\Lambda+1)}{2}-1\right]$$

GLOBAL KINK MODES

$$\begin{pmatrix} \omega & \frac{B_\theta}{a}(m-nq) \\ \frac{2mB_\theta}{\mu_0 a}\left[(m-nq)\frac{(\Lambda+1)}{2}-1\right] & \omega\rho \end{pmatrix}\begin{pmatrix}\tilde{\varphi}_a \\ \tilde{\chi}_a\end{pmatrix} = 0$$

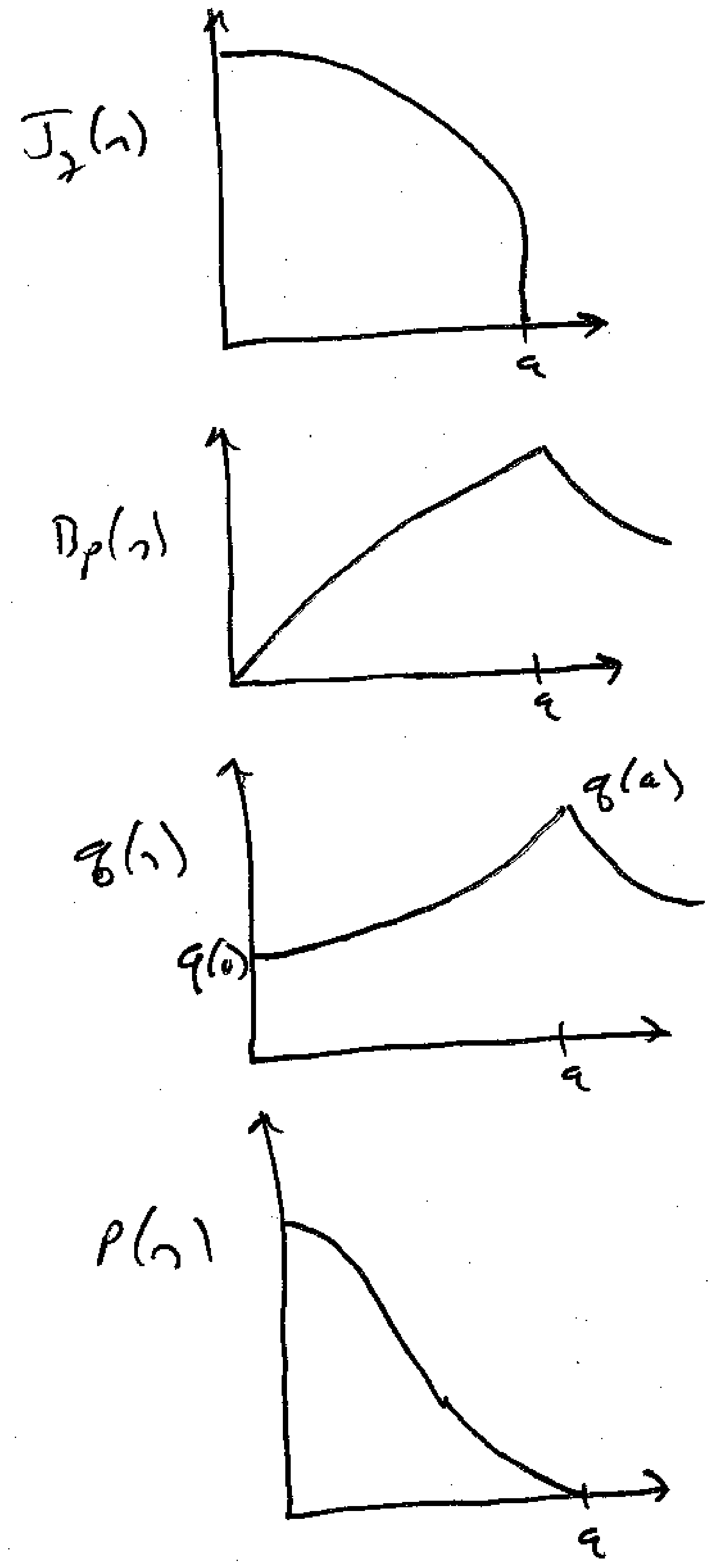


$$\boxed{\Delta'(a) = -\frac{m}{a}(\Lambda+1)}$$

SHAFRANOV'S FORMULA

Wesson's Cylindrical Equilibrium

REF: WESSON, NUC FUSION 18 (1978) P. 87



$$J_z(r) = J_0\left(1 - r^2/a^2\right)^{\nu}$$

J_0 = CENTRAL CURRENT DENSITY

$$q(0) = \frac{2B_z}{\mu_0 R J_0} \qquad \mu_0 I_p = \frac{2\pi^2 a^2 B_z}{\mu_0\, q(0)\, R(1+\nu)}$$

$$\nu = \frac{q(a)}{q(0)} - 1$$

$$\ell_i \neq \tfrac{1}{2}$$

$$\langle\beta\rangle \sim \epsilon^2/q_a^2 \qquad \beta_p \sim 1 \qquad \langle\beta_N\rangle \sim 20\,\epsilon/q_a$$

EQUILIBRIUM SET BY $q(0), q(a)$

TWO PARAMETERS

$$\frac{r}{q}\frac{dq}{dr} = s = \text{MAGNETIC SHEAR} \;{**} \neq 0$$

PRESSURE PROFILE CAN BE MORE PEAKED

$B\cdot\nabla \to 0$ AT INTERNAL RESONANT LAYER

$$(m - n/q(r)) \simeq 0$$

Wesson's Kink Modes

LINEARIZED EQUATIONS FOR PERTURBED STREAM FUNCTION (χ)
AND PERTURBED POLOIDAL FLUX (ψ)

$$-\rho\,\omega\,\nabla_\perp^2\tilde{\chi} = -\frac{m}{r}\frac{\partial J_z}{\partial r}\tilde{\psi} + \frac{B_\theta}{\mu_0 r}(m - nq)\nabla_\perp^2\tilde{\psi} \qquad \text{(MHD)}$$

$$-\omega\tilde{\psi} = \frac{B_\theta}{r}(m-nq)\tilde{\chi} \qquad \text{(INDUCTION)}$$

LET'S TAKE $\rho \simeq$ UNIFORM, WITH A SHARP JUMP AT THE PLASMA'S EDGE:

$$\omega\,\rho\,\frac{\partial \chi}{\partial r}\bigg|_{a^-} = \frac{B_\theta(a)}{\mu_0 a}(m - nq_a)\,\tilde{\psi}_a \quad \boxed{\Delta'(a)}$$

PERTURBED SURFACE CURRENT AT PLASMA'S EDGE

WITH INDUCTION EQUATION:

$$\omega^2 = -\,\omega_A^2\,(m-nq_a)^2\,\Delta'(a)\,\frac{\chi_a}{(\partial\chi/\partial r)_{a^-}}$$

BUT, HOW TO FIGURE OUT $\tilde{\psi}(r)$?

INSTABILITY REQUIRES $\Delta'(a) > 0$

Wesson's Kink Modes

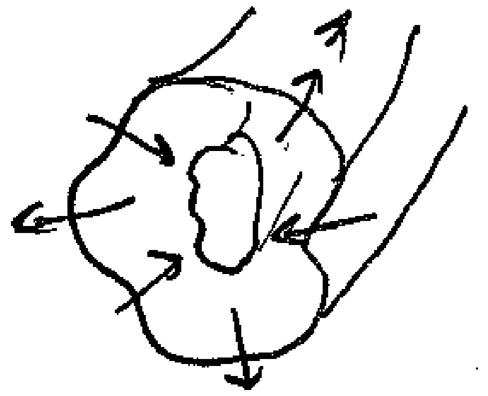
SINCE $|\omega| < \omega_A$, THE KINK MODE CAUSES THE "INTERNAL" PLASMA TO RESPOND "QUICKLY", SO QUICKLY THAT WE CAN IGNORE THE TIME CT TERMS TO FORM A DISTORTED, 3D, QUASI-EQUILIBRIUM.

INSIDE, THE PLASMA IS A "FORCED" EQUILIBRIUM

$$0 \approx -\frac{m}{r}\frac{\partial J_z}{\partial r}\tilde{\Psi} + \frac{B_\theta}{\mu_0 r}(n - mq)\nabla_\perp^2\tilde{\Psi}$$

OUTSIDE, THE RESPONSE IS THE "VACUUM" RESPONSE.

WITH $J_z(r)$, WE HAVE TO SOLVE FOR $\tilde{\Psi}$ USING A COMPUTER. (THIS IS VERY EASY FOR THE CYLINDRICAL "TOKAMAK")



← THE SURFACE CURRENT "PUSHES"/"PULLS" PLASMA, AND THE "DISTORTED" PLASMA IS MEASURED BY $\tilde{\Psi}(r, \theta, z)$

Next Lecture:

- Examining the properties of kink modes in the (straight) reduced MHD formalism.