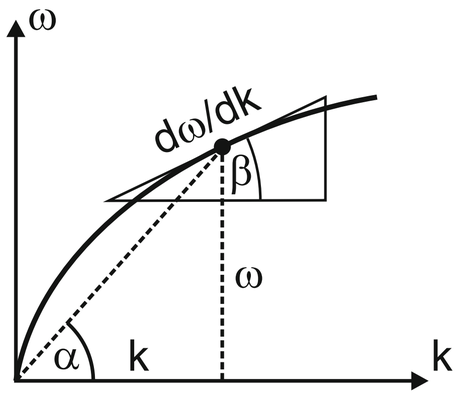
Lecture12: Quiz 2 Review
Plasma Waves

APPH E6101x
Columbia University

Key Concepts

$$\mathbf{E} = \hat{\mathbf{E}}\ \exp[\mathrm{i}(\mathbf{k}\cdot\mathbf{r} - \omega t)]$$
$$\mathbf{B} = \hat{\mathbf{B}}\ \exp[\mathrm{i}(\mathbf{k}\cdot\mathbf{r} - \omega t)]$$
$$\mathbf{j} = \hat{\mathbf{j}}\ \exp[\mathrm{i}(\mathbf{k}\cdot\mathbf{r} - \omega t)]$$

- Linearized equations of “fluid” motion
- Plane wave representation (“phasors”)
- “Cold” plasma without pressure: describes the simplest plasma wave properties
- “Warm” plasma includes (usually) electron pressure: easiest for ion acoustic waves
- Electrostatic (longitudinal) and Electromagnetic waves
- Polarization (and wave energy density)
- Dispersion relations, ω vs. k, phase-velocity, group-velocity
- Wave propagation through a plasma as a diagnostic



Basic Equations Review

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{B} = \mu_0 \left(\mathbf{j} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right)$$

$$\begin{aligned} \nabla \times (\nabla \times \mathbf{E}) &= -\nabla \times \frac{\partial \mathbf{B}}{\partial t} \\ &= -\frac{\partial}{\partial t}(\nabla \times \mathbf{B}) \\ &= -\mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} - \mu_0 \frac{\partial \mathbf{j}}{\partial t} \end{aligned}$$

Maxwell's Equations

nonlinear (!)

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\mathbf{u}) = 0$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} = 0$$

"cold"

(plus "collisions"…)

$$nm \left[\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} \right] = nq(\mathbf{E} + \boxed{\mathbf{u} \times \mathbf{B}})$$

"magnetized" or "unmagnetized"

"warm"

$$nm \left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} \right) = nq(\mathbf{E} + \boxed{\mathbf{u} \times \mathbf{B}}) - \nabla p$$

Plasma Fluid Dynamics

Review of EM Waves in Medium

$$\varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} + \mathbf{j} = \varepsilon_0 \bar{\varepsilon}(\omega) \frac{\partial \mathbf{E}}{\partial t}$$

$$\mathbf{j}(\omega) = \bar{\sigma}(\omega) \cdot \mathbf{E}(\omega)$$

$$\boldsymbol{\varepsilon}_\omega = \mathbf{I} + \frac{\mathrm{i}}{\omega \varepsilon_0} \boldsymbol{\sigma}_\omega$$

All of the plasma physics here

$$\left\{ \mathbf{kk} - k^2 \boldsymbol{I} + \frac{\omega^2}{c^2} \boldsymbol{I} + \mathrm{i}\omega\mu_0 \boldsymbol{\sigma}(\omega) \right\} \cdot \hat{\mathbf{E}} = 0$$

$$\left\{ \mathbf{kk} - k^2 \boldsymbol{I} + \frac{\omega^2}{c^2} \boldsymbol{\varepsilon}(\omega) \right\} \cdot \hat{\mathbf{E}} = 0$$

EM Waves without Magnetic Field

$$\varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} + \mathbf{j} = \varepsilon_0 \bar{\varepsilon}(\omega) \frac{\partial \mathbf{E}}{\partial t} \qquad \mathbf{j}(\omega) = \bar{\sigma}(\omega) \cdot \mathbf{E}(\omega)$$

$$\boldsymbol{\varepsilon}_\omega = \mathbf{I} + \frac{\mathrm{i}}{\omega \varepsilon_0} \boldsymbol{\sigma}_\omega$$

FOR HIGH FREQUENCY WAVES, $\omega \gg \omega_{pi}$, THEN ONLY ELECTRON CURRENTS

$$\bar{J} = -e n_0 \bar{v}_e$$

$$m_e \frac{dv}{dt} = -eE \Rightarrow -j\omega \bar{v} = -\frac{e}{m_0} \bar{E}$$

$$\bar{v} = j\left(-\frac{e}{m_0}\right) \bar{E}/\omega$$

$$\bar{J} = j \frac{e^2 n}{m_e \omega} \bar{E}$$

$$\bar{J} = j \epsilon_0 (\omega_{pe}^2/\omega) \bar{E}$$

$$\bar{\bar{\sigma}} = j \epsilon_0 \omega \begin{pmatrix} \omega_p^2/\omega^2 & 0 & 0 \\ 0 & \omega_p^2/\omega^2 & 0 \\ 0 & 0 & \omega_p^2/\omega^2 \end{pmatrix} \quad \text{FOR } \bar{E} = (E_x, E_y, E_z)$$

$$\bar{\bar{\epsilon}} = \bar{\bar{I}} + \frac{i}{\omega \epsilon_0} \bar{\bar{\sigma}} = \bar{\bar{I}} \left(1 - \omega_p^2/\omega^2\right)$$

"Cold" EM Waves without Magnetic Field

FOR ELECTROMAGNETIC WAVES

$$\mathbf{j}(\omega) = \bar{\sigma}(\omega) \cdot \mathbf{E}(\omega)$$

$$\left\{ \bar{N}\bar{N} - N^2 \bar{\bar{I}} + \bar{\bar{\epsilon}} \right\} \cdot \bar{E} = 0$$

$$\bar{N} = \frac{c\bar{k}}{\omega}$$

INDEX OF REFRACTION

TAKE $\bar{k} = \hat{z} k$ OR $\bar{N} = \hat{z} N$

$$\boxed{\boldsymbol{\varepsilon}_\omega = \mathbf{I} + \frac{\mathrm{i}}{\omega \varepsilon_0} \boldsymbol{\sigma}_\omega}$$

$$\begin{Bmatrix} 1 - N^2 - \frac{\omega_p^2}{\omega^2} & 0 & 0 \\ 0 & 1 - N^2 - \frac{\omega_p^2}{\omega^2} & 0 \\ 0 & 0 & 1 - \frac{\omega_p^2}{\omega^2} \end{Bmatrix} \cdot \begin{Bmatrix} E_x \\ E_y \\ E_z \end{Bmatrix}$$

*** (!!)

POLARIZATION:

TRANSVERSE WAVES $\bar{E} \perp \bar{k}$

LONGITUDINAL WAVES $\bar{E} \parallel \bar{k}$

FARADAY'S LAW

$$\bar{k} \times \bar{E} = \omega \bar{B}$$

SO $B = 0$ FOR LONGITUDINAL WAVES

"Warm" EM Waves without Magnetic Field

WITH ELECTRON PRESSURE...

$$\left.\begin{aligned} \frac{\partial n}{\partial t} + \bar{\nabla}\cdot(n\bar{v}) &= 0 \\ m\frac{\partial v}{\partial t} &= q\bar{E} - \nabla p \end{aligned}\right\} \Rightarrow \quad \begin{aligned} \tilde{n} &= n_0 \frac{\bar{k}\cdot\bar{v}}{\omega} \\ -j\omega\bar{v} &= \frac{q}{m}\bar{E} - jk\gamma v_{TH}^2 \tilde{n} \end{aligned}$$

WHEN $\bar{k}\cdot\bar{v} = 0 \Rightarrow$ NO CHANGE OF DENSITY

WHEN $\bar{k}\cdot\bar{v} \neq 0 \Rightarrow$ COMPRESSION/DECOMPRESSION

$\therefore$ FOR LONGITUDINAL WAVES

$$-j\omega\bar{v} = \frac{q}{m}\bar{E} - j\gamma\frac{k^2\lambda_D^2}{\omega}\omega_p^2\bar{v}$$

OR

$$-j\omega\bar{v}\left(1 - \gamma k^2\lambda_D^2\frac{\omega_p^2}{\omega^2}\right) = \frac{q}{m}\bar{E}$$

ANS

$$\bar{J} = qn\bar{v} \Rightarrow \quad \sigma = j\omega\epsilon_0\frac{\omega_p^2/\omega^2}{1 - \gamma k^2\lambda_D^2(\omega_p^2/\omega^2)}$$

$$\Rightarrow \quad \omega^2 = \omega_p^2(1 + \gamma k^2\lambda_D^2)$$

ELECTRON PLASMA WAVES

Waves in Magnetized Plasma

$$\frac{\partial \mathbf{v}^{(\alpha)}}{\partial t} = \frac{q_\alpha}{m_\alpha}\left(\mathbf{E}_1 + \mathbf{v}^{(\alpha)} \times \mathbf{B}_0\right) \qquad \alpha = \mathrm{e}, \mathrm{i}$$

$$\hat{v}^{\pm} = \mathrm{i}\frac{q}{\omega m}(\hat{E}^{\pm} \mp \mathrm{i}\hat{v}^{\pm} B_0)$$

$$\hat{v}_x = \mathrm{i}\frac{q}{\omega m}(\hat{E}_x + \hat{v}_y B_0)\,, \qquad \hat{v}_y = \mathrm{i}\frac{q}{\omega m}(\hat{E}_y - \hat{v}_x B_0)$$

$$\boxed{\hat{v}^{\pm} = \mathrm{i}\frac{q}{m}\hat{E}^{\pm}\frac{1}{\omega \mp s\omega_c}}$$

$$\hat{v}^{\pm} = \hat{v}_x \pm \mathrm{i}\hat{v}_y\,, \qquad \hat{E}^{\pm} = \hat{E}_x \pm \mathrm{i}\hat{E}_y$$

Waves in Magnetized Plasma

$$\begin{pmatrix} \hat{v}_x \\ \hat{v}_y \\ \hat{v}_z \end{pmatrix} = \mathrm{i}\frac{q}{\omega m}\begin{pmatrix} \dfrac{\omega^2}{\omega^2-\omega_\mathrm{c}^2} & \mathrm{i}\dfrac{s\omega\omega_\mathrm{c}}{\omega^2-\omega_\mathrm{c}^2} & 0 \\ -\mathrm{i}\dfrac{s\omega\omega_\mathrm{c}}{\omega^2-\omega_\mathrm{c}^2} & \dfrac{\omega^2}{\omega^2-\omega_\mathrm{c}^2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \hat{E}_x \\ \hat{E}_y \\ \hat{E}_z \end{pmatrix}$$

$$\boldsymbol{\sigma}(\omega) = \mathrm{i}\omega\varepsilon_0 \begin{pmatrix} \sum_\alpha \dfrac{\omega_{\mathrm{p}\alpha}^2}{\omega^2-\omega_{\mathrm{c}\alpha}^2} & \mathrm{i}\sum_\alpha s_\alpha \dfrac{\omega_{\mathrm{p}\alpha}^2}{\omega^2-\omega_{\mathrm{c}\alpha}^2}\dfrac{\omega_{\mathrm{c}\alpha}}{\omega} & 0 \\ -\mathrm{i}\sum_\alpha s_\alpha \dfrac{\omega_{\mathrm{p}\alpha}^2}{\omega^2-\omega_{\mathrm{c}\alpha}^2}\dfrac{\omega_{\mathrm{c}\alpha}}{\omega} & \sum_\alpha \dfrac{\omega_{\mathrm{p}\alpha}^2}{\omega^2-\omega_{\mathrm{c}\alpha}^2} & 0 \\ 0 & 0 & \sum_\alpha \dfrac{\omega_{\mathrm{p}\alpha}^2}{\omega^2} \end{pmatrix}$$

$$\boldsymbol{\varepsilon}_\omega = \mathbf{I} + \frac{\mathrm{i}}{\omega\varepsilon_0}\boldsymbol{\sigma}_\omega$$

(These formula will be on your quiz!)

Waves in Magnetized Plasma

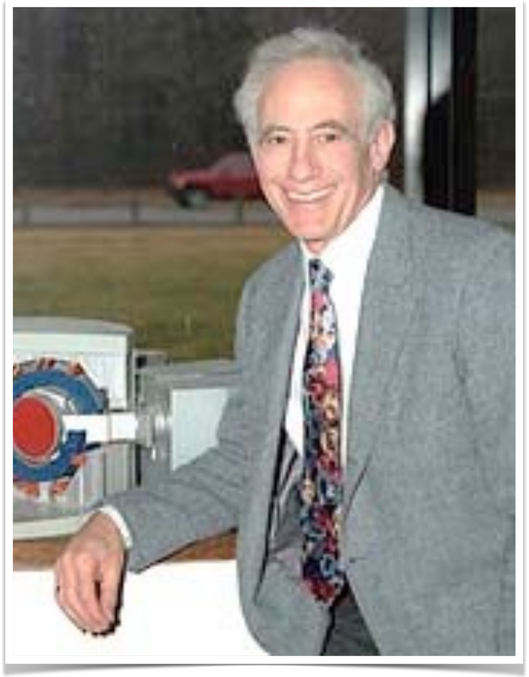
$$\boldsymbol{\varepsilon}_\omega = \mathbf{I} + \frac{\mathrm{i}}{\omega\varepsilon_0}\boldsymbol{\sigma}_\omega$$

$$\boldsymbol{\varepsilon}(\omega) = \begin{pmatrix} S & -\mathrm{i}D & 0 \\ \mathrm{i}D & S & 0 \\ 0 & 0 & P \end{pmatrix}$$

$$S = 1 - \sum_\alpha \frac{\omega_{\mathrm{p}\alpha}^2}{\omega^2 - \omega_{\mathrm{c}\alpha}^2}$$

$$D = \sum_\alpha s_\alpha \frac{\omega_{\mathrm{p}\alpha}^2}{\omega^2 - \omega_{\mathrm{c}\alpha}^2}\frac{\omega_{c\alpha}}{\omega}$$

$$P = 1 - \sum_\alpha \frac{\omega_{\mathrm{p}\alpha}^2}{\omega^2}\,.$$



Working both in the laboratory and with theoretical calculations, he found many ways to put waves to work in fusion research in succeeding decades, and his 1962 book, "The Theory of Plasma Waves," codified the subject in mathematical form for the first time.

http://www.nytimes.com/2001/04/18/nyregion/thomas-h-stix-plasma-physicist-dies-at-76.html

$$S = 1 - \sum_\alpha \frac{\omega_{\mathrm{p}\alpha}^2}{\omega^2 - \omega_{\mathrm{c}\alpha}^2}$$

$$D = \sum_\alpha s_\alpha \frac{\omega_{\mathrm{p}\alpha}^2}{\omega^2 - \omega_{\mathrm{c}\alpha}^2} \frac{\omega_{c\alpha}}{\omega}$$

$$P = 1 - \sum_\alpha \frac{\omega_{\mathrm{p}\alpha}^2}{\omega^2}.$$

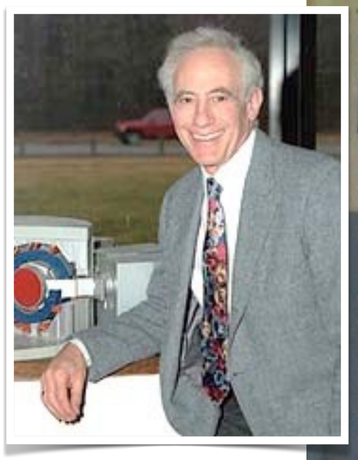
$$R \equiv 1 - \sum_s \frac{\omega_{ps}^2}{\omega^2}\left(\frac{\omega}{\omega \pm \Omega_s}\right)$$

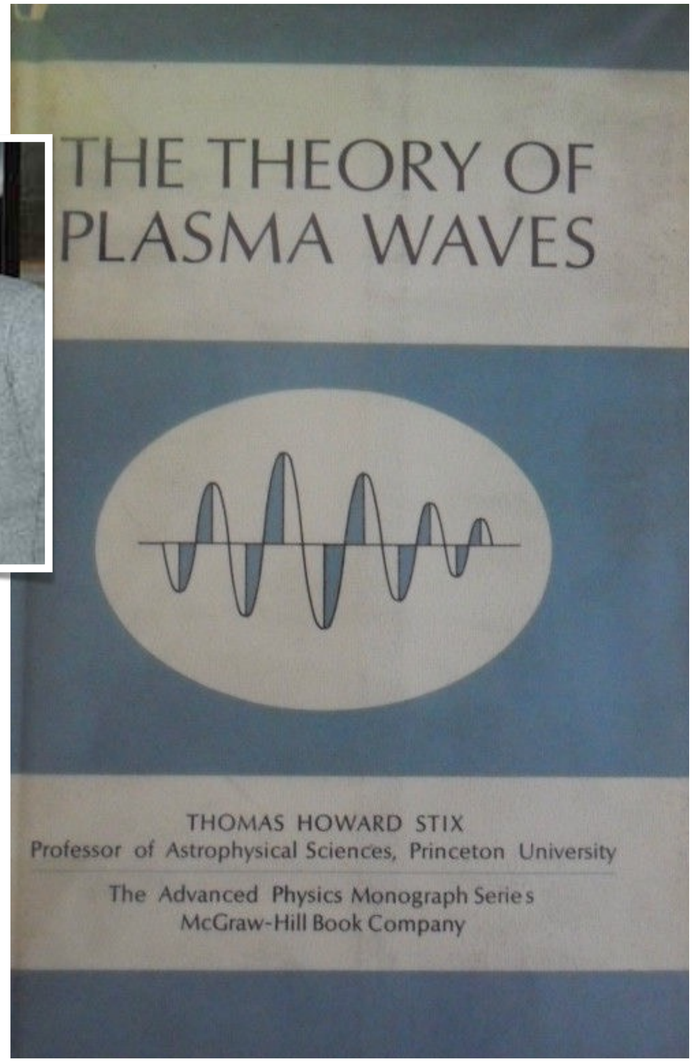
$$L \equiv 1 - \sum_s \frac{\omega_{ps}^2}{\omega^2}\left(\frac{\omega}{\omega \mp \Omega_s}\right)$$

$$S \equiv \frac{1}{2}(R+L) \qquad D \equiv \frac{1}{2}(R-L)$$

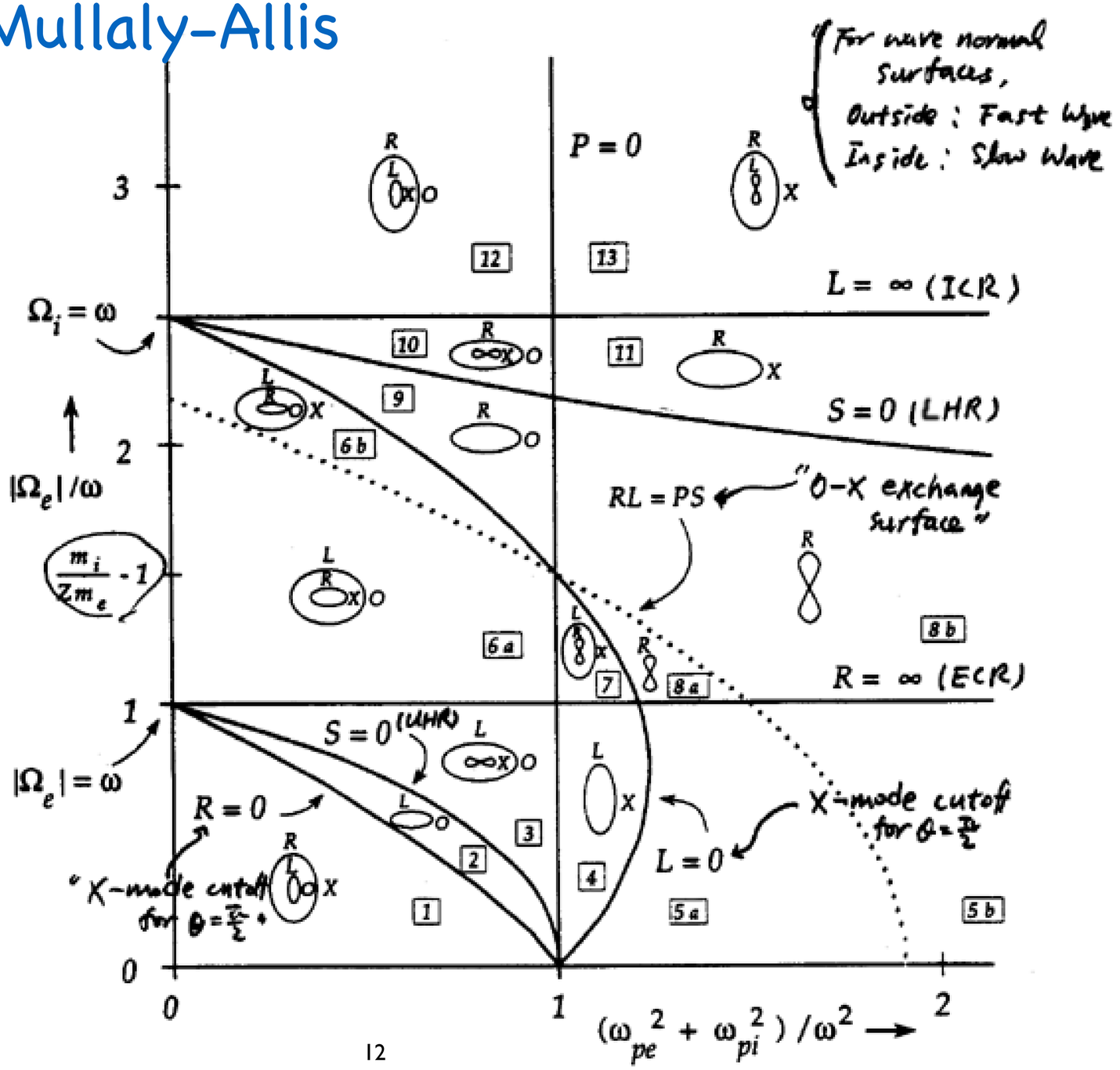
$$P \equiv 1 - \sum_s \frac{\omega_{ps}^2}{\omega^2}.$$

$$\boldsymbol{\varepsilon}(\omega) = \begin{pmatrix} S & -\mathrm{i}D & 0 \\ \mathrm{i}D & S & 0 \\ 0 & 0 & P \end{pmatrix}$$





Clemmow-Mullaly-Allis



Lecture 12:
CMA Diagram

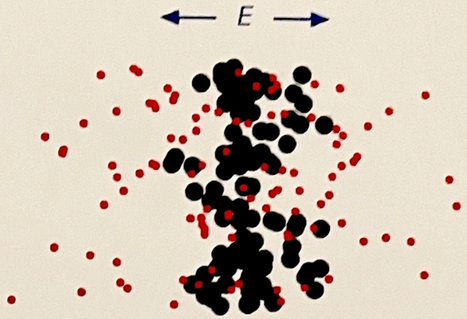
$\mathbf{u} \equiv \omega\mathbf{k}/k^2c$

Recent milestones from the APEX Collaboration, on the path toward confined e+e- pair plasmas

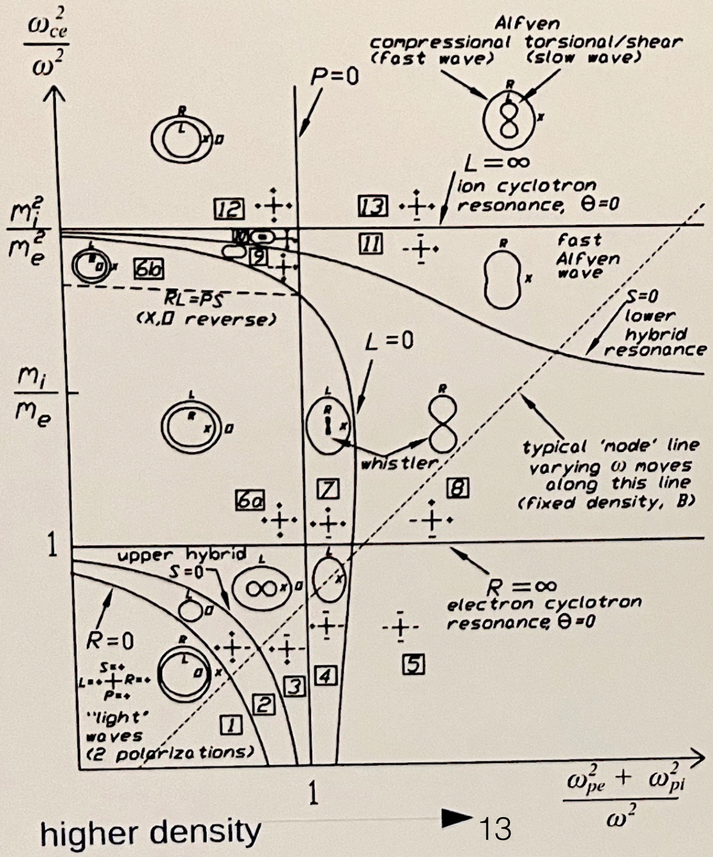
Eve V Stenson (Max Planck Institute for Plasma Physics)

Recall that mass asymmetry causes inherent coupling of *p* and E perturbations (which can then cause drift waves, and in turn turbulent transport) . . .

In an electron-ion plasma, density gradients (left) lead to electric field gradients (right), because the ions are slower to respond than the electrons.

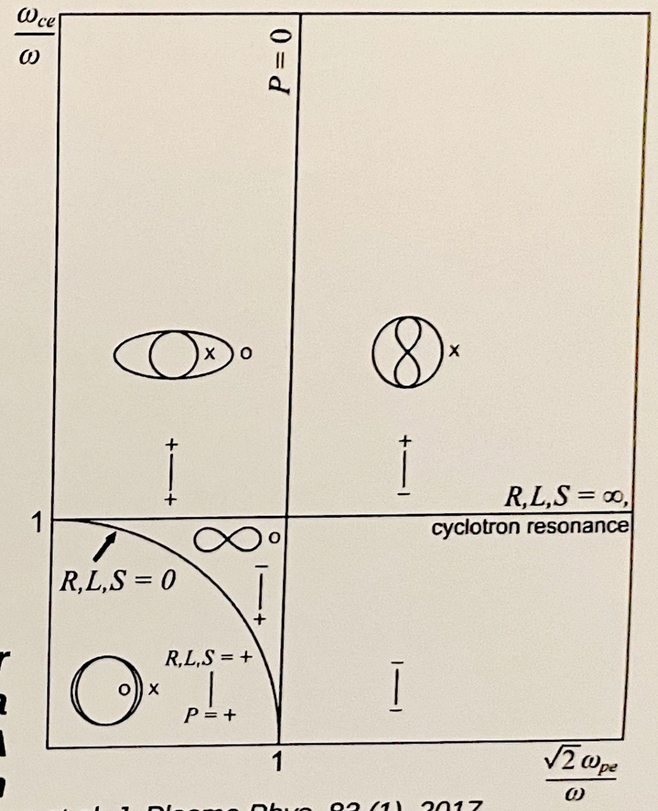


. . . and significantly increases the number of available modes, even in simple systems.



standard CMA diagram *(Bellan)*

larger magnetic field



pair plasma CMA diagram

Stenson et al. J. Plasma Phys. 83 (1), 2017

$$\mathcal{N} = \frac{kc}{\omega}$$

Waves in Magnetized Plasma

$$\begin{pmatrix} S - \mathcal{N}^2 \cos^2\psi & -\mathrm{i}D & \mathcal{N}^2 \cos\psi \sin\psi \\ \mathrm{i}D & S - \mathcal{N}^2 & 0 \\ \mathcal{N}^2 \cos\psi \sin\psi & 0 & P - \mathcal{N}^2 \sin^2\psi \end{pmatrix} \cdot \begin{pmatrix} \hat{E}_x \\ \hat{E}_y \\ \hat{E}_z \end{pmatrix} = 0$$

$$\mathbf{k} = (k \sin\psi, 0, k \cos\psi)$$

$$S = 1 - \sum_\alpha \frac{\omega_{\mathrm{p}\alpha}^2}{\omega^2 - \omega_{\mathrm{c}\alpha}^2}$$

$$D = \sum_\alpha s_\alpha \frac{\omega_{\mathrm{p}\alpha}^2}{\omega^2 - \omega_{\mathrm{c}\alpha}^2} \frac{\omega_{c\alpha}}{\omega}$$

$$P = 1 - \sum_\alpha \frac{\omega_{\mathrm{p}\alpha}^2}{\omega^2} .$$

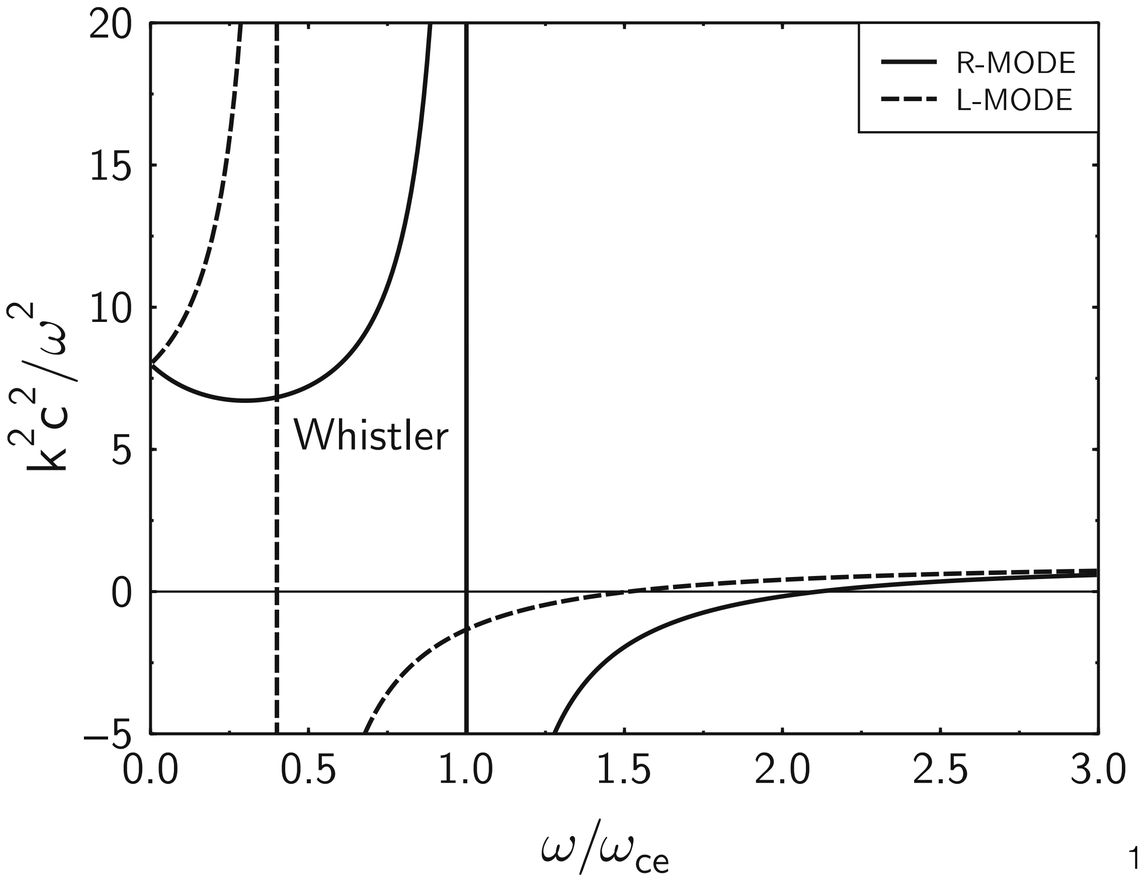
Good places to start:

Propagation along B (ψ = 0)

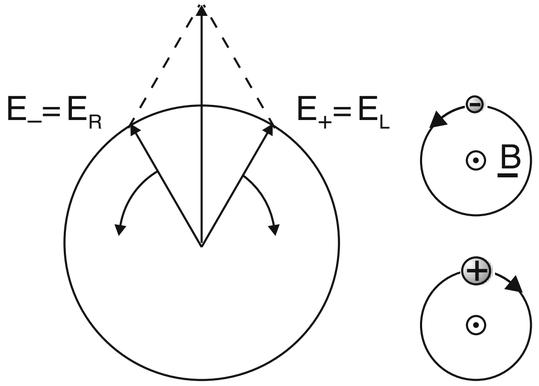
Propagation ⊥ to B (ψ = π/2)

Waves k || B

$$\begin{pmatrix} S-\mathcal{N}^2 & -\mathrm{i}D & 0 \\ \mathrm{i}D & S-\mathcal{N}^2 & 0 \\ 0 & 0 & P \end{pmatrix} \cdot \begin{pmatrix} \hat{E}_x \\ \hat{E}_y \\ \hat{E}_z \end{pmatrix} = 0$$



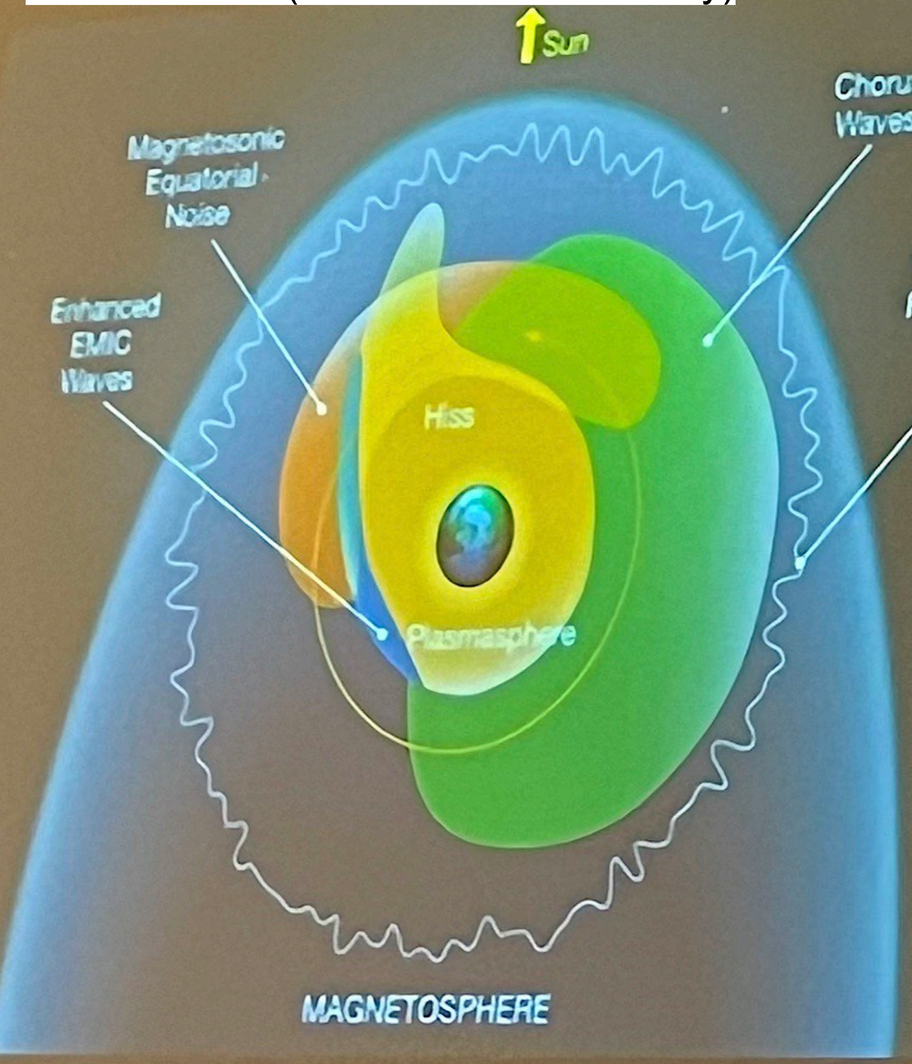
Faraday Rotation



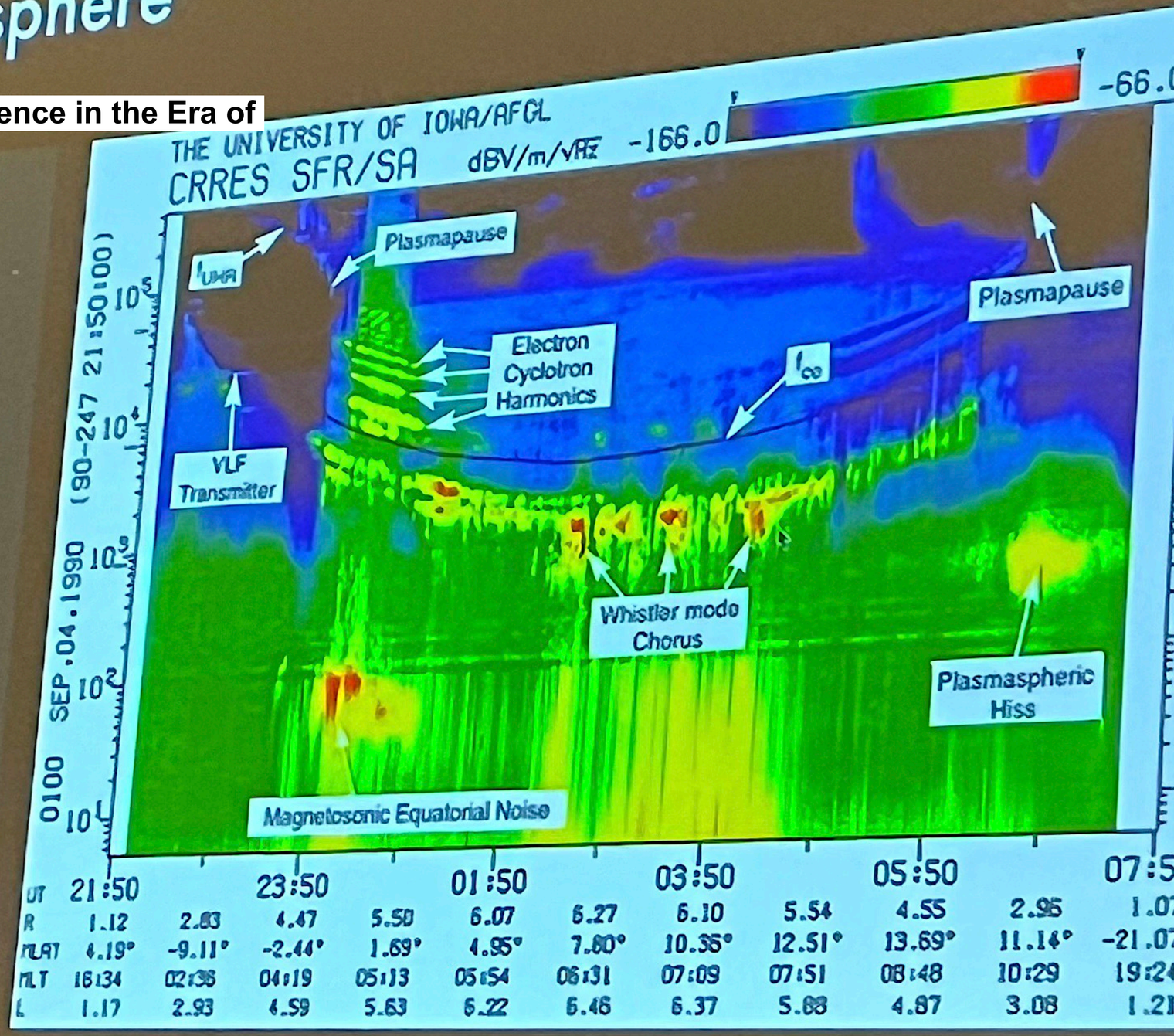
Advances in the Study of Collisionless Plasma Turbulence in the Era of NASA's Magnetospheric Multiscale Mission

Julia Stawarz (Northumbria University)

Plasma waves in the magnetosphere



om NASA GSFC, adapted from Thorne et al. 2005



Tsurutani and Smith, 1974

Waves k ⊥ B

Extra-Ordinary Mode

$$\begin{pmatrix} S & -\mathrm{i}D \\ \mathrm{i}D & (S - \mathscr{N}^2) \end{pmatrix} \cdot \begin{pmatrix} \hat{E}_x \\ \hat{E}_y \end{pmatrix} = 0$$

$$\mathscr{N}_{\mathrm{X}} = \left(\frac{S^2 - D^2}{S} \right)^{1/2}$$

$$\omega_{\mathrm{uh}} = (\omega_{\mathrm{ce}}^2 + \omega_{\mathrm{pe}}^2)^{1/2}$$

S = 0 at the "upper hybrid resonance" and the "lower hybrid resonance" w ions

Plus: "Ordinary" Mode

$$\omega^2 = \omega_{\mathrm{pe}}^2 + k^2 c^2$$

Ordinary Mode only depends upon electron density ⇒

Interferometry (!)

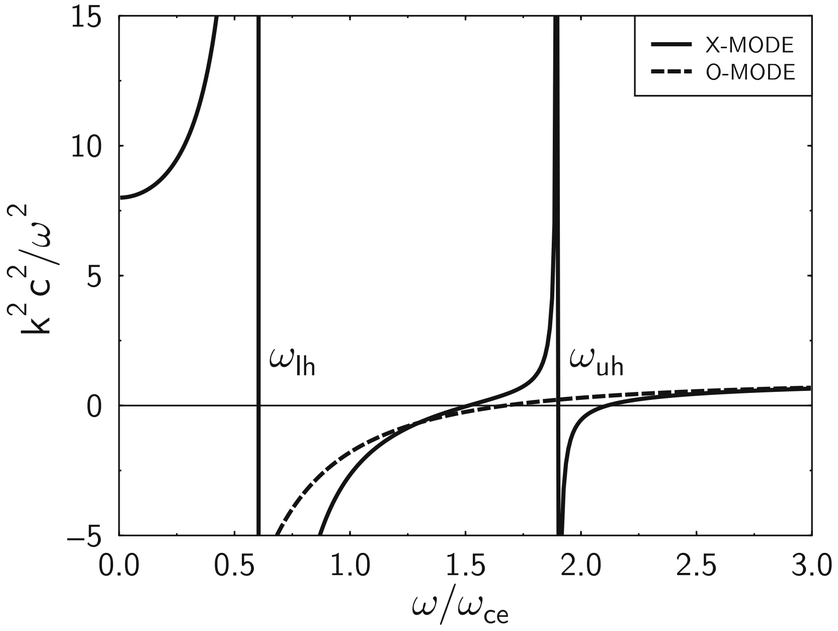
Waves k ⊥ B

Extra-Ordinary Mode

$$\begin{pmatrix} S & -\mathrm{i}D \\ \mathrm{i}D & (S-\mathcal{N}^2) \end{pmatrix} \cdot \begin{pmatrix} \hat{E}_x \\ \hat{E}_y \end{pmatrix} = 0$$

Plus: Ordinary Mode

$$\omega^2 = \omega_{\mathrm{pe}}^2 + k^2 c^2$$



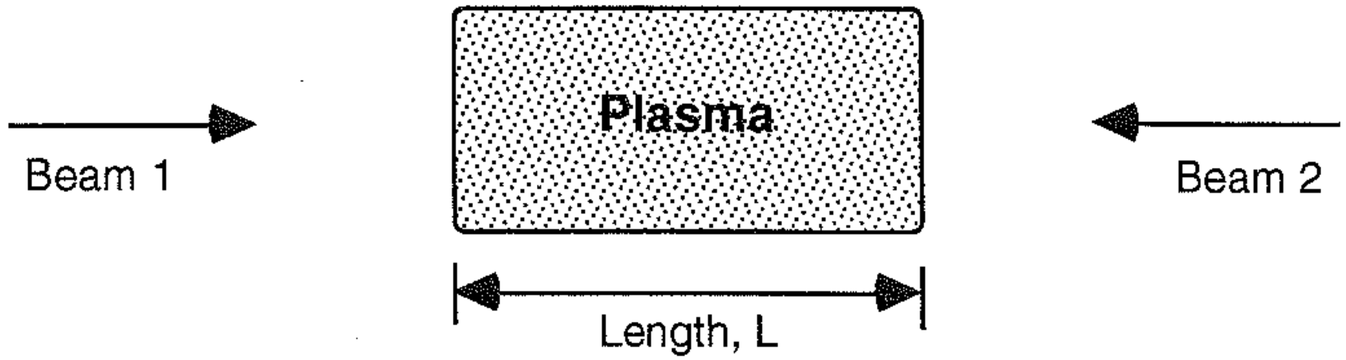
Example Problems and Discussion

- Plasma waves with electron flow
- Finding simple limits to plasma wave dispersion
- Plasma wave damping due to collisions ($\omega = \omega_r + i\,\omega_i$ or $k = k_r + i\,k_i$)
- Whistler waves
- Interferometry and Faraday Rotation

Part 6: Two beams and a Plasma

(25 points total)

Consider a finite-sized plasma into which two counter-streaming beams are injected.



Each beam has the same velocity, V_0, and the same plasma density, and the density of the stationary plasma is equal to the densities of each beam.

The dispersion relation for electrostatic plasma waves in this system is given by

$$D(\omega,k) = 1 - \frac{\omega_p^2}{\omega^2} - \frac{\omega_p^2}{\left(\omega - kV_0\right)^2} - \frac{\omega_p^2}{\left(\omega + kV_0\right)^2}$$

If we assume that the longest wavelength that can exist within the plasma is given by the plasma length, L, then prove that two-stream instabilities are stabilized when the density is sufficiently small. Indeed, the stability requirement is $\omega_p/V_0 < K/L$, where K is a constant of the order of 3.

Consider a uniformly magnetized and a fully-ionized plasma made from carbon. There would be six times the density of electrons than of ions (but the plasma would still be approximately charge-neutral.)

Describe the Alfvén wave, the electron whistler wave, and the ordinary wave in this plasma. How do these waves (in a fully-ionized carbon plasma) compare to the same waves in a plasma made from singly-ionized carbon having the same mass density (*i.e.* the same density of carbon)?

Neglecting plasma resistivity, Ohm's Law is $\mathbf{E} + \mathbf{V} \times \mathbf{B} = 0$, where $\mathbf{V}$ is the plasma mass flow velocity.

Using Maxwell's equations, derive the dispersion relation for a shear Alfvén wave, propagating along an otherwise unperturbed, straight magnetic field, $\mathbf{B}_0$. Consider only perturbations such that the perturbed plasma motion, $\mathbf{V}_1$, and the perturbed magnetic field, $\mathbf{B}_1$, both perpendicular to $\mathbf{B}_0$.

Describe the polarization of the Alfvén wave by relating the perturbed electric field and plasma velocity to the perturbed magnetic field. Are these perturbations oscillating in-phase, out-of-phase, or with a phase-shift?

Show that in the limit $\omega \to 0$,

$$R = L = S = 1 + \frac{\Pi_i^2}{\Omega_i^2} + \frac{\Pi_e^2}{\Omega_e^2},$$

$$D = 0,$$

$$P = -\frac{\Pi_i^2}{\omega^2} - \frac{\Pi_e^2}{\omega^2}.$$

Consider the effect of electron collisions on plasma waves in a uniform cold plasma. The equation for electron motion is

$$m_e \frac{d\mathbf{v}_e}{dt} = -e\mathbf{E} - e\mathbf{v}_e \times \mathbf{B} - m_e \nu \mathbf{v}_e$$

where $\mathbf{E}$ and $\mathbf{B}$ are the electric and magnetic field of the electromagnetic wave. (There is *no equilibrium magnetic field* in this problem.)

Part A

Show that the effect of collisions can be represented by the substitution

$$m_e \to m_e \left(1 + \frac{i\nu}{\omega}\right)$$

where ω is the wave frequency.

Part B

Find the linear dispersion relation for longitudinal electron plasma oscillations including the effects of collisions. Briefly discuss the dissipation of these oscillations when $\nu \ll \omega_{pe}$.

Part C

For electromagnetic waves in a cold plasma, find approximate expressions for the real and imaginary parts of the wave number ($k = k_r + ik_i$) when $\nu \ll \omega$ and $\omega_{pe} \ll \omega$.

The whistler wave propagates along the magnetic field with a frequency range, $\omega_{ci} \ll \omega < \omega_{ce}$. An approximate dispersion relation for the whistler wave is

$$\frac{k^2c^2}{\omega^2} = \frac{\omega_{pe}^2}{\omega(\omega_{ce} - \omega)}$$

where the sign of all frequencies are positive in the expression above.

Consider the propagation of whistler's in the magnetospheres of planets caused by atmospheric lightning. At the moment of the lightning strike, a broad band of electromagnetic frequencies are excited in a single pulse. Some of this wave energy couples to whistler waves that propagate from one pole to the other. Because the waves have different group velocities, waves at different frequencies arrive at different times.

Part A

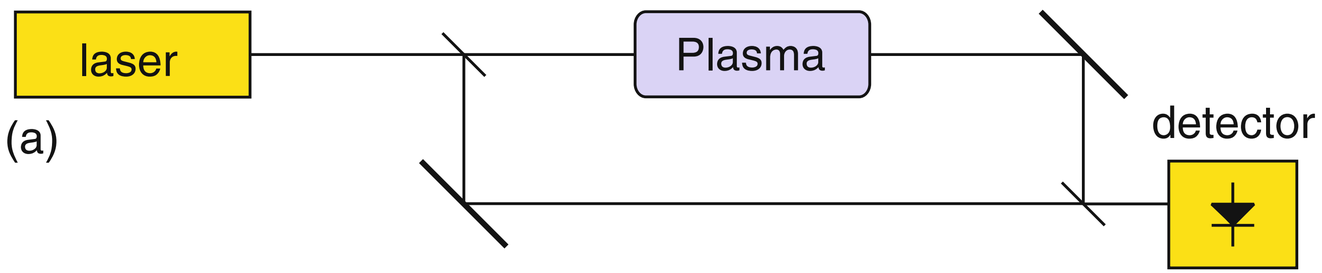
Describe the steps needed to derive the whistler wave dispersion relation in a cold magnetized plasma.

Part B

Since the group velocity is $v_g = \partial\omega/\partial k$, given an expression for the time of arrival of whistler pulse as a function of the wave frequency.

Part C

At a "cut-off" frequency, the wavelength becomes very long. At a "resonance", the wavelength tends toward zero. Is there a "cut-off" or "resonance" for whistler waves? If so, what happens at this "cut-off" or "resonance"?



The figure above shows an optical inferometer. A beam of light passes through the plasma, and the phase of the light beam through the plasma is measured relative to the phase of a light beam that does not pass through the plasma. This relative phase shift is labeled as $\Delta\phi$.

Part 1

What is the relationship between the measured relative phase shift, $\Delta\phi$, and various characteristic properties of the plasma? When the optical beam has a frequency much greater than the maximum plasma frequency, what is the quantitative relationship between $\Delta\phi$ and the plasma?

Part 2

The interferometer only works provided the plasma density is not too high. Describe the plasma condition that prevents operation of the interferometer.

Part 3

If a magnetic field is applied perpendicular to the path of the optical beam while keeping all other average plasma parameters constant, uncer what conditions will $\Delta\phi$ remain constant? Under what conditions will $\Delta\phi$ change?

In this problem, you are to derive expressions for the phase shift of the laser passing through the plasma as described in Question 6.

For typical magnetized laboratory plasmas, the laser frequency is much greater than the electron plasma and cyclotron frequencies. With the z-axis aligned with the magnetic field, the plasma dielectric tensor is

$$\bar{\bar{\epsilon}} = \begin{pmatrix} S & -iD & 0 \\ iD & S & 0 \\ 0 & 0 & P \end{pmatrix}$$

where

$$S = 1 - \frac{\omega_{pe}^2}{\omega^2 - \omega_{ce}^2}; \quad D = -\left(\frac{\omega_{ce}}{\omega}\right)\frac{\omega_{pe}^2}{\omega^2 - \omega_{ce}^2}; \quad \text{and } P = 1 - \frac{\omega_{pe}^2}{\omega^2}\,.$$

Maxwell's equations describe the electromagnetic wave of the laser with the three equations

$$\left[\mathbf{nn} - n^2\bar{\bar{I}} + \bar{\bar{\epsilon}}\right] \cdot \mathbf{E} = 0\,,$$

where $\mathbf{n} = \mathbf{k}c/\omega$ is the vector index of refraction.

Interferometry

- For the first case, the laser beam propagates *across* the plasma cylinder and *perpendicular* to the magnetic field. For this case, the total change in the phase-shift of the laser beam integrated across the plasma is

  $$\Delta\phi = \int dx\,(n-1)(\omega/c)\,.$$

  Find the correct expression for the phase shift when $\omega \gg \omega_{pe}$.

- For the second case, the laser beam propagates *along* the plasma cylinder and *parallel* to the magnetic field. For this case, an initially linearly polarized laser beam must be decomposed into the two propagating electromagnetic waves that propagate along $\mathbf{B}$. These waves are called the "right-hand circularly polarized" and "right-hand circularly polarized" waves.

  Find expressions for the polarizations of the right-hand and left-hand waves and show that the indices of refraction for these waves are (right-hand) $n^2 = S + D = 1 - \omega_{pe}^2/\omega(\omega - \omega_{ce}) \approx 1 - (\omega_{pe}^2/\omega^2)(1 + \omega_{ce}/\omega + \ldots)$ and (left-hand) $n^2 = S - D = 1 - \omega_{pe}^2/\omega(\omega + \omega_{ce}) \approx 1 - (\omega_{pe}^2/\omega^2)(1 - \omega_{ce}/\omega - \ldots)$.

  What is the relative difference per unit distance of propagation between the phase shifts of the right and left-handed waves?

  How quickly does the polarization of the laser beam change per unit distance of propagation when $\omega \gg \omega_{pe}$ and $\omega \gg \omega_{ce}$.

Faraday Rotation

Consider a magnetized plasma cylinder of radius a. You are to measure the line-density of the plasma by measuring the phase change of a polarized laser beam passing through the plasma.

- If the plasma is confined by an axial plasma current so that the magnetic field is entirely azimuthal (*i.e.* $\mathbf{B} = \hat{\theta} B_\theta$, of the so-called z-pinch), how must the laser be polarized so that the interferometer's phase-shift is proportional to the integral of the density along the laser's path?

- If the plasma is confined by a magnetic field that is directed along the $\hat{z}$-direction (*i.e.* the so-called θ-pinch), how must the laser be polarized so that the interferometer's phase-shift is proportional to the integral of the density along the laser's path?

- For the θ-pinch (with the magnetic field directed along the $\hat{z}$-direction), what happens to the polarization of a laser beam, which is linearly polarized and propagating along the axial magnetic field in the $\hat{z}$-direction?