Plasma 2
Lecture 21:
Reduced MHD

APPH E6102y
Columbia University

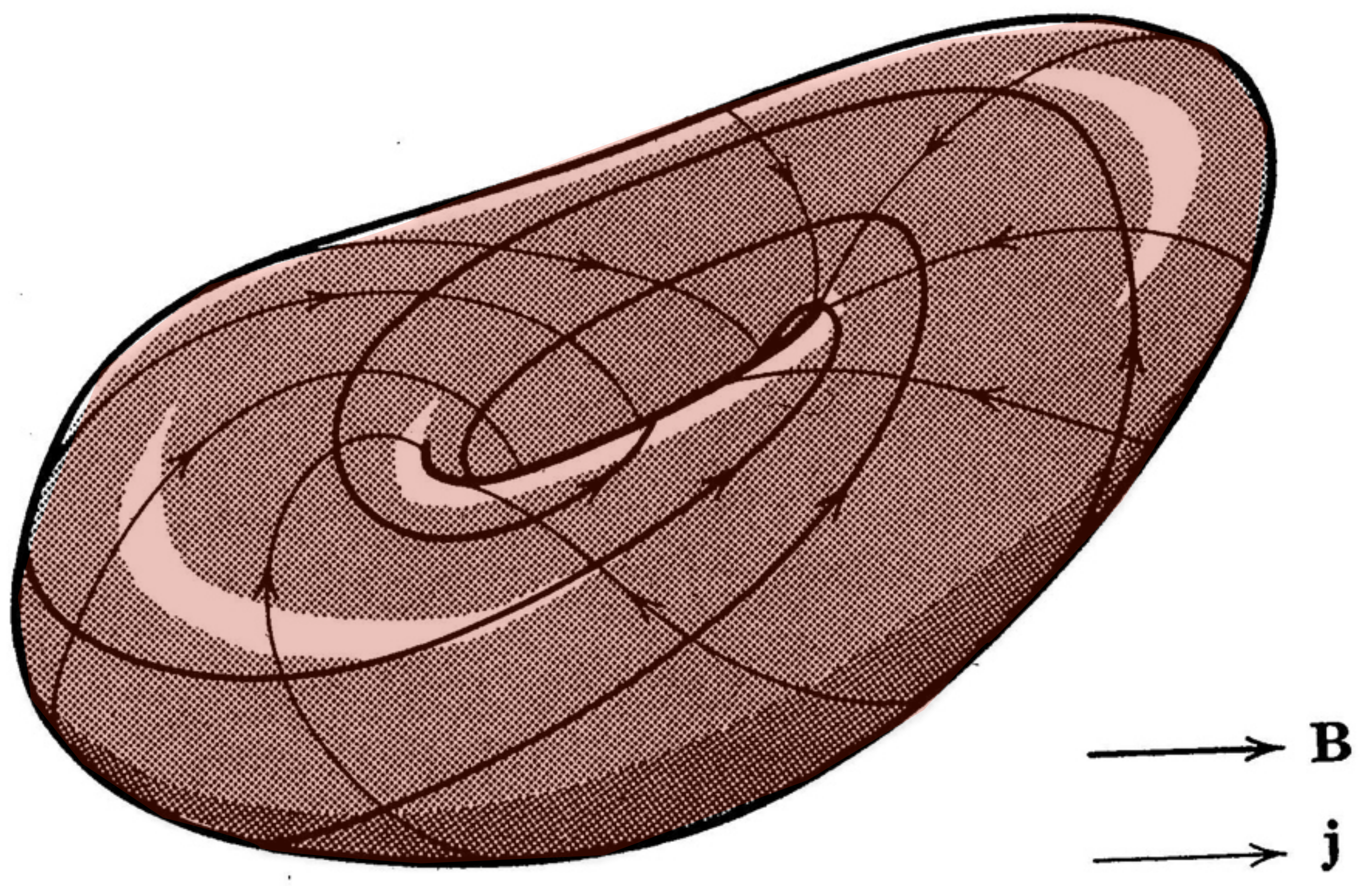
Toroidal Magnetic Confinement (and Instabilities)

Plasma Pressure

Plasma Current

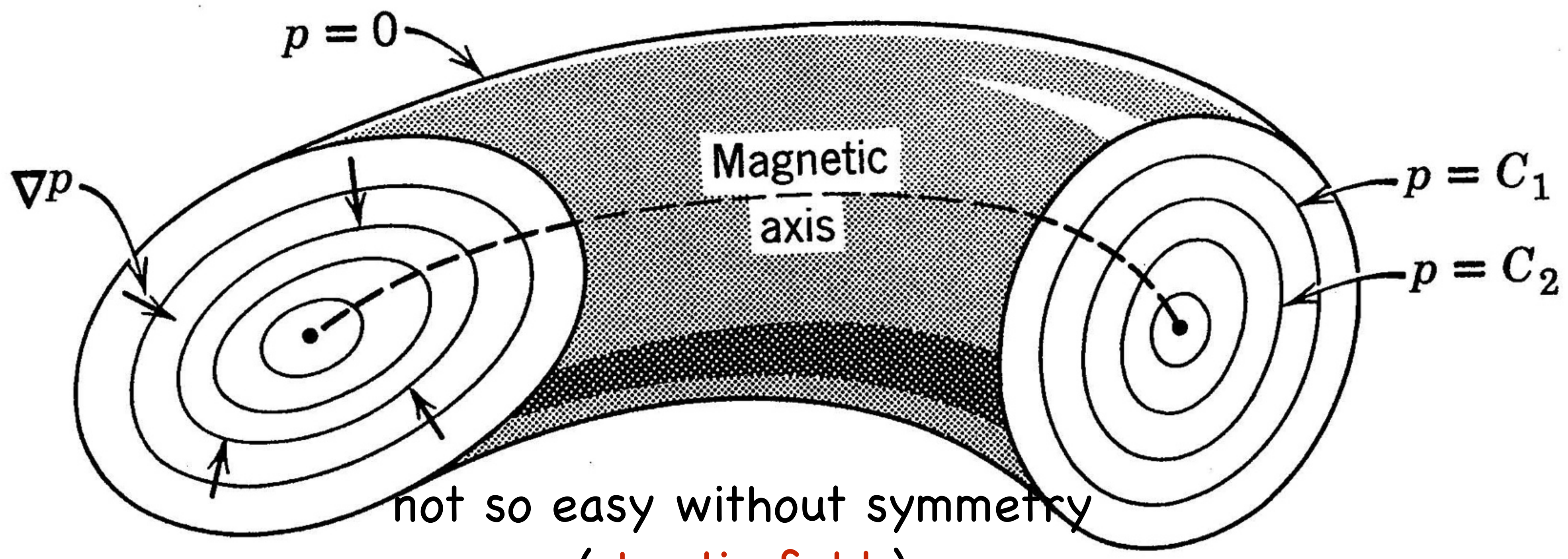
$$
\begin{aligned}
\text{(No monopoles)} \quad \nabla \cdot \mathbf{B} &= 0 \\
\text{(No charge accumulation)} \quad \nabla \cdot \mathbf{J} &= 0 \\
\text{(No unbalanced forces)} \quad 0 &= -\nabla P + \mathbf{J} \times \mathbf{B} \\
\text{(Magnetostatics)} \quad \nabla \times \mathbf{B} &= \mu_0 \mathbf{J}
\end{aligned}
$$

Magnetic Torus



$$
\begin{aligned}
\mathbf{J} \times \mathbf{B} &= \nabla P \\
\mathbf{B} \cdot \nabla P &= 0 \\
\mathbf{J} \cdot \nabla P &= 0
\end{aligned}
$$

Surfaces of constant plasma pressure form nested tori



not so easy without symmetry (chaotic fields)

MHD Stability/Instability (1957)

ANNALS OF PHYSICS: 1, 120–140 (1957)

Stability of Plasmas Confined by Magnetic Fields[1]

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In this paper, we examine the question of the stability of plasmas confined by magnetic fields. Whereas previous studies of this problem have started from the magnetohydrodynamic equations, we pay closer attention to the motions of individual particles. Our results are similar to, but more general than, those which follow from the magnetohydrodynamic equations.

I. INTRODUCTION

The problem of the behavior of highly ionized plasmas in electromagnetic fields has recently become the object of considerable interest (*1*). Although there is little more involved in the problem than Newton's laws and Maxwell's equations, there are many questions one can ask to which the answers have been by no means obvious or even easily calculable. Two of the several rather broad areas into which these questions fall are the following.

(a) The existence and properties of stationary solutions of the equations. Here, "stationary" is not meant to imply that fields and particle positions or velocities are absolutely constant, but that averages of these quantities over times longer than the Larmor period and over the statistical particle distribution are constant. Collisions between the particles are to be ignored. Effects to be considered are the diamagnetic and electric effects of the charged particles on the fields, and, conversely, the effects of the fields in influencing the particle distribution function.

(b) The stability of these stationary solutions under arbitrary perturbations of the plasma configuration. Here again collisions are to be ignored. It is known

Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences, Volume 244, Issue 1236 (Feb. 25, 1958), 17-40.

An energy principle for hydromagnetic stability problems

BY I. B. BERNSTEIN, E. A. FRIEMAN, M. D. KRUSKAL AND R. M. KULSRUD

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(*Communicated by S. Chandrasekhar, F.R.S.—Received* 18 *April* 1957—*Revised* 26 *August* 1957)

The problem of the stability of static, highly conducting, fully ionized plasmas is investigated by means of an energy principle developed from one introduced by Lundquist. The derivation of the principle and the conditions under which it applies are given. The method is applied to find complete stability criteria for two types of equilibrium situations. The first concerns plasmas which are completely separated from the magnetic field by an interface. The second is the general axisymmetric system.

1. INTRODUCTION

The investigation of hydromagnetic systems and their stability is of interest in such varied fields as the study of sunspots, interstellar matter, terrestrial magnetism, auroras and gas discharges. An excellent summary and bibliography of these applications has been given by Elsasser (1955, 1956). The stability of hydromagnetic systems has been extensively investigated in a fundamental series of papers by Chandrasekhar (1952 to 1956).

The present work is concerned with those hydromagnetic equilibria in which the fluid velocity at each point is assumed to vanish. It is divided into two parts. The first is a development of an energy principle, originally stated by Lundquist (1951, 1952), for investigating the stability of such systems. The second part consists of the application of this principle to obtain a number of specific results for such systems.

The 'normal mode' technique is the usual method for the investigation of stability in many systems, mechanical, electrical, etc. It consists of solving the linearized equations of motion for small perturbations about an equilibrium state. The system is said to be unstable if any solution increases indefinitely in time; if no such solution exists, the system is stable.

7

MHD Equilibria and Stability

This chapter is devoted to the analysis of MHD equilibria and stability. By equilibria, we mean a plasma state that is time-independent. Such states may or may not have equilibrium flows. When the states do not have equilibrium flows, that is, $\mathbf{U} = \mathbf{0}$ in some appropriate frame of reference, the equilibria are called magnetostatic equlibria. When the states have flows that cannot be simply eliminated by a Galilean transformation, the equilbria are called magnetohydrodynamic equilibria. When we introduce small perturbations in a particular equilibrium which is itself time-independent, the time dependence of the perturbations determines the stability of the system. If an equilibrium is unstable, the instability typically grows exponentially in time. The mathematical problem for the stability of magnetostatic equilibria is made tractable due to the formulation of the so-called energy principle. It turns out that when MHD equilibria contain flows that are spatially dependent, the power of the energy principle is weakened significantly, and there has been a general tendency to rely on the normal mode method, for which we provide simple examples.

$$\delta W = \frac{1}{2}\left(\frac{\partial^2 W}{\partial x^2}\right)\Delta x^2, \tag{7.3.1}$$

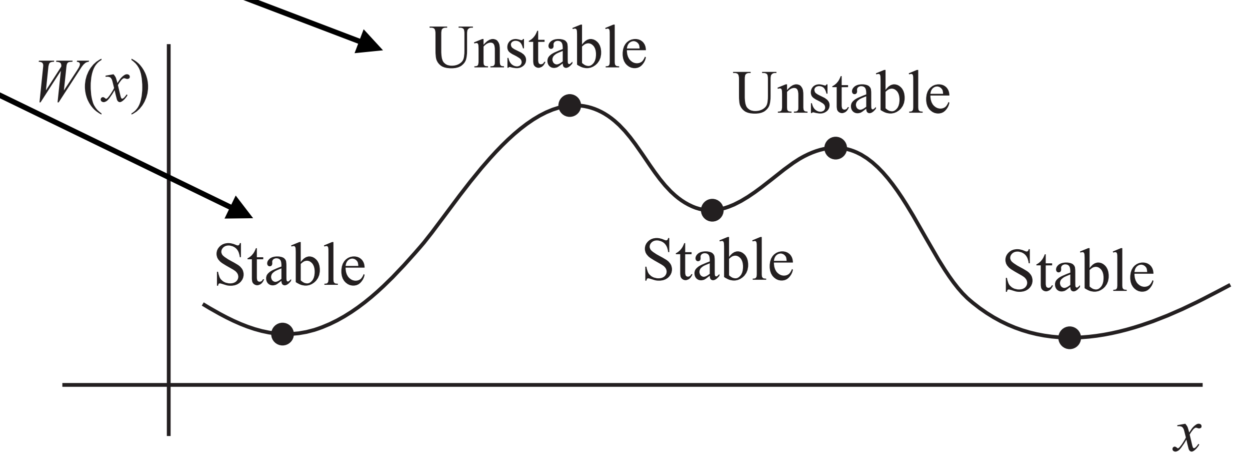


Figure 7.8 The potential energy of a mechanical system has points of stable equilibrium, characterized by $\partial^2 W/\partial x^2 > 0$, and points of unstable equilibrium, characterized by $\partial^2 W/\partial x^2 < 0$.

7.3.3 The Linear Force Operator for Magnetostatic Equilibria

$$\mathbf{U} = \frac{\partial \boldsymbol{\xi}(\mathbf{r}, t)}{\partial t}. \tag{7.3.12}$$

$$\rho_{m0} \frac{\partial^2 \boldsymbol{\xi}}{\partial t^2} = \frac{1}{\mu_0}[(\nabla \times \mathbf{B}) \times \mathbf{B}_0 + (\nabla \times \mathbf{B}_0) \times \mathbf{B}] - \nabla P, \tag{7.3.13}$$

$$\mathbf{B} = \nabla \times (\boldsymbol{\xi} \times \mathbf{B}_0). \tag{7.3.14}$$

$$\frac{\partial \rho_m}{\partial t} + \rho_{m0} \nabla \cdot \mathbf{U} + \mathbf{U} \cdot \nabla \rho_{m0} = 0, \tag{7.3.15}$$

With the linearized form of the adiabatic equation of state (6.1.34),

$$\frac{\partial P}{\partial t} + \mathbf{U} \cdot \nabla P_0 - \frac{\gamma P_0}{\rho_{m0}} \frac{\partial \rho_m}{\partial t} - \frac{\gamma P_0}{\rho_{m0}} \mathbf{U} \cdot \nabla \rho_{m0} = 0, \tag{7.3.16}$$

to obtain

$$\frac{\partial P}{\partial t} + \mathbf{U} \cdot \nabla P_0 + \gamma P_0 (\nabla \cdot \mathbf{U}) = 0. \tag{7.3.17}$$

$$P = -\boldsymbol{\xi} \cdot \nabla P_0 - \gamma P_0 \nabla \cdot \boldsymbol{\xi}. \tag{7.3.18}$$

$$\rho_{m0} \frac{\partial^2 \boldsymbol{\xi}}{\partial t^2} = \mathbf{F}(\boldsymbol{\xi}), \tag{7.3.19}$$

where $\mathbf{F}(\boldsymbol{\xi})$, the linear force operator, is given by

$$\begin{aligned} \mathbf{F}(\boldsymbol{\xi}) = & \frac{1}{\mu_0}[(\nabla \times \{\nabla \times (\boldsymbol{\xi} \times \mathbf{B}_0)\}) \times \mathbf{B}_0 + (\nabla \times \mathbf{B}_0) \times \{\nabla \times (\boldsymbol{\xi} \times \mathbf{B}_0)\}] \\ & + \nabla[\boldsymbol{\xi} \cdot \nabla P_0 + \gamma P_0 (\nabla \cdot \boldsymbol{\xi})]. \end{aligned} \tag{7.3.20}$$

7.3.4 The Normal Mode Method

$$\boldsymbol{\xi}(\mathbf{r}, t) = \sum_n \boldsymbol{\xi}_n(\mathbf{r}) \exp(-\mathrm{i}\omega_n t). \tag{7.3.22}$$

$$-\rho_{\mathrm{m}0}\, \omega_n^2\, \boldsymbol{\xi}_n = \mathbf{F}(\boldsymbol{\xi}_n). \tag{7.3.23}$$

The normal mode method is a brute force method for determining the eigenmodes and eigenfrequencies of the system using Eq. (7.3.23). Once these are known, the general solution (7.3.22) can be constructed by superposition. Next, we discuss the energy principle, which is a more subtle and even more powerful method of testing the stability of a magnetostatic equilibrium.

7.3.5 The Energy Principle

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \left[\frac{1}{2}\rho_{\mathrm{m}0}\left|\frac{\partial \boldsymbol{\xi}}{\partial t}\right|^2 - \frac{1}{2}\boldsymbol{\xi}^* \cdot \mathbf{F}(\boldsymbol{\xi})\right]\mathrm{d}^3x = 0, \qquad (7.3.33)$$

By simple inspection, one can see that Eq. (7.3.33) is an energy conservation equation for the perturbed system. It implies that

$$\delta K + \delta W = C = \text{constant}, \qquad (7.3.36)$$

where

$$\delta K = \frac{1}{2}\int_V \rho_{\mathrm{m}0}\left|\frac{\partial \boldsymbol{\xi}}{\partial t}\right|^2 \mathrm{d}^3x \qquad (7.3.37)$$

is the perturbed kinetic energy and

$$\delta W = -\frac{1}{2}\int_V \boldsymbol{\xi}^* \cdot \mathbf{F}(\boldsymbol{\xi})\,\mathrm{d}^3x \qquad (7.3.38)$$

is the perturbed potential energy. Note that this equation differs from Eq. (6.4.21) for the total energy, W, because it represents the energy change, δW, relative to the equilibrium state, due to the linearized small-amplitude perturbation.

7.3.6 A More Useful Form for δW

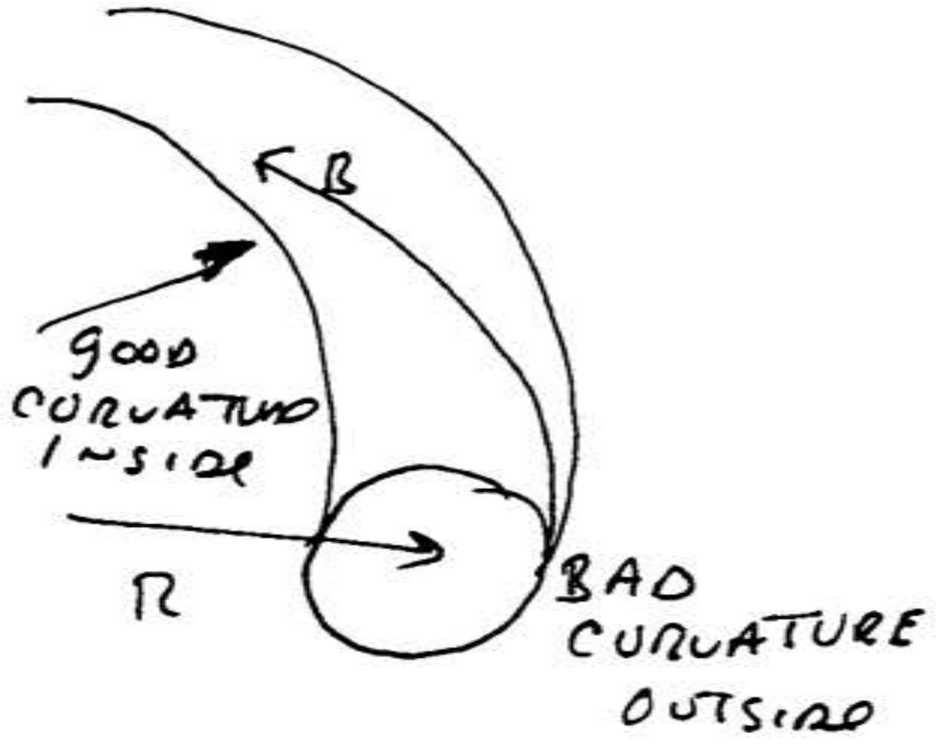
$$\delta W = \frac{1}{2}\int_V \left[\frac{1}{\mu_0}|\nabla\times(\boldsymbol{\xi}\times\mathbf{B}_0)|^2 + \gamma P_0|\nabla\cdot\boldsymbol{\xi}|^2 \right.$$

$$\left. -\,\boldsymbol{\xi}^*\cdot\mathbf{J}_0\times\{\nabla\times(\boldsymbol{\xi}\times\mathbf{B}_0)\} - \boldsymbol{\xi}^*\cdot\nabla(\boldsymbol{\xi}\cdot\nabla P_0)\right] d^3x. \tag{7.3.52}$$

The first two terms on the right-hand side of the above equation are positive and stabilizing, while the third and fourth terms can be destabilizing. The first term represents the energy required to bend field lines. The second term represents the energy required to compress a plasma with non-zero equilibrium pressure. The third term, which depends explicitly on the equilibrium current density, $\mathbf{J}_0$, can potentially drive instabilities of the “kink” type. The fourth term, which depends explicitly on the equilibrium pressure gradient, can potentially drive instabilities of the “ballooning” or “interchange” type. For an ideal MHD plasma, an instability always arranges its eigenfunction in such a way as to minimize the stabilizing contributions to δW (the first and second terms). For example, in an infinite cylinder or a torus, the marginally stable ($\omega_n = 0$) eigenfunctions are incompressible and obey the condition $\nabla\cdot\boldsymbol{\xi} = 0$, which reduces the second stabilizing term in δW to zero.

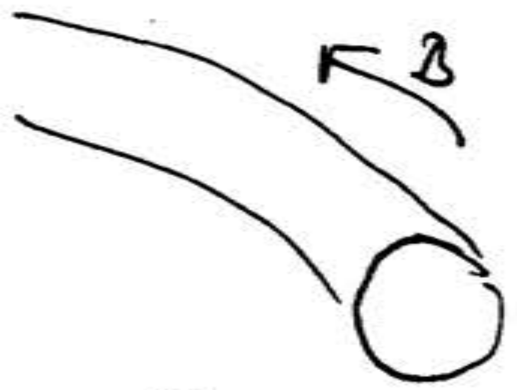
Ballooning Modes

EXAMPLE



TOKAMAK

($k_\parallel \neq 0$)



PURE TOROIDAL FIELD

$k_\parallel = 0$

WHEN $k_\parallel \neq 0$ (LIKE A TOKAMAK)

$$\nabla_\perp \cdot J_\perp = -\nabla_\parallel \cdot J_\parallel$$

$$\nabla_\perp \cdot J_{POL} + \underbrace{\nabla_\parallel \cdot J_\parallel}_{\text{FIELD LINE BENDING}} + \nabla_\perp \cdot J_{CURVATURE} = 0$$

$$j\frac{M_i n_0}{B^2}\nabla_\perp \omega \nabla_\perp \tilde{\Phi} - \nabla_\parallel \frac{1}{\mu_0}\nabla^2 \tilde{A}_\parallel$$

$$\tilde{A}_\parallel = \frac{k_\parallel}{\omega}\tilde{\Phi}$$

(SINCE $\tilde{E} = 0$)

$$-j\frac{M_i n_0}{B^2}\omega k_\perp^2 \tilde{\Phi}\left(1 - \frac{k_\parallel^2 V_A^2}{\omega^2}\right) \dots$$

$$V_A^2 = \frac{B^2}{\mu_0 n_0 M_i}$$

Ballooning Stability

Drive: Plasma Pressure

$(R_c = R)$

$$\omega^2 - k_\parallel^2 V_A^2 = -(\mathrm{DRIVE})\,\frac{k_y^2}{k^2}$$

$$\mathrm{DRIVE} = -\frac{p'}{p}\,\frac{2(T_e+T_i)}{M_i R_c}$$

PRESSURE DRIVE

IF $k_\parallel$ COULD GO TO ZERO (Z-PINCH)
THEN $\omega^2 = -(\mathrm{DRIVE})\, k_y^2/k^2$ UNSTABLE WITH (DRIVE) > 0

BUT WITH <u>FINITE</u> $k_\parallel$, FIELD-LINE BENDING IS <u>STABILIZING</u>

<u>EXAMPLE</u>

TOKAMAK: $k_\parallel \sim \frac{1}{qR} \sim$ CONNECTION LENGTH FROM OUTSIDE TO INSIDE

UNSTABLE WHEN DRIVE $> k_\parallel^2 V_A^2 \sim V_A^2/(qR)^2$

$$\mathrm{DRIVE} \sim \frac{2(T_e+T_i)}{M_i\, a_c R_c} > \frac{V_A^2}{q^2R^2} \Rightarrow \frac{2n(T_e+T_i)}{B^2/\mu_0} > \frac{a}{Rq^2}$$

beta limit

Kink Modes
(The most dangerous instabilities for current-carrying plasma.)

ION POLARIZATION

FIELD LINE BENDING

PRESSURE DRIVE

$$\nabla\cdot J = 0 = \nabla\cdot J_{POL} + \nabla_{\parallel}\cdot J_{\parallel} + \nabla\cdot J_{CUR} + \nabla\cdot\left(J_0\frac{\delta \bar{B}_{\perp}}{B_{\parallel}}\right) = 0$$

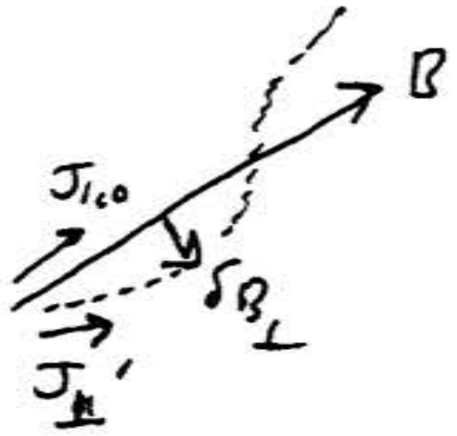
DRIFT DUE TO PERTURBED FIELD

Drive: Plasma Current Gradient

$$\nabla\cdot\left(J_0\frac{\delta B_{\perp}}{B_{\parallel}}\right) = \frac{\delta \bar{B}_{\perp}}{B_{\parallel}}\cdot\nabla J_{\parallel 0}$$

CURRENT GRADIENT DRIVE

$$J_{\perp\delta B} \sim J_{\parallel 0}\frac{\delta \bar{B}_{\perp}}{B}$$



"ELECTROMAGNETIC DRIFT"

$$\bar{B} = B_0\hat{z} + \delta\bar{B}_{\perp}$$

$$\hat{z}\times(\bar{v}\times\bar{B}) = \bar{v}B_{\parallel} - v_{\parallel}\bar{B} = B_{\parallel}\bar{v}_{\perp} - v_{\parallel}\delta\bar{B}_{\perp}$$

$$\omega^2 - k_{\parallel}^2 V_A^2 = -(\text{DRIVE})\frac{k_y^2}{k_{\perp}^2} + \frac{B_0}{n_0 m_i}\frac{k_{\parallel}k_y}{k_{\perp}^2}\left(\frac{dJ_{\parallel 0}}{dx}\right)$$

PRESSURE DRIVE

CURRENT GRADIENT DRIVE

$\frac{dJ_{\parallel 0}}{dx} < 0$ (USUALLY)

current limit

$$k_{\parallel} = \frac{1}{qR}(m - nq)$$

SO IF $nq > m$ STABLE !!

Reduced MHD (Simple, but Powerful Theory from 1970's)

- Cylindrical Reduced MHD
- Ideal instabilities
- Tearing instabilities
- RWMs and FWMs

- M. Rosenbluth, D. Monticello, H. Strauss, and R. White, Phys Fluids 19, 1987 (1976).
- H. Strauss, D. Monticello, M. Rosenbluth, and R. White, Phys Fluids 20, 390 (1977).
- R. Izzo, et al., Phys Fluids 26, 3066 (1983).
- G.T.A. Huysmans, J.P. Goedbloed, and W. KERNER, "Free boundary resistive modes in tokamaks" Phys. Fluids B 5, 1545 (1993).

The Physics of Fluids, Vol. 19, No. 12, December 1976

Numerical studies of nonlinear evolution of kink modes in tokamaks

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(Received 22 December 1975; final manuscript received 22 June 1976)

A set of numerical techniques for investigating the full nonlinear unstable behavior of low-β kink modes of given helical symmetry in tokamaks is presented. Uniform current density plasmas display complicated deformations including the formation of large vacuum bubbles provided that the safety factor q is sufficiently close to integral. Fairly large $m=1$ deformations, but not bubble formation, persist for a plasma with a parabolic current density profile (and hence shear). Deformations for $m \geq 2$ are, however, greatly suppressed.

I. INTRODUCTION

It has been suggested by Kadomtsev[1] that the kink mode plays an instrumental role in the disruptive instability seen in tokamaks. The imagined mechanism is that the nonlinear kink mode development leads to highly distorted shapes with the vacuum on the inside and the plasma on the outside, the so-called bubble state.

The expected large distortions of the plasma led us to treat the problem by numerical methods. A numerical treatment of the nonlinear kink mode in tokamaks in a straightforward way is difficult, however, because of the various time scales involved (Alfvén waves and sound waves, and relatively slow kinks) and because of the free boundary between plasma and vacuum. In Sec.

II. THE REDUCED SET OF EQUATIONS

The energy reservoir for free boundary kinks is very large and the toroidal case is adequately treated by the cylindrical approximation. Hence, we model the tokamak by a cylinder of length $L=2\pi R$, R being the major radius of the plasma.

We also restrict ourselves to following the nonlinear development of perturbations of a fixed helical symmetry. This, together with the fact that the walls and equilibrium are cylindrical, implies that all quantities are functions of τ, r, and t only, where $\tau = m\theta + kz$ and $k=n/R$. Here, m and n are the mode numbers of the original perturbation, which has the form $f(r)\exp[i(m\theta + kz)]$.

Helical symmetry has the obvious advantage of reducing the three-dimensional numerical calculation to a two-dimensional one $[\partial/\partial z = (k/m)(\partial/\partial\theta)]$. In addition, this symmetry, together with $\nabla\cdot\mathbf{B}=0$ implies that B_θ, B_r, and B_z may be related to a scalar ψ by

$$B_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta}, \quad B_\theta = -\frac{\partial\psi}{\partial r} - \frac{kr}{m}B_z \tag{1}$$

or

$$\mathbf{B} = \nabla\psi \times \hat{Z} - (kr/m)B_z\hat{\theta} + B_z\hat{Z}\ . \tag{2}$$

The function ψ is a flux function, i.e., $(B\cdot\nabla)\psi = 0$. It

Cylindrical Reduced MHD

the order of $\epsilon^2 B_0$. To lowest order in ϵ this unknown variation of the toroidal field can be eliminated from the problem by taking the curl of the momentum equation. The resulting equations are the standard low-β tokamak reduced equations that describe free-boundary kink modes[3]:

$$R_0^2 \frac{d\nabla^2 u}{dt} = \boxed{\mathbf{B}\cdot\nabla}(\nabla_\perp^2 \psi),$$

$$\frac{\partial \psi}{\partial t} = R_0^2 \boxed{\mathbf{B}\cdot\nabla} u,$$

important

$$\mathbf{B} = \nabla\psi \times \nabla\zeta + I_0 \nabla\zeta,$$

$$\mathbf{V} = R_0^2 \nabla u \times \nabla\zeta,$$

$$\nabla_\perp^2 = \frac{\partial^2}{\partial R^2} + \frac{\partial^2}{\partial z^2}.$$

Here $I_0 = B_0 R_0$ and $\nabla\zeta = \hat{\zeta}/R_0$.

$$A_\phi = I_0(r/2)$$

$$A_\parallel \approx A_z = \psi_0(r) + \tilde{\psi}(r,\phi)$$

Plasma Physics Series

Tokamak Plasma:

A Complex Physical System

B B Kadomtsev

I V Kurchatov Institute of Atomic Energy, Moscow, Russia

Translation Editor: **Professor E W Laing**

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"In memory of Boris Kadomtsev," by E P Velikhov et al 1998, *Phys.-Usp.* **41** 1155; [https://doi.org/10.1070/PU1998v041n11ABEH000508]

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Cylindrical Reduced MHD

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Here $I_0 = B_0 R_0$ and $\nabla\zeta = \hat{\zeta}/R_0$.

$$A_\phi = I_0 (r/2)$$

$$A_\parallel \approx A_z = \psi_0(r) + \tilde{\psi}(r,\phi)$$

Plasma Physics Series

Plasma Stability 65

Highest Order:
Only Current Gradient Drive

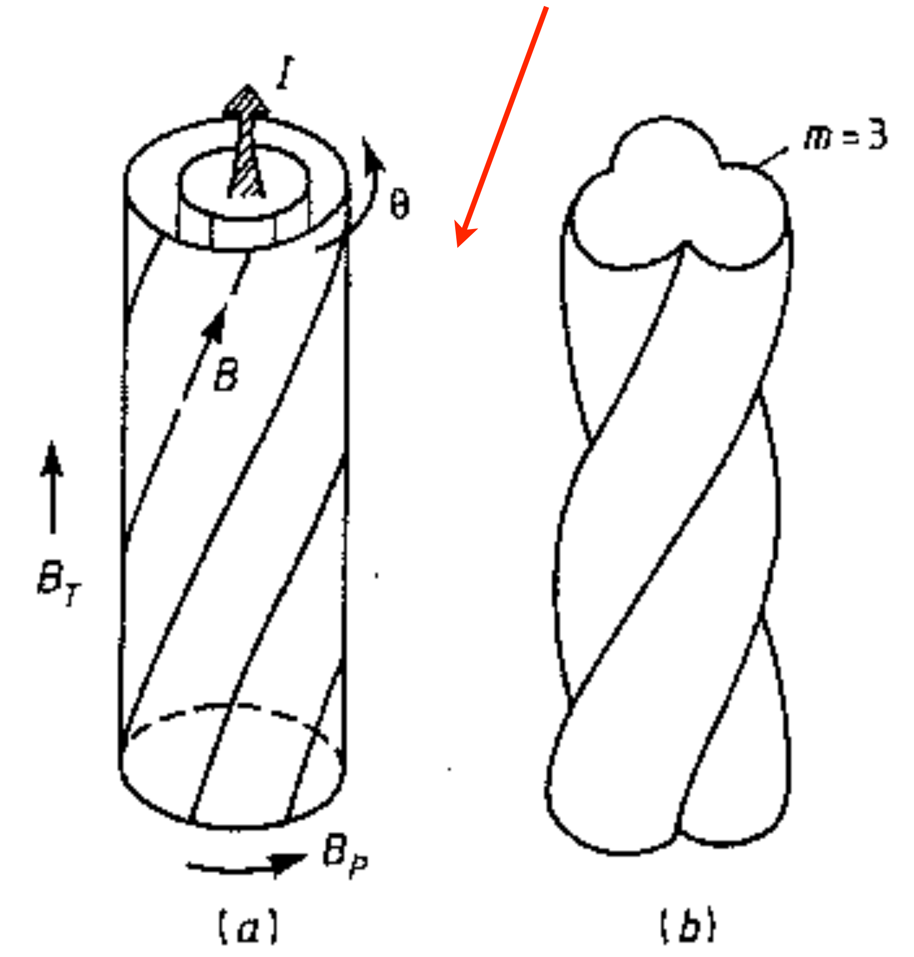

Figure 5.1 Helical perturbation of the tokamak plasma considered as a cylindrical column of length $2R$ with identical ends: (a) initial start; (b) perturbation of the $m = 3$ mode.

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“In memory of Boris Kadomtsev,” by E P Velikhov et al 1998, *Phys.-Usp.* **41** 1155; [https://doi.org/10.1070/PU1998v041n11ABEH000508]

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Reduced MHD

1970's HANK STRAUSS (NYU), MARSHALL ROSENBLUTH (PRINCETON)

THE BASIC SIMPLE ANALYTIC TOOL FOR UNDERSTANDING TOKAMAKS

UNTILL THE "MODERN" AGE OF COMPUTER CODES.

BASIC ASSUMPTIONS :

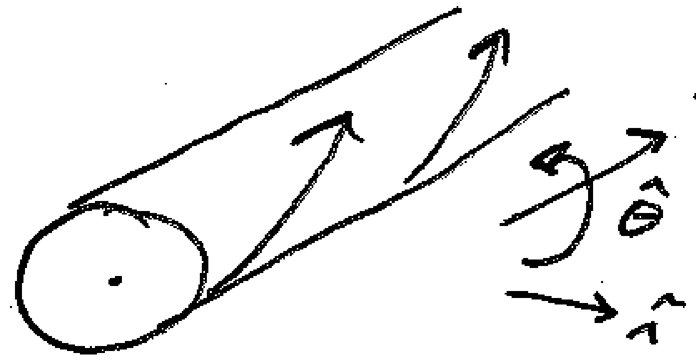
- LOW BETA $\langle\beta\rangle \sim (a/R)^2 \ll 1 \quad \beta_N \sim (a/R)$
- LARGE ASPECT RATIO $\epsilon \equiv a/R \ll 1$
- WITH $q \gtrsim 1$, THEN $B_P/B_T \sim \epsilon/q \ll 1$
- WITH $\epsilon \ll 1$, THEN $B_T \sim B_0\left(\frac{R_0}{R}\right) \sim B_0(1-\epsilon\cos\theta + \ldots) \approx$ CONSTANT

  BUT 1^{ST} ORDER IN ϵ IS <u>IMPORTANT</u>
- LET $\nabla \cdot \hat{\varphi}$ (OR $\nabla \cdot \hat{z}$) $= 0$, ELIMINATING ACOUSTIC MODES (NO "GAMS")
- BASICALLY, A "CYLINDRICAL" TOKAMAK (1D EQUILIBRIUM)

Basic Derivation

REF: KADOMTSEV, "TOKAMAK PLASMA: A COMPLEX PHYSICAL SYSTEM"
IOP (1992)

$\bar{B} = \bar{B}_\perp + \hat{z} B_z$ WITH $B_z \simeq$ CONSTANT



CYLINDRICAL COORDINATES

MHD:

$$\rho \frac{d\bar{v}_\perp}{dt} = -\nabla p + \bar{J} \times \bar{B}$$

$$\bar{E} = -\bar{v} \times \bar{B} \quad \text{(IDEAL)}$$

$$\bar{v} \cdot \hat{z} = 0$$

$$\nabla \cdot \bar{v}_\perp = 0 \quad \text{INCOMPRESSIBLE}$$

MAXWELL'S EM

$$\bar{J} = \frac{1}{\mu_0} \nabla \times \bar{B} \quad \text{(NO DISPLACEMENT CURRENT)}$$

$$\nabla \cdot \bar{J} = 0$$

$$\nabla \cdot \bar{B} = 0 = \nabla \cdot \bar{B}_\perp$$

$$\frac{\partial \bar{B}}{\partial t} = -\nabla \times \bar{E}$$

IDEAL MHD DESCRIBE PLASMA DYNAMICS AT ALFVEN TIME SCALE: $V \sim V_A \sim \sqrt{B^2/\mu_0 \rho}$ (FAST!)

Stream Function and Poloidal Flux

WITH B_z = CONSTANT, THE REDUCED MHD DYNAMICS IS
$v_z = 0$
DESCRIBED BY FOUR UNKNOWN FUNCTIONS OF (r, θ, z, t):

$$\bar{B}_\perp(r, \theta, z, t) \quad \text{AND} \quad \bar{V}_\perp(r, \theta, z, t)$$

TWO POTENTIALS INSTEAD OF TWO VECTOR FIELDS

WE "GREATLY SIMPLIFY" THE MATH BY INTRODUCING THE STREAM FUNCTION, χ, AND THE POLOIDAL FLUX FUNCTION, ψ

$$\bar{B}_\perp(r, \theta, z, t) = \hat{z} \times \nabla \psi \qquad \bar{V}_\perp(r, \theta, z, t) = \hat{z} \times \nabla \chi$$

$$\nabla \cdot \bar{B} = 0 \qquad \nabla \cdot \bar{V} = 0$$

AMPERE'S LAW

$$\bar{J} = \frac{1}{\mu_0} \nabla \times (\hat{z} \times \psi)$$

$$\mu_0 \bar{J} = \hat{z} \nabla^2 \psi - (\hat{z} \cdot \bar{\nabla}) \bar{\nabla} \psi$$

AXIAL VORTICITY

$$\Omega_z \equiv \hat{z} \cdot \nabla \times \bar{V}_\perp = \nabla^2 \chi$$

TWO UNKNOWN POTENTIALS:

$$\psi(r, \theta, z, t) \quad \chi(r, \theta, z, t)$$

Simplifying the MHD Equations

$$\rho \frac{d\bar{V}_\perp}{dt} = -\nabla p + \frac{1}{\mu_0}(\nabla \times B) \times B$$

$$= -\nabla\left(p + B^2/2\mu_0\right) + \frac{1}{\mu_0}(\bar{B}\cdot\bar{\nabla})\bar{B}$$

$$\hat{z}\cdot\nabla\times\left[\text{''} \quad = \quad \text{''}\right]$$

ASSUME $\rho \simeq$ UNIFORM

$$\rho \frac{d}{dt}\left(\hat{z}\cdot\nabla\times V_\perp\right) = \frac{1}{\mu_0}(\bar{B}\cdot\bar{\nabla})\left(\hat{z}\cdot\nabla\times\bar{B}\right)$$

$$\rho \frac{d}{dt}(\nabla_\perp^2 \chi) = \frac{1}{\mu_0}(\bar{B}\cdot\bar{\nabla})\nabla_\perp^2\psi = (\bar{B}\cdot\bar{\nabla})J_z$$

$$\rho \frac{d}{dt}\Omega_z = (\bar{B}\cdot\bar{\nabla})J_z$$

"AXIAL VORTICITY CHANGES ACCORDING TO FIELD-ALIGNED VARIATION OF AXIAL CURRENT"

Simplifying the Induction Equation

$$\frac{\partial \bar{B}}{\partial t} = \nabla \times \left(\bar{V}_\perp \times \bar{B} \right) = \bar{V}_\perp \bar{\nabla}\cdot\bar{B} - \bar{B}\bar{\nabla}\cdot\bar{V} + (\bar{B}\cdot\bar{\nabla})\bar{V} - (\bar{V}\cdot\bar{\nabla})\bar{B}$$

$$= \nabla \times \left(\bar{V}_\perp \times \bar{B}_\perp \right) + B_z \frac{\partial \bar{V}_\perp}{\partial z}$$

$$= (\bar{B}_\perp\cdot\bar{\nabla})\bar{V}_\perp - (\bar{V}_\perp\cdot\bar{\nabla})\bar{B}_\perp + B_z \frac{\partial \bar{V}_\perp}{\partial z}$$

$$\frac{\partial \bar{B}_\perp}{\partial t} + (\bar{V}_\perp\cdot\bar{\nabla})B_\perp = \frac{d\bar{B}_\perp}{dt} = (\bar{B}\cdot\bar{\nabla})\bar{V}_\perp$$

SUBSTITUTING FLUX FUNCTIONS

$$\boxed{\frac{d}{dt}\psi = (\bar{B}\cdot\bar{\nabla})\chi}$$

"POLOIDAL FLUX EVOLVES DYNAMICALLY DUE TO FIELD-ALIGNED CHANGES IN THE STREAM FUNCTION"

"Simplest" Kink Mode Theory

- Reduced MHD (plasma torus with a strong toroidal field)
- Kink modes

$$\frac{d}{dt}\psi = (\mathbf{B}\cdot\nabla)\chi$$

"POLOID FLUX EVOLVES DYNAMICALY DUE TO FIELD-ALIGNED CHANGES IN THE STREAM FUNCTION"

$$\rho \frac{d}{dt}\Omega_z = (\mathbf{B}\cdot\nabla) J_z$$

"AXIAL VORTICITY CHANGES ACCORDING TO FIELD-ALIGNED VARIATION OF AXIAL CURRENT"

Importance of $B\cdot\nabla$

$$\rho \frac{d}{dt} \nabla_\perp^2 \chi = \frac{1}{\mu_0}(B\cdot\bar\nabla)\nabla_\perp^2 \psi \qquad \text{(MHD)}$$

$$\frac{d}{dt}\psi = (\bar B\cdot\bar\nabla)\chi \qquad \text{(INDUCTION)}$$

$$= i\,\bar k\cdot\bar B_0 \qquad \bar k \equiv -\frac{n}{R}\hat z + \frac{m}{r}\hat\theta \quad \text{(PLUS RADIAL TERMS)}$$

LINEAR

$$\bar B\cdot\bar\nabla = B_z\frac{\partial}{\partial z} + B_\perp\cdot\bar\nabla_\perp = -i\frac{n}{R}B_z + i\frac{m}{r}B_\theta = i\,\frac{B_\theta(r)}{r}\left(m - nq(r)\right)$$

$$\text{WITH } q(r) = \frac{r\,B_z}{R\,B_\theta(r)} = \text{SAFETY FACTOR}$$

$\bar B\cdot\bar\nabla \to 0$ WHEN $m/n = q(r)$ {RESONANCE}

WHEN $\bar B\cdot\bar\nabla \neq 0$, THEN IDEAL REDUCED MHD MAKE SENSE

$B\cdot\bar\nabla = 0$, THEN REDUCED MHD <u>DOES</u> NOT DESCRIBE <u>DYNAMICS</u>

(SIDE BAR! $\bar B\cdot\bar\nabla = 0$ DEFINES "INTERCHANGE" MODES. THESE ARE THE <u>DOMINANT</u> MODES IN MAGNETOSPHERES AND DIPOLES, ETC)

First: Equilibrium

APPH 6102 Plasma Physics II: In-Class Worksheet

Answer the following without looking at your notes or textbooks.

Question

In this problem, you are to derive the plasma equilibrium condition for a low-β, very large aspect ratio ($a/R \ll 1$), magnetized plasma cylinder. This equilibrium condition is expressed as the relationship between the plasma pressure, $P(\psi)$, and the plasma flux function $\psi(r)$, which will be a function of radius, r.

(First,) Equilibrium

$V_\perp = 0, \ \frac{\partial}{\partial t} = 0,$

$$0 = -\nabla P + J \times B$$
$$= -\nabla P + \bar{J} \times (\hat{z} \times \nabla\psi)$$
$$= -\nabla P - J_z \nabla \psi$$

ALL EQUILIBRIUM VARIATION IS RADIAL, IN $\bar{\nabla}\psi$ DIRECTION

SO

$$\frac{\nabla\psi \cdot \nabla P}{(\nabla\psi)^2} = -J_z \implies \frac{\partial P}{\partial \psi} = -J_z$$

(= CONSTANT) WHEN J_z = CONSTANT SHAFRANOV'S

$$q(r) = \frac{r B_z}{R B_p(r)}$$

← EQUILIBRIUM EQUATION FOR $\psi(r)$

$$\mu_0 J_z = \nabla \times B_p = \nabla_\perp^2 \psi = -\mu_0 \frac{\partial P}{\partial \psi}$$

← EQUILIBRIUM FUNCTIONS

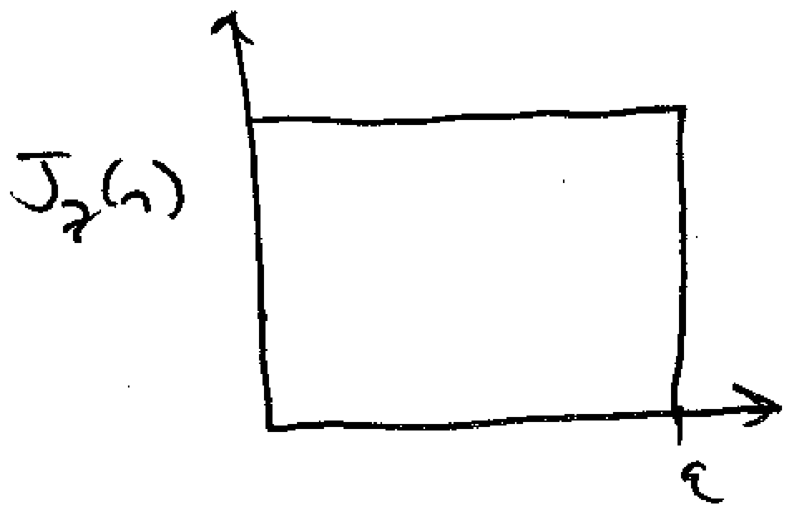
$$= \frac{1}{r}\frac{\partial}{\partial r}(r B_p)$$

$P(\psi), J_z(\psi), q(\psi)$

$$= \frac{B_z}{rR}\frac{\partial}{\partial r}\left(r^2 / q(r)\right)$$

$P(r), J_z(r), q(r)$

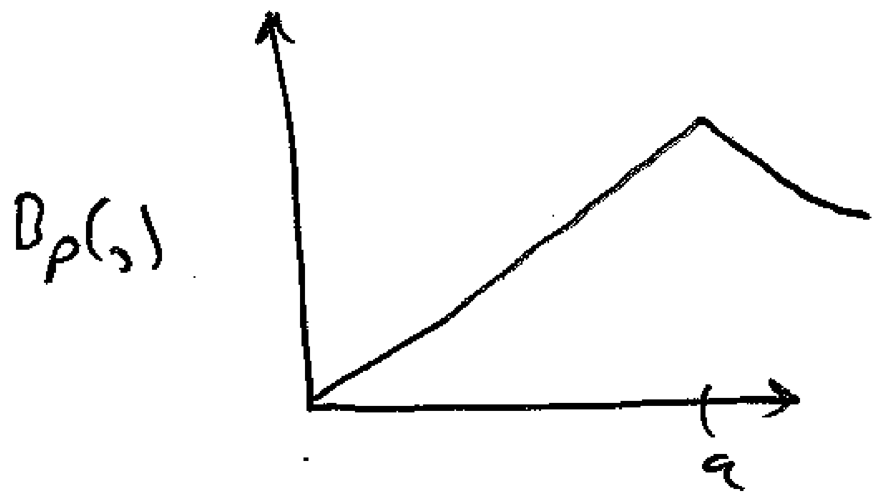
Step 1: Equilibrium (Shafranov's Simplest Case)



CONSTANT J_z

$\ell_i = \frac{1}{2}$

$$B_p(a) = \frac{a\,\mu_0 J_z}{2} \qquad B_p(r) = \frac{r\,\mu_0 J_z}{2}$$

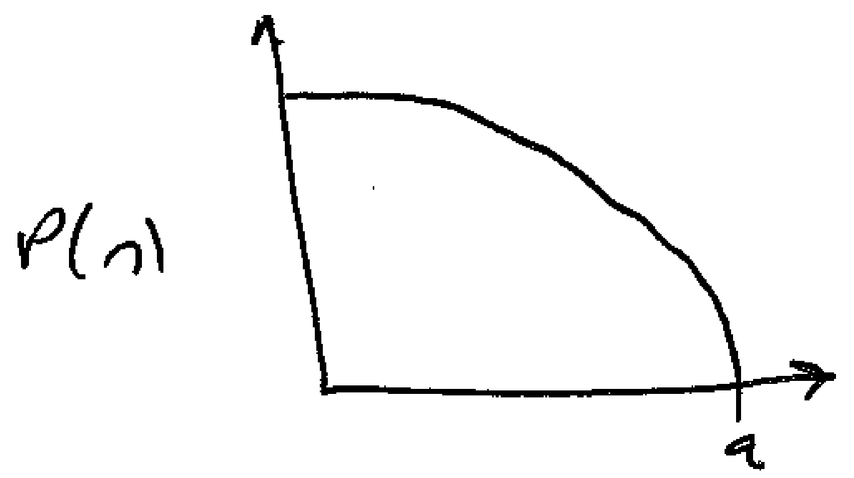


$$q(r) = q(a) = \text{CONSTANT} \;{**} = \frac{2B_z}{\mu_0 R J_z}$$

$$B_p(r) = \hat{z} \times \nabla\psi$$

OR

$$\psi(r) = \frac{r^2 B_p(a)}{2a} = \frac{r}{2} B_p(r) = \frac{r^2}{a^2}\left(\frac{a B_p}{2}\right)$$



$$\frac{\partial P}{\partial \psi} = -J_z = -\frac{2B_p(a)}{a\mu_0}$$

OR

$$P(\psi) = \left(\psi(a) - \psi(r)\right)\frac{2B_p(a)}{a\mu_0} = \left(1 - r^2/a^2\right)\frac{B_p^2(a)}{\mu_0}$$

$$\langle\beta\rangle = \frac{2\pi R \int_0^a 2\pi r\,dr\,P(r)}{2\pi R\,\pi a^2\; B_z^2/2\mu_0} = \frac{B_p^2(a)}{B_z^2} = \epsilon^2/q^2(a) \ll 1$$

$$\ell_i = \frac{\text{POLOIDAL FIELD ENERGY}}{\frac{1}{4}\mu_0 R I_p^2} = \frac{2\pi R \int_0^a 2\pi r\,dr\,\frac{B_p^2(r)}{2\mu_0}}{\frac{1}{4}\mu_0 R\left(\pi^2 a^4 J_z^2\right)} = \frac{1}{2}$$

$$\beta_p = \langle\beta\rangle\frac{B_z^2}{B_p^2} = 1 \;({**})$$

$$\langle\beta_N\rangle \equiv \frac{\langle\beta\rangle\,\cancel{B}}{I/aB\;(\text{MKS})\;(\text{MA})} = \left(\frac{2\epsilon}{q(a)}\right)\left(\frac{\cancel{a}}{R}\right)$$

NOT TINY (OF ORDER 1)

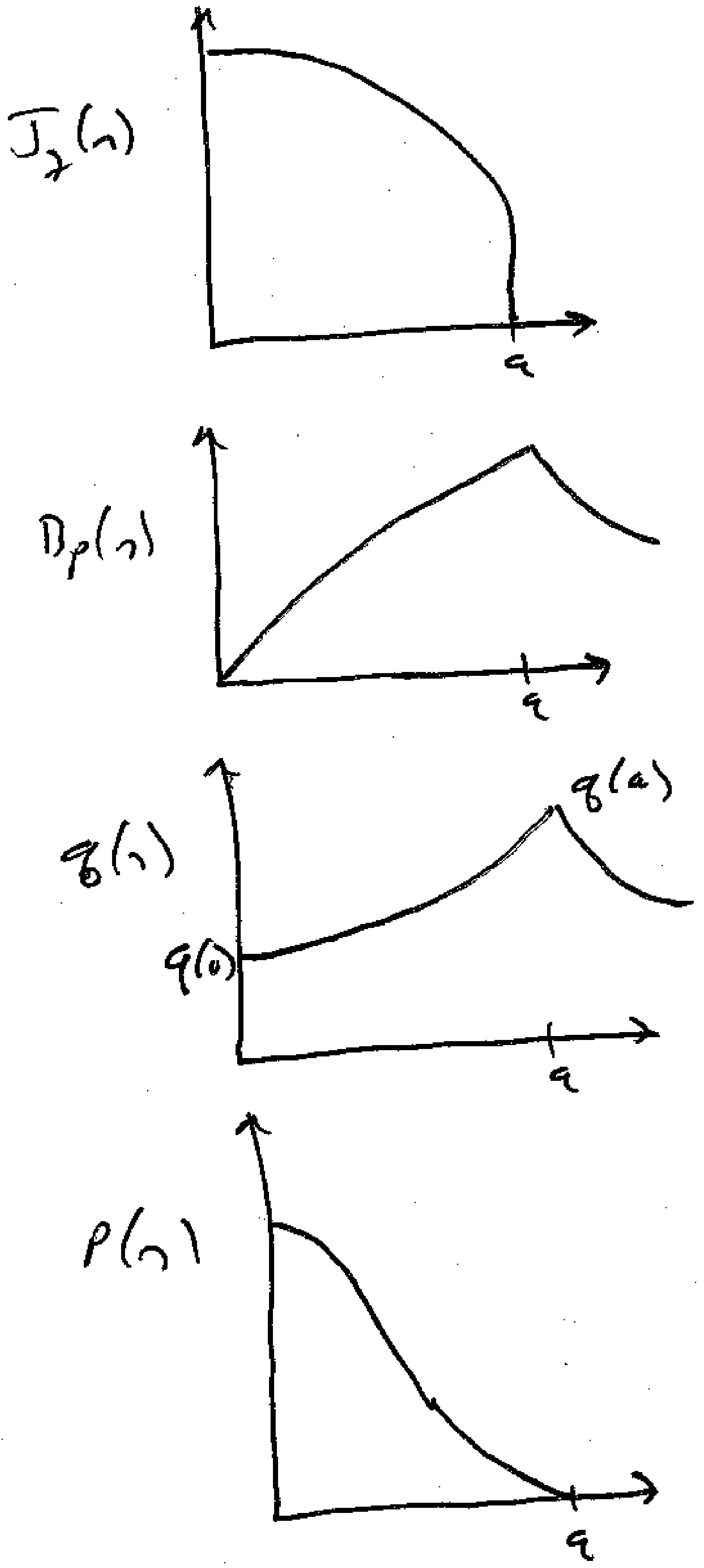
PLASMA IS NOT DIAMAGNETIC

IS NOT PARAMAGNETIC

NO CHANGE IN B_z

Wesson's Cylindrical Equilibrium

REF: WESSON, NUC FUSION 18 (1978) P. 87



$$J_z(r) = J_0\left(1 - r^2/a^2\right)^{\nu}$$

J_0 = CENTRAL CURRENT DENSITY

$$q(0) = \frac{2B_z}{\mu_0 R J_0} \qquad \mu_0 I_p = \frac{2\pi^2 a^2 B_z}{\mu_0\, q(0)\, R(1+\nu)}$$

$$\nu = \frac{q(a)}{q(0)} - 1$$

$$\ell_i \neq \tfrac{1}{2}$$

$$\langle\beta\rangle \sim \epsilon^2/q_a^2 \qquad \beta_p \sim 1 \qquad \langle B_n\rangle \sim 20\,\epsilon/q_a$$

EQUILIBRIUM SET BY $q(0), q(a)$
TWO PARAMETERS

$$\frac{r}{q}\frac{dq}{dr} = s = \text{MAGNETIC SHEAR **} \neq 0$$

$B\cdot\nabla \to 0$ AT INTERNAL RESONANT LAYER

$$\left(m - n/q(r)\right) \simeq 0$$

PRESSURE PROFILE CAN BE MORE PEAKED

Linearized Reduced MHD

MHD: $\rho \frac{d}{dt} \nabla_\perp^2 \chi = \frac{1}{\mu_0} (\vec{B} \cdot \nabla) \nabla_\perp^2 \psi$ $\qquad \frac{d}{dt} = -i\omega$

INDUCTION: $\frac{d\psi}{dt} = (B \cdot \nabla) \chi$ $\qquad B_0 \cdot \nabla \rightarrow i \frac{B_p(r)}{r} (m - n q(r))$

$$\boxed{(\vec{B} \cdot \nabla) \nabla_\perp^2 \psi = \underbrace{(\tilde{B} \cdot \nabla) \nabla_\perp^2 \psi_0}_{\frac{\mu_0 J_z(r)}{\text{[illegible]}}} + (\vec{B}_0 \cdot \nabla) \nabla_\perp^2 \tilde{\psi}} + \text{NONLINEAR TERMS}$$

$\leftarrow$ WHEN EQUILIBRIUM CURRENT DENSITY VARIES WITHIN PLASMA

$$\tilde{B} \cdot \nabla J_z(r) = (\hat{z} \times \nabla \psi) \cdot \hat{r} \frac{\partial J_z}{\partial r} = -\hat{\theta} \cdot \nabla \psi \frac{\partial J_z^0}{\partial r}$$

LINEAR MHD:

$$\boxed{\underbrace{-\rho \omega \nabla_\perp^2 \tilde{\chi}}_{\text{PLASMA INERTIA}} = -\frac{m}{r} \frac{\partial J_z}{\partial r} \tilde{\psi} + \frac{B_p}{\mu_0 r} (m - n q(r)) \nabla_\perp^2 \tilde{\chi}}$$

LINEAR INDUCTION:

$$-\omega \tilde{\psi} = \frac{B_p}{r} (m - n q(r)) \tilde{\chi}$$

Alfvén Waves in Shafranov's Equilibrium

WITHIN PLASMA

$$-\rho\,\omega\,\nabla_\perp^2 \hat{X} = \frac{B_p}{\mu_0 r}(m - nq)\,\nabla_\perp^2 \hat{\varphi}$$

$$-\omega\,\hat{\varphi} = \frac{B_p}{\mu_0 r}(m - nq)\,X$$

NOTE: $\frac{B_p}{r} = \text{CONSTANT} = \frac{B_p(a)}{a}$

$$\omega_A^2 \equiv \frac{B_p^2/\mu_0\rho}{a^2} = \frac{B_z^2/\mu_0\rho}{(\varepsilon_a R)^2}$$

$$= V_A^2/(qR)^2$$ ALFVEN TRANSIT TIME!

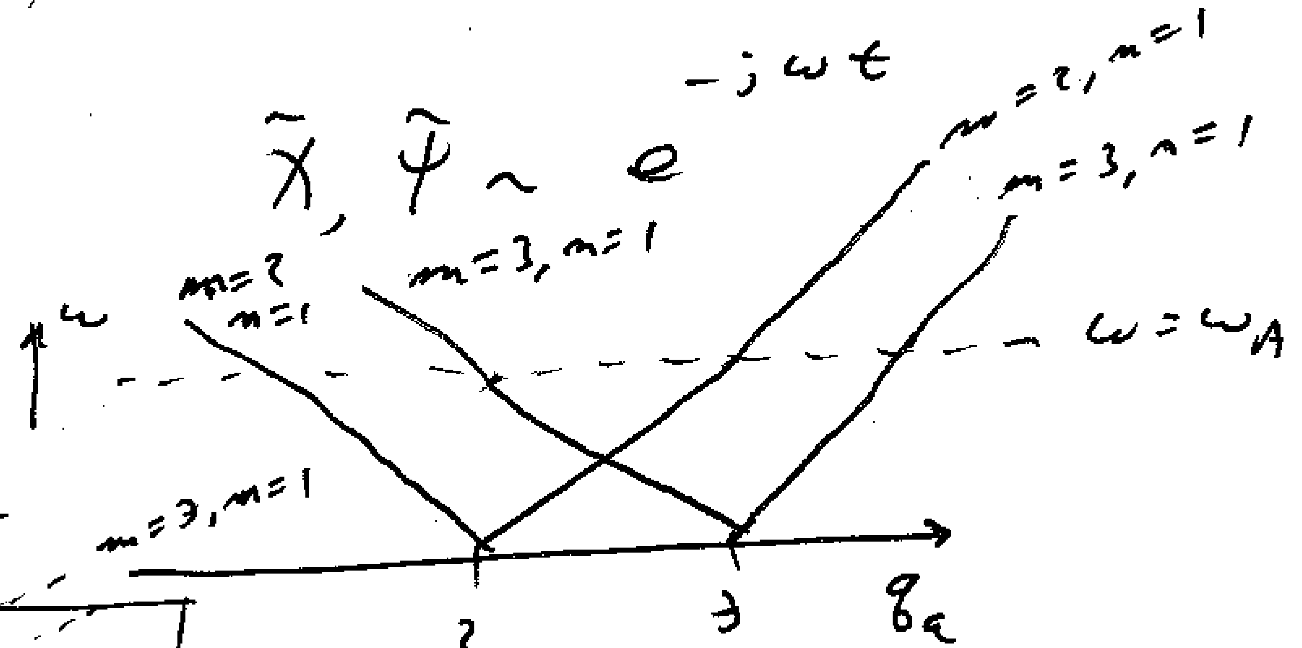
NORMAL MODES (ALFVEN WAVES)

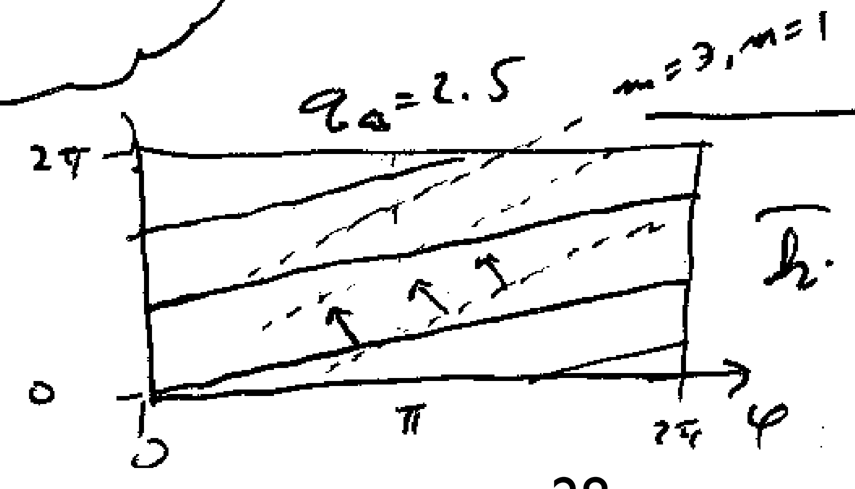
$$\begin{pmatrix} \rho\,\omega & \frac{B_p}{\mu_0 a}(m - nq) \\ \frac{B_p}{\mu_0 a}(m - nq) & \omega \end{pmatrix} \cdot \begin{pmatrix} \hat{X}(r) \\ \tilde{\varphi}(r) \end{pmatrix}$$

WITH $\rho \approx$ CONSTANT

$$\tilde{X}, \tilde{\varphi} \sim e^{-j\omega t}$$

$$\boxed{\omega^2 = \omega_A^2 (m - nq_a)^2}$$



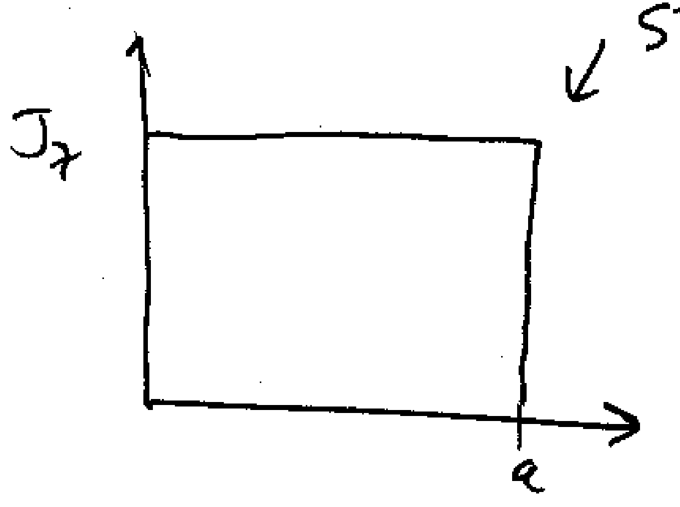


$\vec{k} \cdot \vec{B}_0 \neq 0$

RADIAL STRUCTURE NOT SPECIFIED

Global Kink Eigenmodes

$$\nabla_\perp \cdot \vec{A} = \frac{1}{r}\frac{\partial}{\partial r}(r A_r) + \dots$$



STEP DISCONTINUITY AT PLASMA EDGE

$$\frac{d}{dt}\nabla_\perp\cdot(\rho\nabla_\perp\chi) = (\hat{z}\times\nabla_\perp\tilde\psi)\cdot\hat{r}\,\frac{\partial J_z}{\partial r} + i\frac{B_\theta}{\mu_0 r}(m-nq)\nabla_\perp^2\tilde\psi$$

VERY BIG AT EDGE

$$\int_{a^-}^{a^+} 2\pi r\,dr \left\{ \text{" MHD "} \right\} \Rightarrow$$

$$\omega\rho\frac{\partial\chi}{\partial r}\Big|_{a^-} = \frac{2mB_\theta(a)}{\mu_0 a^2}\hat\psi_a + \frac{B_\theta(a)}{\mu_0 a}(m-nq)\hat\psi_a\underbrace{\left[\frac{1}{\tilde\psi}\frac{\partial\tilde\psi}{\partial r}\Big|_{a^+} - \frac{1}{\tilde\psi}\frac{\partial\tilde\psi}{\partial r}\Big|_{a^-}\right]}$$

$$\boxed{\Delta'(a) \equiv \frac{1}{\psi}\left(\frac{\partial\tilde\psi}{\partial r}\Big|_{a^+} - \frac{\partial\psi}{\partial r}\Big|_{a^-}\right)}$$

THIS MEASURES PERTURBED SURFACE CURRENT ON PLASMA

IMPORTANT:

$$\omega\rho\frac{\partial\chi}{\partial r}\Big|_{a^-} = \frac{2mB_\theta(a)}{\mu_0 a^2}\hat\psi_a\left[(m-nq)\left(\frac{-\Delta'(a)}{2m/a}\right) - 1\right]$$

$$\Delta'(a)\hat\psi_a = \mu_0\tilde{K}_z(\theta,\varphi)/a$$

Global Kink Eigenmodes

WHAT ARE $(\psi(r), \chi(r))$ INSIDE AND OUTSIDE PLASMA?

Boundary conditions

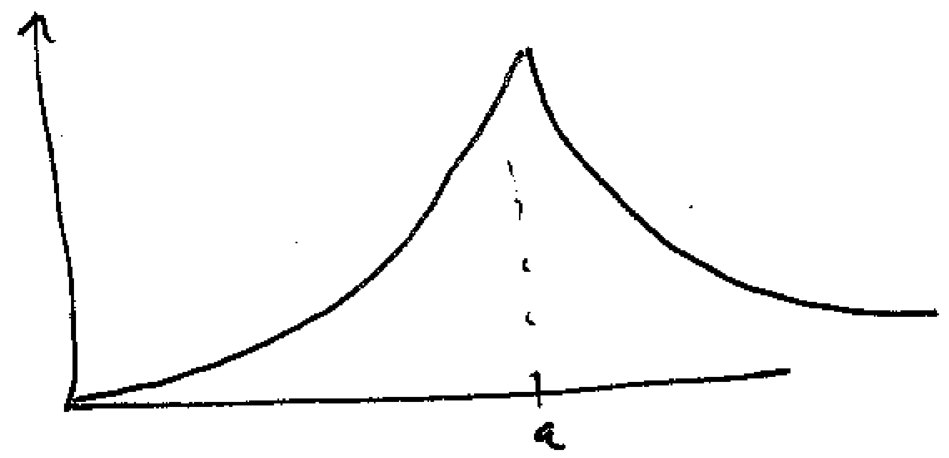
$\nabla^2 \psi = 0$ (NO CURRENT) OUTSIDE PLASMA

$\nabla^2 \chi = 0$ (NO FLOW, VORTICITY, NO SLIP)

$\nabla^2 \psi = 0$ (NO CURRENTS INSIDE PLASMA TOO)

$\nabla^2 \chi = 0$ (NO VORTICITY WITHIN PLASMA)

PERTURBED FIELDS + PLASMA MOTION
BUT NO CURRENTS OR VORTICITY



NO WALL

$$\psi(r) \sim \left(\frac{r}{a}\right)^m \quad r<a$$

$$\sim \left(\frac{a}{r}\right)^m \quad r>a$$

WITH WALL

$$\sim \left(\frac{r}{a}\right)^m \quad r<a$$

$$\sim \frac{\left(\frac{b}{r}\right)^m - \left(\frac{r}{b}\right)^m}{\left(\frac{b}{a}\right)^m - \left(\frac{a}{b}\right)^m} \quad (a<r<b)$$

$$V_\perp = \hat{z} \times \nabla\chi = \hat{\theta}\,\frac{m}{r}\left(\frac{r}{a}\right)^m - \hat{r}\,\frac{im}{r}\left(\frac{r}{a}\right)^m \quad \text{INSIDE}$$

$$= -\hat{\theta}\,\frac{m}{r}\left(\frac{a}{r}\right)^m - \hat{r}\,\frac{im}{r}\left(\frac{a}{r}\right)^m \quad \text{OUTSIDE}$$

Kink Mode

$$-\omega \tilde{\varphi}_a = \frac{B_p}{r}(m-nq)\tilde{\chi}_a$$

$$\omega \rho \frac{\partial \chi}{\partial r}\Big|_{a^-} = \frac{2mB_p}{\mu_0 a^2}\tilde{\varphi}_a \left[(m-nq)\left(\frac{-\Delta'(a)}{2m/a}\right) - 1\right]$$

$$\Delta'(a) = -\frac{2m}{a}\frac{(b/a)^m}{(b/a)^m - \left(\frac{a}{b}\right)^m}$$

FOR A WALL AT $r=b$

$$\frac{\tilde{\chi}_a}{\tilde{\varphi}_a} = -\frac{\omega q_a R}{B_0(m-nq_a)}$$

EIGENVECTOR

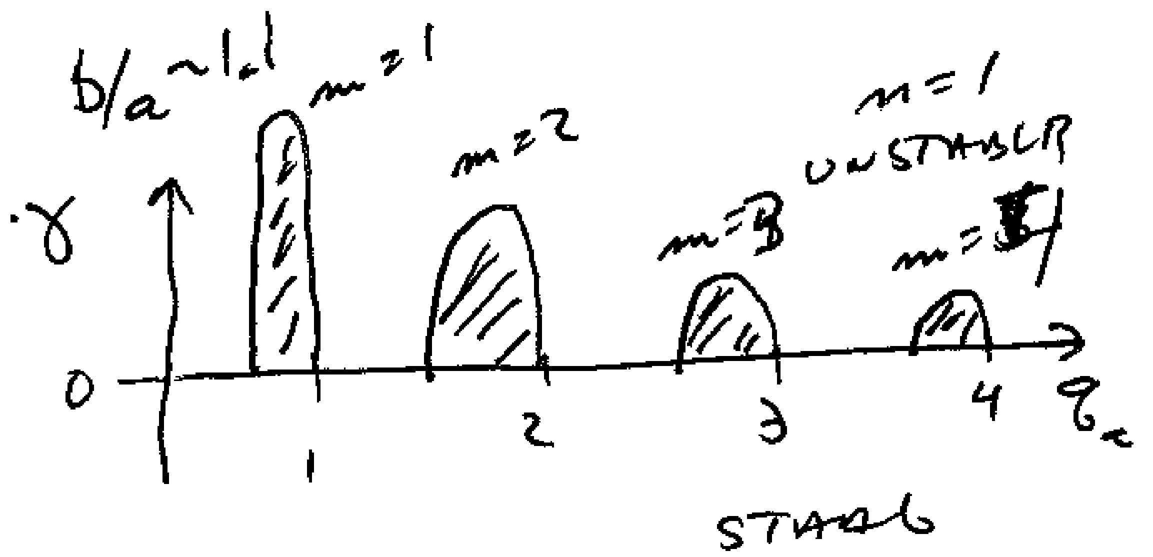
$$\frac{\partial \chi}{\partial r}\Big|_{a^-} = -\frac{m}{a}\tilde{\chi}_a$$

EIGENVALUE

$$\omega^2 = 2\omega_A^2(m-nq) \times \left[(m-nq)\frac{(\Lambda+1)}{2} - 1\right]$$

GLOBAL KINK MODES

$$\begin{pmatrix} \omega & \frac{B_p}{a}(m-nq) \\ \frac{2mB_p}{\mu_0 a}\left[(m-nq)\frac{(\Lambda+1)}{2} - 1\right] & \omega\rho \end{pmatrix}\begin{pmatrix}\tilde{\varphi}_a \\ \tilde{\chi}_a\end{pmatrix} = 0$$

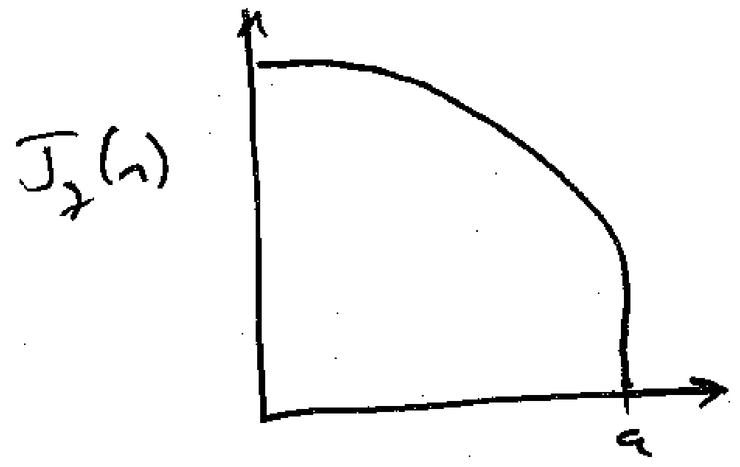


$$\boxed{\Delta'(a) = -\frac{m}{a}(\Lambda+1)}$$

SHAFRANOV'S FORMULA

Wesson's Cylindrical Equilibrium

REF: WESSON, NUC FUSION 18 (1978) P. 87

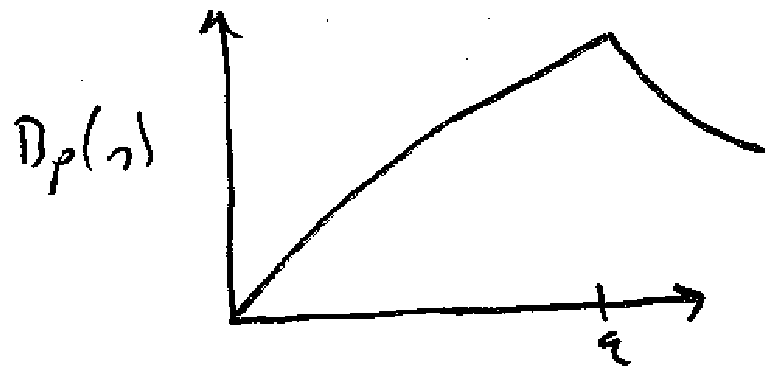


$$J_z(r) = J_0 \left(1 - r^2/a^2\right)^{\nu}$$

J_0 = CENTRAL CURRENT DENSITY

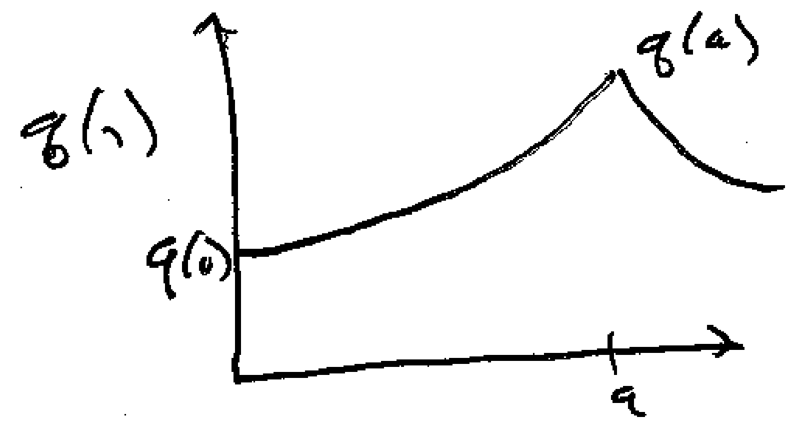
$$q(0) = \frac{2B_z}{\mu_0 R J_0} \qquad \mu_0 I_p = \frac{2\pi^2 a^2 B_z}{\mu_0 q(0) R (1+\nu)}$$

$$\nu = \frac{q(a)}{q(0)} - 1$$



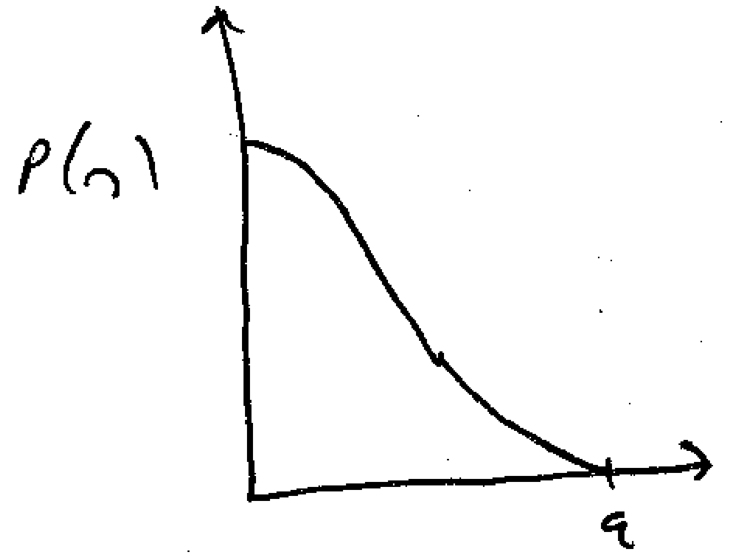
$$\ell_i \neq \tfrac{1}{2}$$

$$\langle \beta \rangle \sim \epsilon^2/q_a^2 \qquad \beta_p \sim 1 \qquad \langle \beta_N \rangle \sim 20\,\epsilon/q_a$$



EQUILIBRIUM SET BY $q(0), q(a)$
<u>TWO</u> PARAMETERS

$$\frac{r}{q}\frac{dq}{dr} = s = \text{MAGNETIC SHEAR} \neq 0$$



PRESSURE PROFILE CAN BE MORE PEAKED

$B\cdot\nabla \to 0$ AT <u>INTERNAL</u> RESONANT LAYER

$$\left(m - n/q(r)\right) \cong 0$$

Wesson's Kink Modes

LINEARIZED EQUATIONS FOR PERTURBED STREAM FUNCTION (χ)
AND PERTURBED POLOIDAL FLUX (ψ)

$$-\rho\,\omega\,\nabla_\perp^2\tilde{\chi} = -\frac{m}{r}\frac{\partial J_z}{\partial r}\tilde{\psi} + \frac{B_\theta}{\mu_0 r}(m - nq)\nabla_\perp^2\tilde{\psi} \qquad \text{(MHD)}$$

$$-\omega\tilde{\psi} = \frac{B_\theta}{r}(m - nq)\tilde{\chi} \qquad \text{(INDUCTION)}$$

LET'S TAKE $\rho \simeq$ UNIFORM, WITH A SHARP JUMP AT THE PLASMA'S EDGE:

$$\omega\rho\left.\frac{\partial\chi}{\partial r}\right|_{a^-} = \frac{B_\theta(a)}{\mu_0 a}(m - nq_a)\,\tilde{\psi}_a \, \Delta'(a)$$

$\Delta'(a)$ ← PERTURBED SURFACE CURRENT AT PLASMA'S EDGE

WITH INDUCTION EQUATION:

$$\omega^2 = -\omega_A^2(m - nq_a)^2\,\Delta'(a)\,\frac{\chi_a}{(\partial\chi/\partial r)_{a^-}}$$

BUT, HOW TO FIGURE OUT $\tilde{\psi}(r)$?

INSTABILITY REQUIRES $\Delta'(a) > 0$

Wesson's Kink Modes

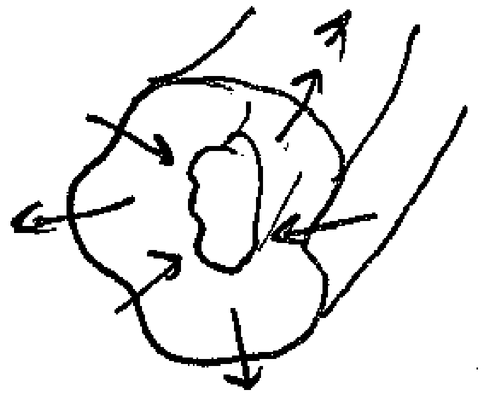
SINCE $|\omega| < \omega_A$, THE KINK MODE CAUSES THE "INTERNAL" PLASMA TO RESPOND "QUICKLY", SO QUICKLY THAT WE CAN IGNORE THE TIME CT TAKES TO FORM A DISTORTED, 3D, QUASI-EQUILIBRIUM.

INSIDE, THE PLASMA IS A "FORCED" EQUILIBRIUM

$$0 \approx -\frac{m}{r}\frac{\partial J_z}{\partial r}\tilde{\psi} + \frac{B_\theta}{\mu_0 r}(n - mq)\nabla_\perp^2\tilde{\psi}$$

OUTSIDE, THE RESPONSE IS THE "VACUUM" RESPONSE.

WITH $J_z(r)$, WE HAVE TO SOLVE FOR $\tilde{\psi}$ USING A COMPUTER. (THIS IS VERY EASY FOR THE CYLINDRICAL "TOKAMAK")



← THE SURFACE CURRENT "PUSHES"/"PULLS" PLASMA, AND THE "DISTORTED" PLASMA IS MEASURED BY $\tilde{\psi}(r, \theta, z)$

Next Lecture:

- Examining the properties of kink modes in the (straight) reduced MHD formalism.