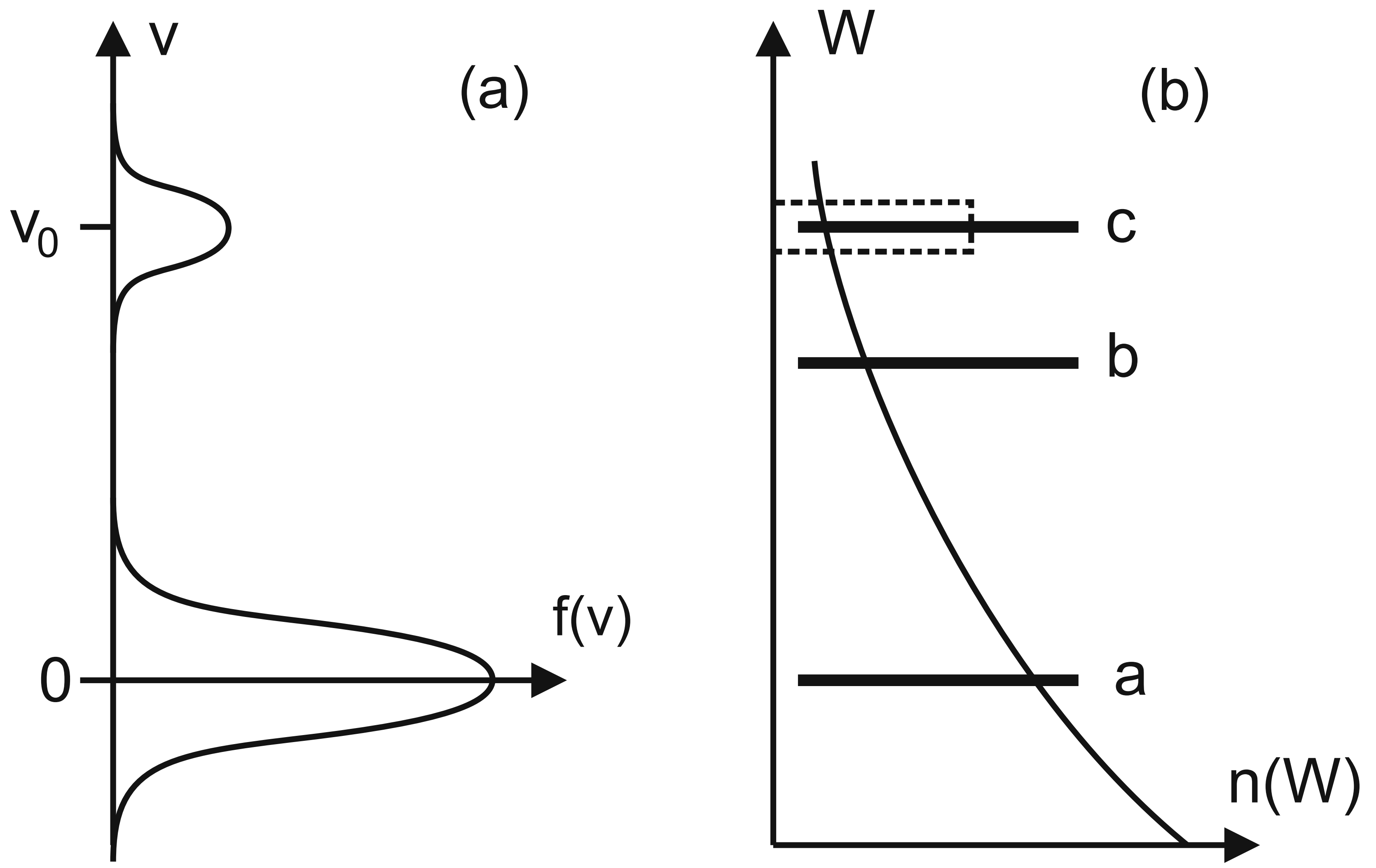
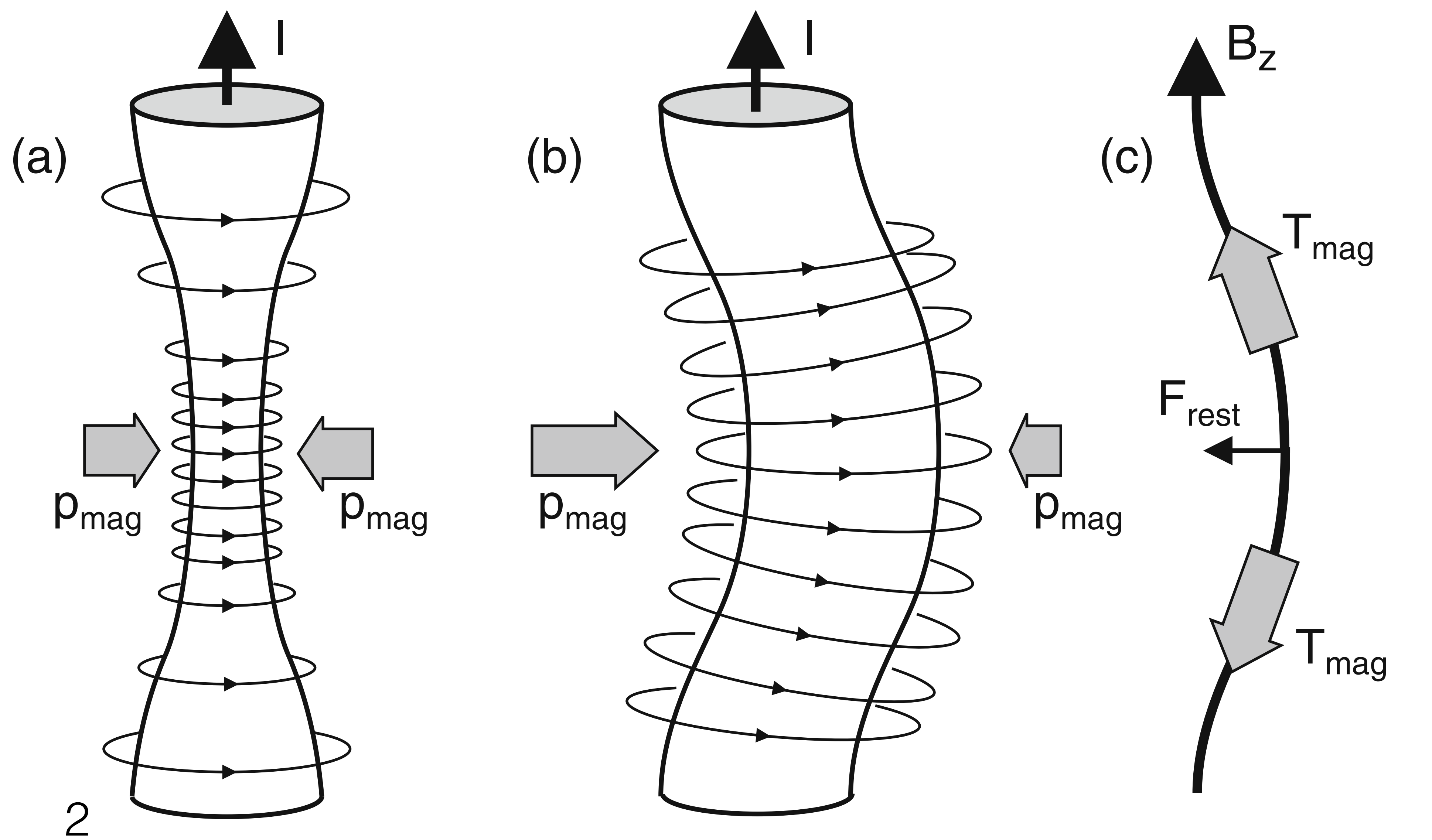
Lecture19: Beam-Plasma Instability
Plasma Physics 1

APPH E6101x
Columbia University

Instabilities

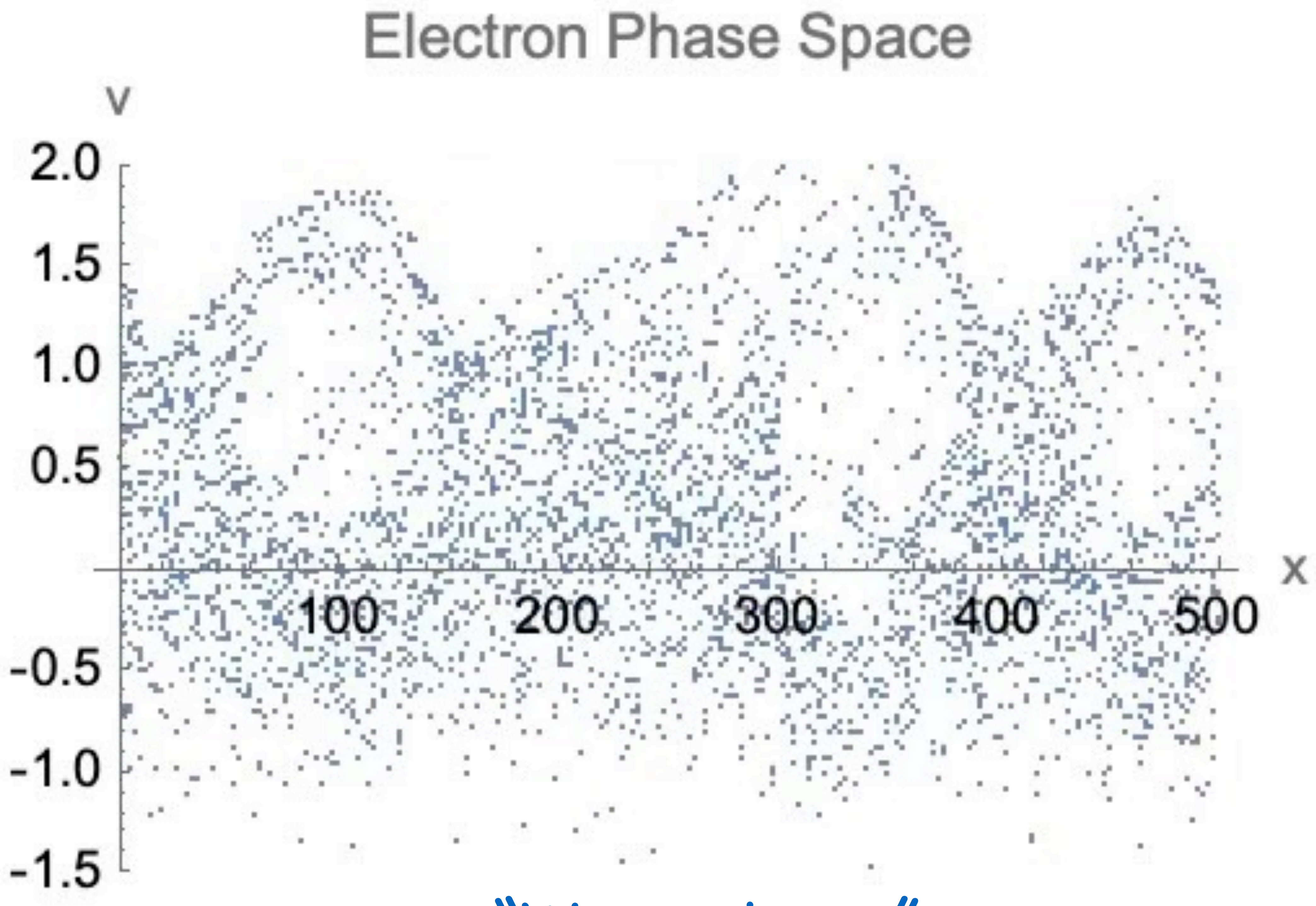
- "Micro-instabilities"
  - Velocity-space
  - Drift wave instabilities
  - …
- "Macro-instabilities"
  - Gravitational
  - Kink, Sausage, etc
  - Ballooing
  - Resistive, tearing modes
  - …





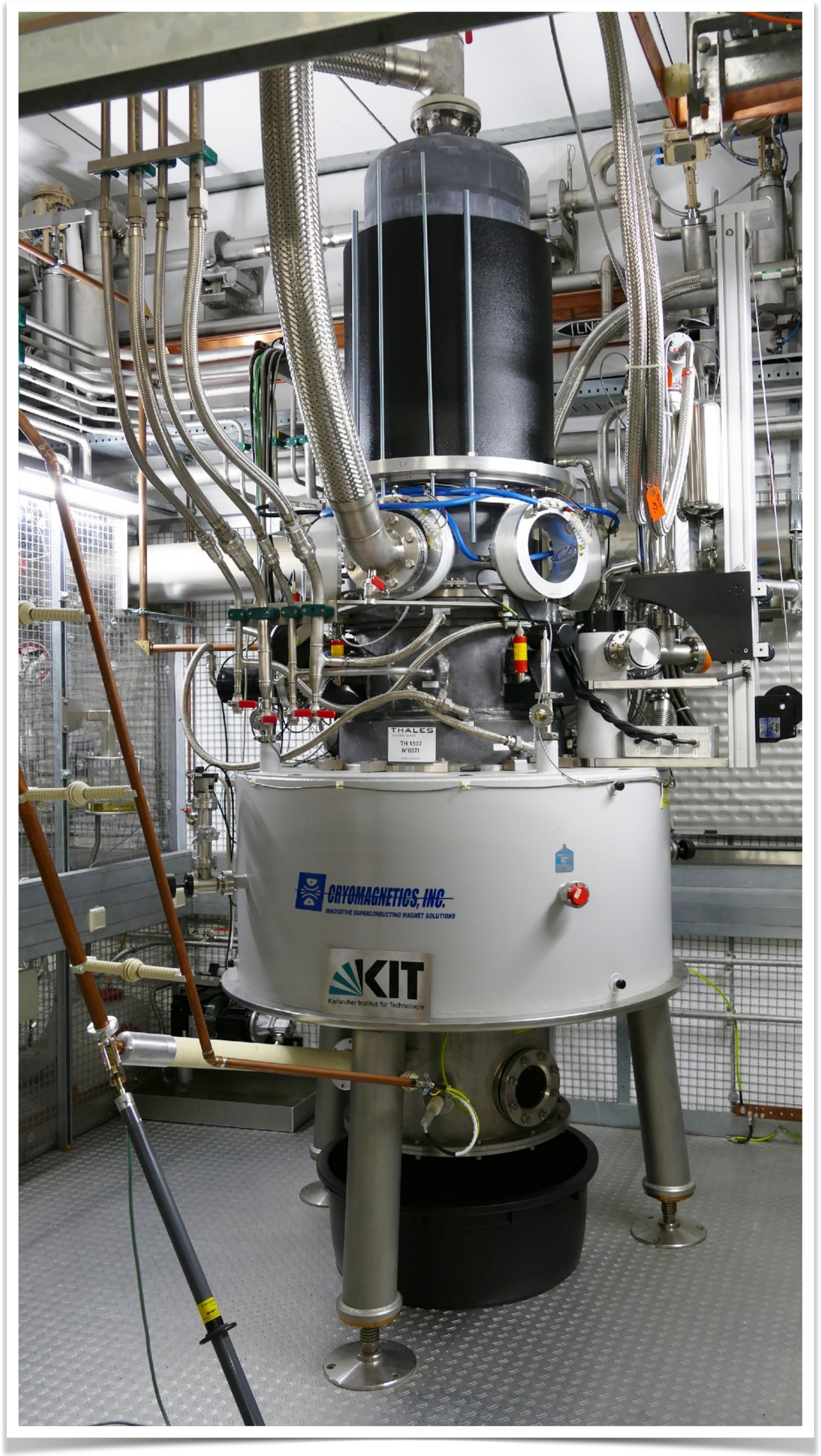
This Lecture
(Non-Relativistic, Cold-Plasma) Electron-Beam Plasma Instability

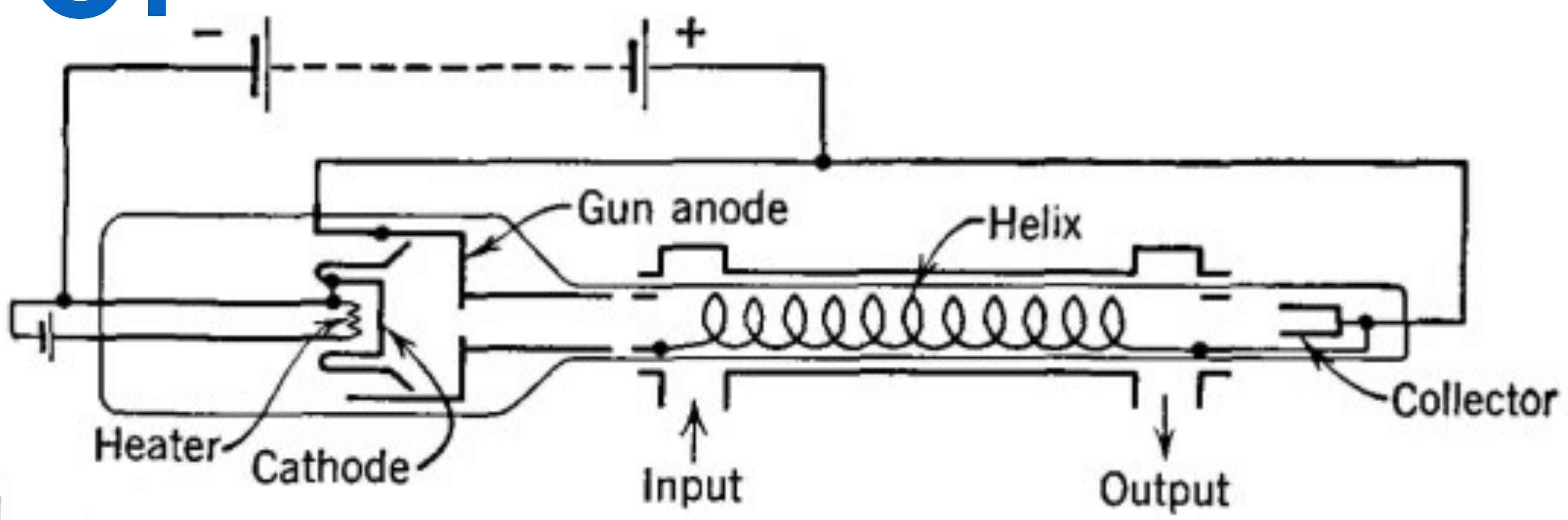
- Linearized electron fluid equations
- Dispersion relation
- Complex eigenfrequencies, wavenumber
- Energetics



"Warm-plasma"
Kinetics
Nonlinear

e-beam power





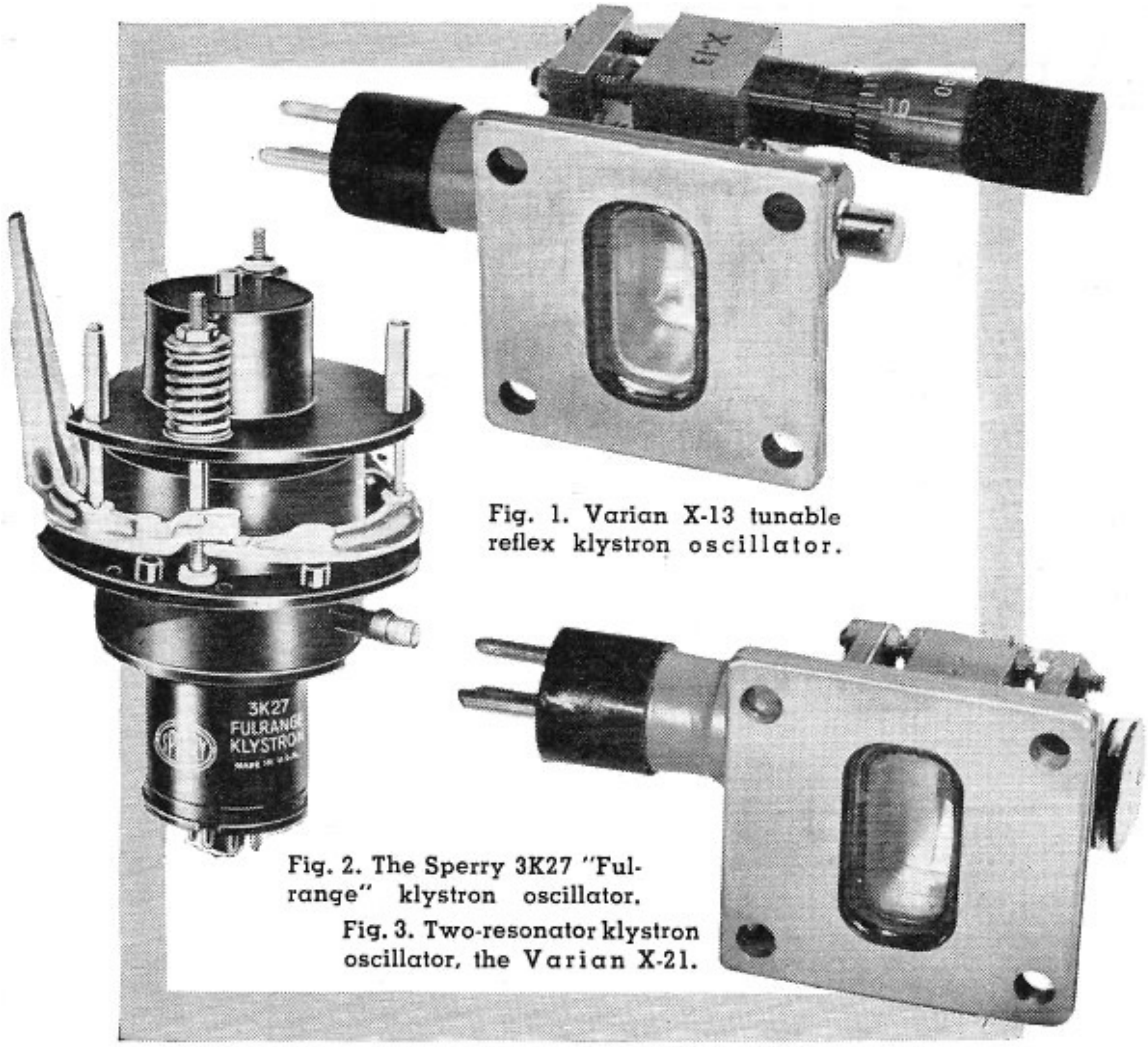
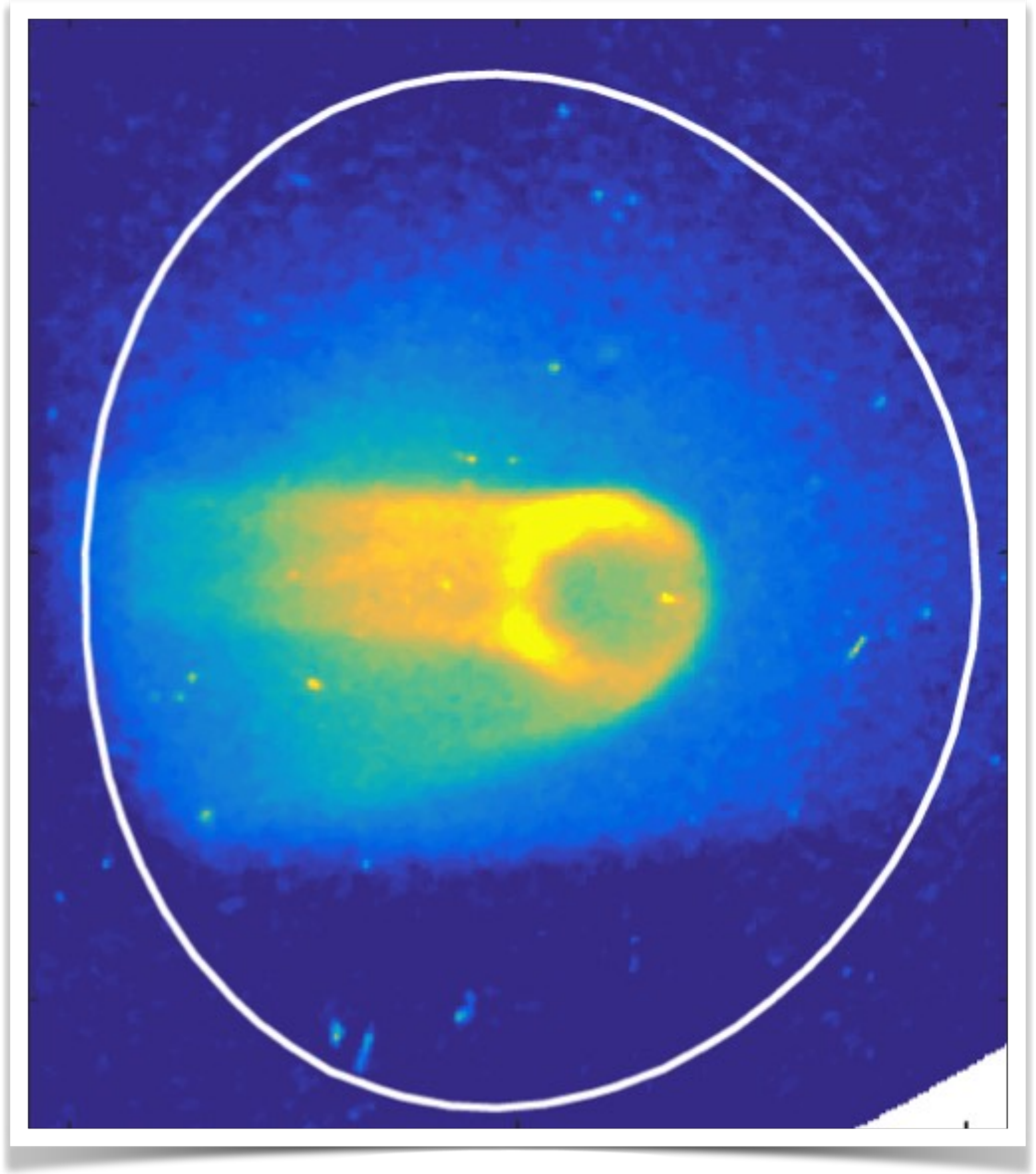


Fig. 1. Varian X-13 tunable reflex klystron oscillator.

Fig. 2. The Sperry 3K27 "Ful-range" klystron oscillator.

Fig. 3. Two-resonator klystron oscillator, the Varian X-21.



A. Lvovskiy *et al* 2020 *Nucl. Fusion* **60** 056008

THE PHYSICS OF FLUIDS VOLUME 14, NUMBER 12 DECEMBER 1971

Observations of the Beam-Plasma Instability

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(Received 30 June 1971)

The nonlinear limit of the instability driven by a low density cold electron beam in a collisionless plasma is experimentally found to be determined by the trapping of the beam by the most rapidly growing wave.

The apparatus has previously been described,[5] where it was used for a similar experiment to test quasilinear theory. The apparatus and techniques are also very similar to those used by Malmberg and Wharton[6] for the problem. The present work complements and extends the results of that paper in the nonlinear regime. For these experiments, the two-meter column contained a 3-cm diam plasma with a density near 10^9 and a temperature of 20 eV. The magnetic field of 1 kG was sufficient to render the dynamics one dimensional and the background pressure of 8×10^{-6} Torr precluded collisional effects. The plasma is quiet, with low-frequency density fluctuations of only a few percent. The axial density uniformity is excellent. Although the density drops approximately 20% over the 40 cm near the gun, no measurable gradient exists over the remainder of the column. A gradient of more than a few percent would be readily detectable. The density gradient near the gun is not significant. The waves are still far from saturation when they enter the uniform region, and all the important physics occurs in that region.

Typical results for the development of the instability are shown in Fig. 1. The wave power is measured with a loosely coupled, calibrated (−32 dB) coaxial probe connected to a broad band amplifier and rf voltmeter. The probe may be moved the length of the machine, and the figure shows the result for the absolute power in the waves. The qualitative behavior is precisely that predicted: linear growth to a peak, followed by slow oscillation. Although the theory was presented for infinite geometry and an initial value problem, the argument can easily be applied to finite geometry and growth in space. The peak wave power should be

$$P_w = 2^{2/3}\eta^{1/3}P_B, \tag{1}$$

where $P_b = I_b V_b$, the input beam power, and

$$\eta = \int_0^a n_b(r)r\,dr\left(\int_0^a n_p(r)r\,dr\right)^{-1}. \tag{2}$$

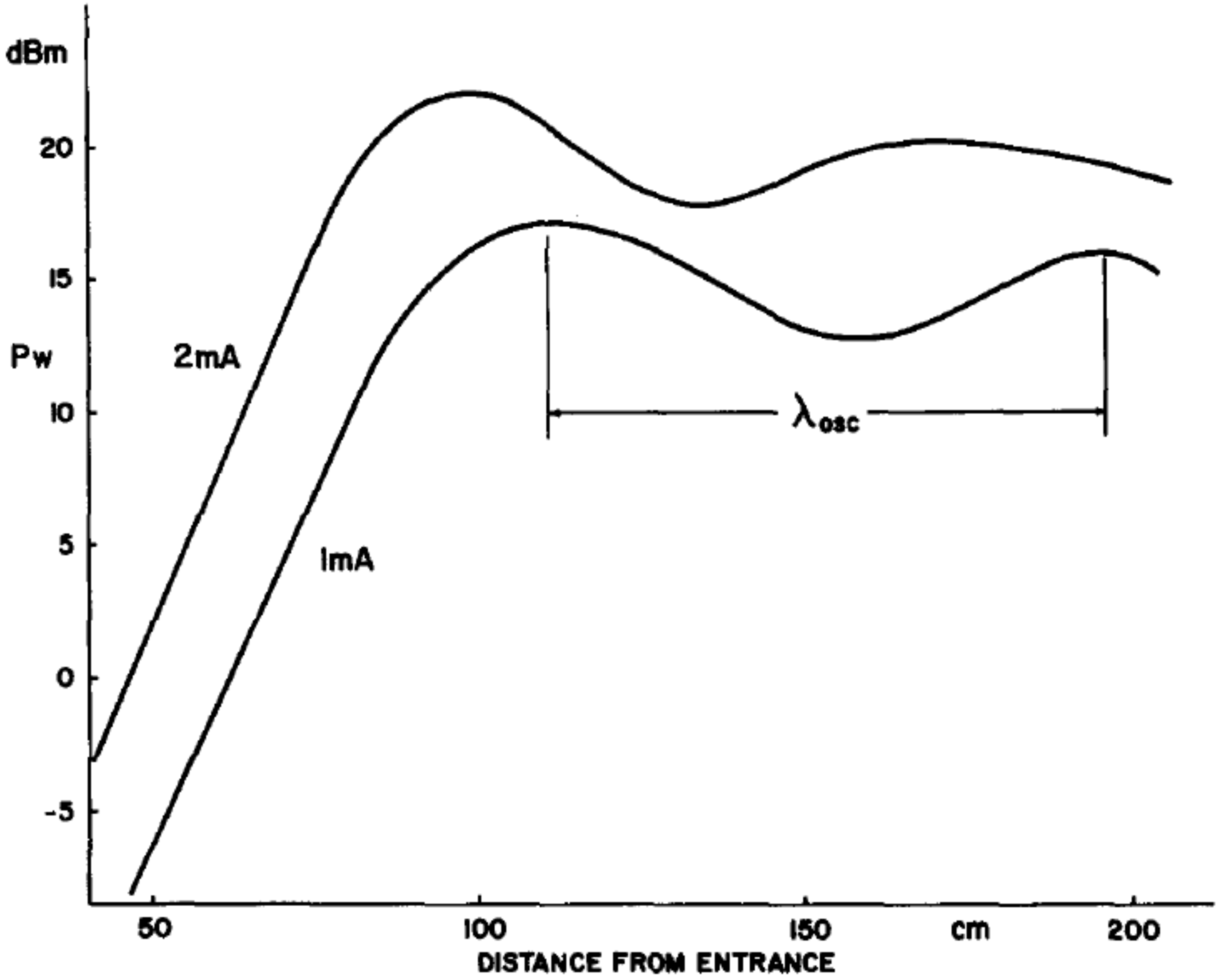

FIG. 1. Total wave power as a function of distance from the point of injection of the beam into the plasma column. The background plasma had a density of 7×10^8 and a temperature of 18 eV.

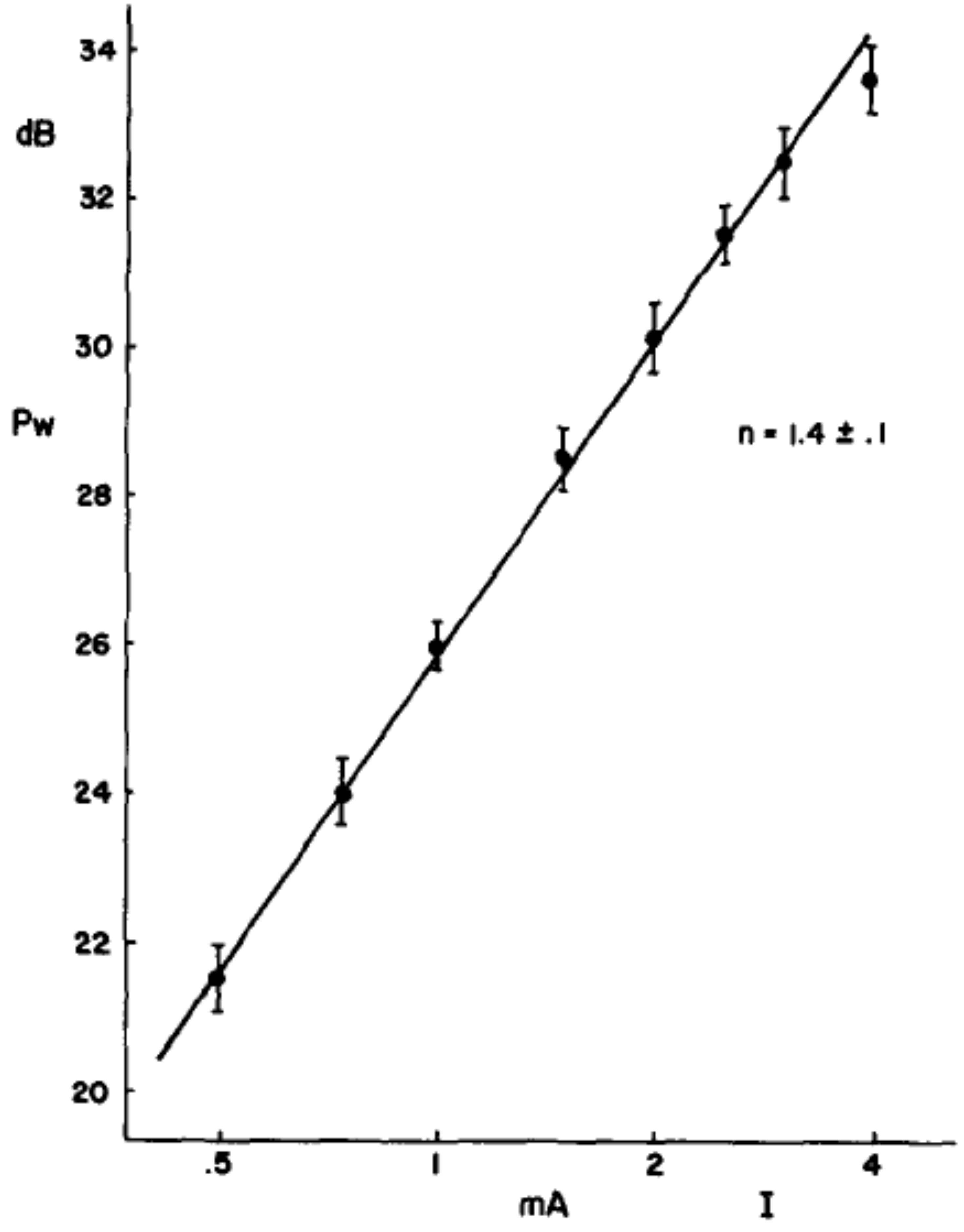

FIG. 2. Total wave power measured at the spatial maximum as a function of beam current. The wave power is in relative units.

growth in space. The peak wave power should be

$$P_w = 2^{2/3}\eta^{1/3}P_B, \qquad (1)$$

where $P_b = I_b V_b$, the input beam power, and

tions in the potential well. In space, the oscillations will appear with a wavelength determined by the particle oscillation frequency in the well and the propagation velocity, the beam velocity u. If the potential is sinusoidal and the particles are trapped near the bottom of the well, the oscillation length is given by

$$\lambda_{osc} = 2\pi u(m/ekE)^{1/2}. \qquad (3)$$

Making use of Eq. (1), we note that this implies that

$$\lambda_{osc} \propto I_b^{-1/3}. \qquad (4)$$

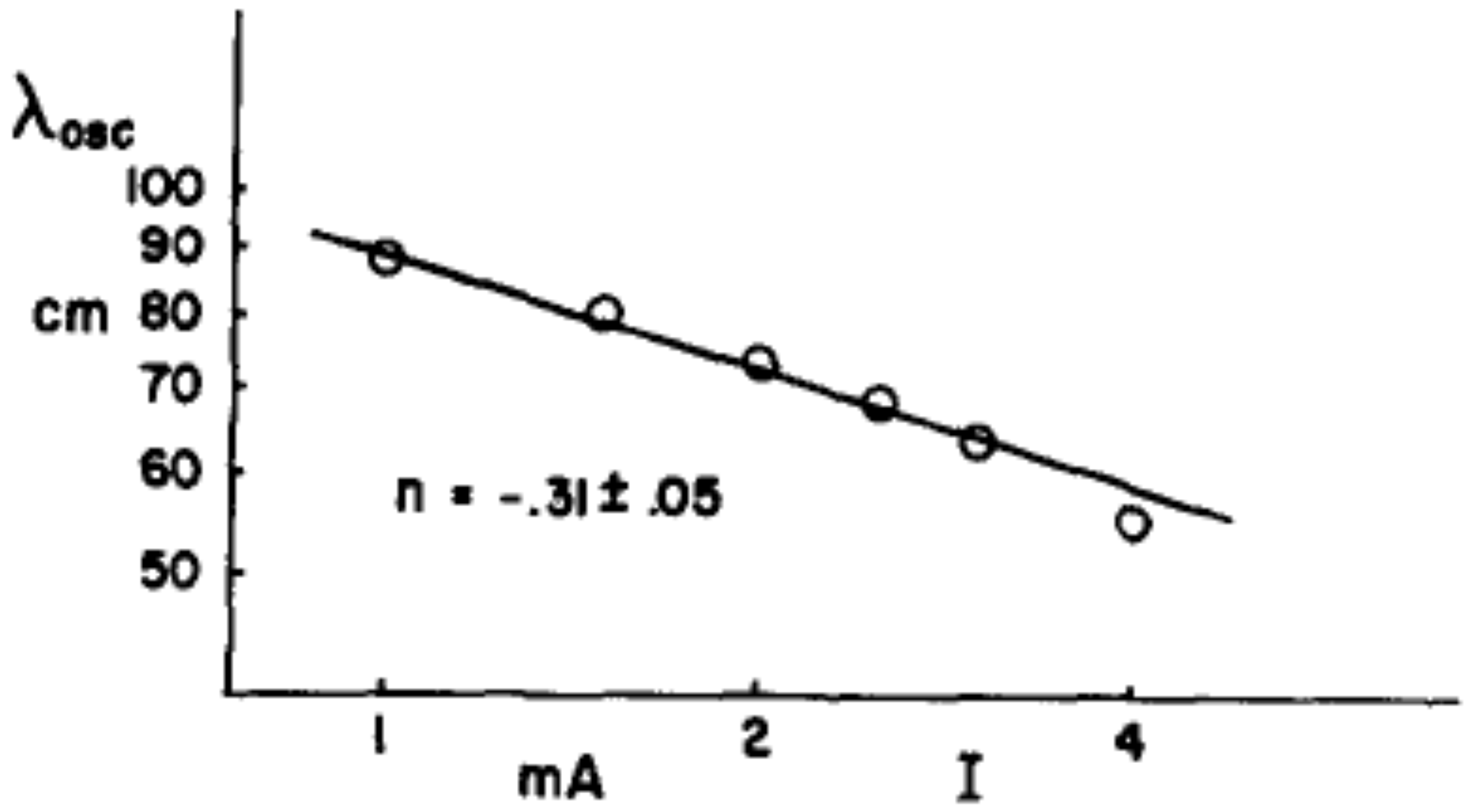

FIG. 3. Oscillation wavelength of the wave energy as a function of beam current.

NONLINEAR EVOLUTION OF A TWO-STREAM INSTABILITY*

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(Received 12 June 1967)

Calculations of a two-stream instability have been made by following the motion of the phase-space boundaries of an incompressible and constant-density phase-space fluid. Because of the condensation of holes, which to a good approximation act as gravitational particles, large-scale nonlinear pulses develop.

$$\frac{\partial f}{\partial t}+v\frac{\partial f}{\partial x}-\frac{\partial \varphi}{\partial x}\frac{\partial f}{\partial v}=0.$$

$$\frac{dx}{dt}=v,\quad \frac{dv}{dt}=-\frac{\partial\varphi}{\partial x},$$

$$\frac{\partial^2\varphi}{\partial x^2}=\omega_p{}^2\left[\int f\frac{dv}{v_0}-1\right],$$

The example to be discussed is a two-stream instability, in which the electron plasma is slightly perturbed at $t=0$ from an equilibrium characterized by four straight lines in phase space: $f=1$ for $\frac{1}{2}v_0<|v|<v_0$ and $f=0$ elsewhere. Periodic boundary conditions are imposed at $x=(0,L)$ and the parameters of the problem are $v_0\Delta t/\Delta x=0.25$, $\omega_p\Delta t=1/20$, and $\Delta x=L/64$, where Δx is the grid used for evaluating Poisson's equation. The unstable wave numbers are $k=2\pi n/L$ with $n=(1,2)$, and the linear growth rates are $\gamma/\omega_p=0.30,\ 0.315$.

The most striking feature of the calculation is the behavior of the $f = 0$ "cavity" which initially occupies the strip ($|v| < \frac{1}{2}v_0$) between the two plasma layers. This must preserve constant area as it deforms, and it is seen in Fig. 1 to coalesce into holes of roughly elliptical shape, so that a large-amplitude electrostatic wave is set up. Superimposed on this wave are coherent oscillations due to rotation of the holes in phase space, and also random fluctuations due to the motion of smaller elements of the hole "fluid." The two outer curves adjust almost adiabatically to the instantaneous potential function.

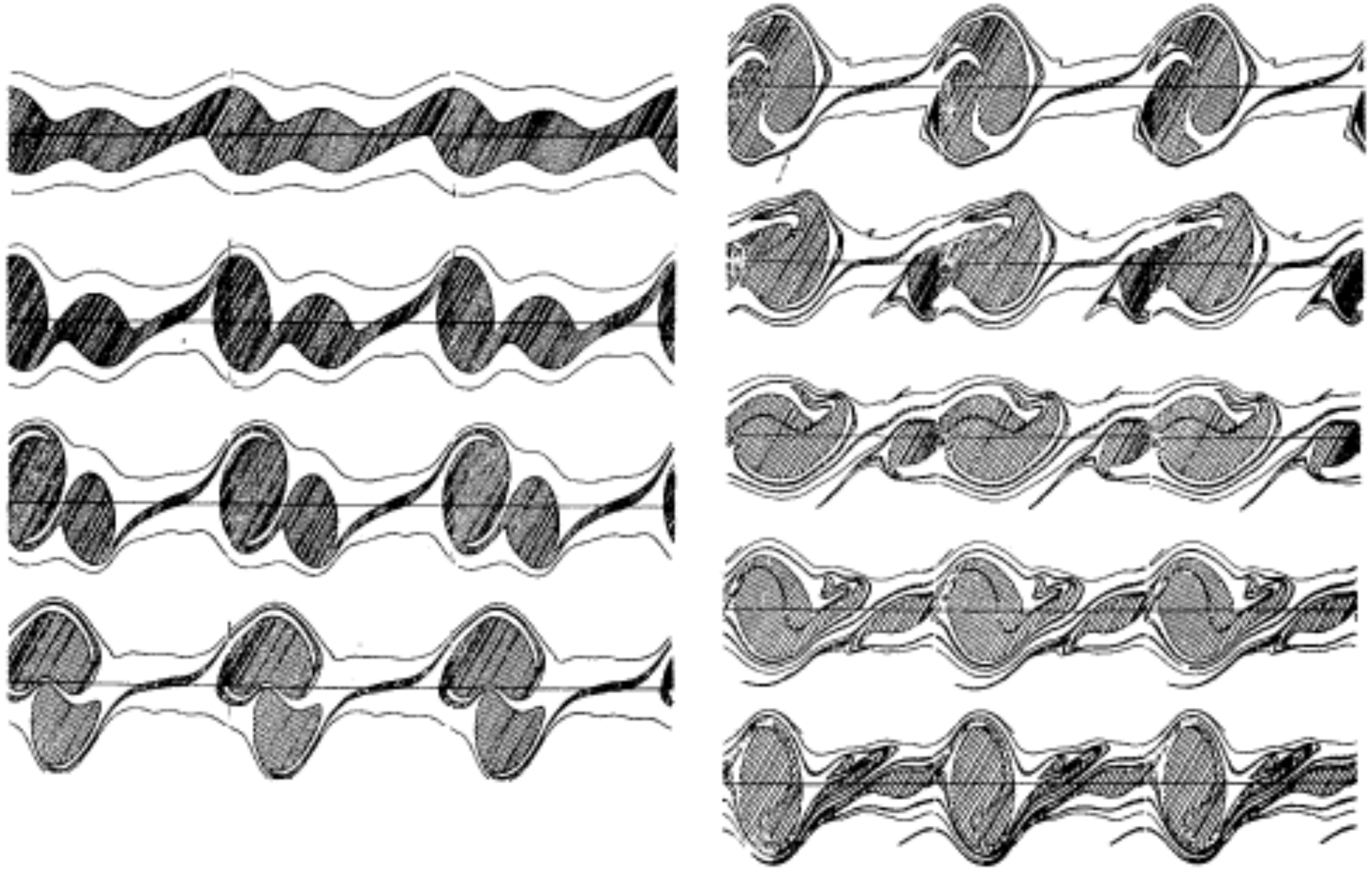


FIG. 1. Evolution in phase space of a two-stream instability from time steps 200 to 600 at intervals of 50 steps. Each step is 1/20 of a plasma period, and the horizontal and vertical coordinates are x, v, respectively. Periodic boundaries have been imposed and three identical periods are shown along each row. The shaded area represents the $f=0$ region enclosed by the plasma fluid.

Linearize Beam-Plasma

ELECTRON BEAM, n_b V_0
COLD PLASMA, n_p
} LINEARIZE FOR ELECTROSTATIC PLASMA WAVES

STATIONARY IONS $n_i = n_b + n_p$

• CONTINUITY: $\frac{\partial n}{\partial t} + \nabla\cdot(nv) = 0$

$$\begin{cases} -j\omega\tilde{n}_p + i k\cdot\tilde{v}_p n_{p0} = 0 \\ -j\omega\tilde{n}_b + i k\cdot\tilde{v}_b n_{b0} + i k\cdot V_0 \tilde{n}_b = 0 \end{cases}$$

NOTE: $nv = n_0 v_0 + n_0 v_1 + v_1 v_0 + n_1 v_1$

• MOMENTUM: $-j\omega\tilde{V}_p = -\frac{q}{m_e}\tilde{E}$

$$-j\omega\tilde{v}_p + j k\cdot V_0\tilde{V}_p = -\frac{q}{m_e}\tilde{E}$$

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \bar{v}\cdot\bar{\nabla}$$
$$= -j\omega + j k\cdot V_0$$

• POISSON'S EQUATION:

$$-k^2\tilde{\Phi} = -\frac{q}{\epsilon_0}(\tilde{n}_p + \tilde{n}_b) \qquad \tilde{E} = -j k\tilde{\Phi}$$

$$\therefore \tilde{n}_p = n_{p0}\frac{\bar{k}\cdot\tilde{V}_p}{\omega} \quad \& \quad \tilde{V}_p = -j\frac{q}{m_0}\frac{\tilde{E}}{\omega} = -\frac{q}{m_e\omega}k\tilde{\Phi}$$

$$\tilde{n}_b = n_{b0}\frac{k\cdot\tilde{V}_b}{\omega - kV_0} \qquad \tilde{V}_b = -j\frac{q}{m_0}\frac{\tilde{E}}{\omega - kV_0} = -\frac{q}{m_0}\frac{k\tilde{\Phi}}{(\omega - kV_0)}$$

Linear Beam-Plasma Dispersion Relation

LINEAR DISPERSION RELATION

$$-k^2\tilde{\Phi} = -\frac{q^2}{\epsilon_0}\left[n_{p0}\frac{k^2\tilde{\Phi}}{m_0\omega} + n_{b0}\frac{k^2\tilde{\Phi}}{m_0(\omega-kv_0)^2}\right]$$

OR $\quad D(k,\omega)\,\tilde{\Phi} = 0$

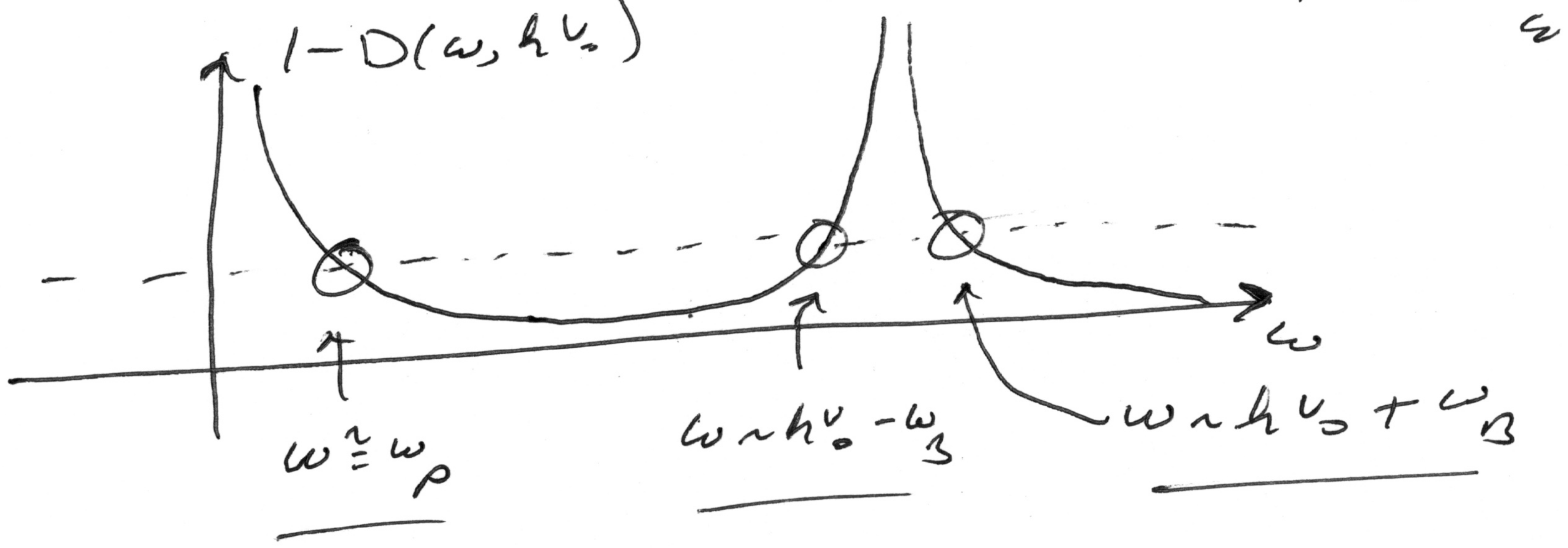
$$D(k,\omega) = 1 - \frac{\omega_p^2}{\omega^2} - \frac{\omega_b^2}{(\omega-kv_0)^2}$$

A BEAM-PLASMA SYSTEM HAS THREE (3) ROOTS

THREE EIGENMODES / EIGENVALUES

$$1 - D = \frac{\omega_p^2}{\omega^2} + \frac{\omega_B^2}{(\omega-kv_0)^2}$$

DEPENDS UPON k AND v_0



Energetics of Linear Modes

EXAMINE THESE MODES . . .

WHAT IS THE TIME-AVERAGED "WAVE ENERGY"?

$$(\text{WAVE ENERGY}) = \frac{1}{2}\epsilon_0 \left|\tilde{E}\right|^2 + \frac{1}{2}\left|(n_0+\tilde{n})\, m_e (V_0+\tilde{V})^2\right|_{time}$$

$$\simeq \frac{1}{2}\epsilon_0\left|\tilde{E}\right|^2 + \frac{1}{2} n_{p0} m_e \left|\tilde{V}_p^2\right|$$

$$+ \frac{1}{2} n_{b0} m_e \left|\tilde{V}_b^2\right| + m_e V_0 \left|\tilde{n}_b \tilde{V}_b\right| + \ldots$$

FIRST STEP: WHAT ARE THE WAVE ENERGY FOR THE "FAST BEAM" LIMIT $kV_0 \gg \omega_p$; $\omega_p \gg \omega_b$

Energetics of Plasma Wave

WAVE #1 : PLASMA WAVE $\omega \approx \omega_p$

$$\hat{V}_p = -j \frac{q}{m_0} \frac{\tilde{E}}{\omega_p} \qquad \hat{n}_p = -j \frac{q n_{p0}}{m_0} \frac{k \cdot \tilde{E}}{\omega_p^2}$$

$$\hat{V}_B = -j \frac{q}{m_0} \frac{\tilde{E}}{(-kU_0)} \qquad \hat{n}_B = -j \frac{q n_{b0}}{m_0} \frac{k \cdot \tilde{E}}{(-kU_0)^2}$$

$$\therefore \; (\text{WAVE ENERGY}) = \frac{\epsilon_0}{2} |\tilde{E}|^2 + \frac{1}{2} \epsilon_0 \frac{\omega_p^2}{\omega_p^2} |\tilde{E}|^2 + \text{BEAM ENERGY}$$

$$= 2 \times \left(\frac{\epsilon_0}{2} |\tilde{E}|^2 \right)$$

BEAM ENERGY SMALL BECAUSE $\omega_B^2/(kU_0)^2 \ll 1$

ELECTROSTATIC PLASMA WAVES *** HAVE TWICE (2×) ENERGY OF ELECTRIC FIELD

Energetics of "Fast" Beam Mode

WAVE #2: FAST BEAM MODE $\omega \simeq \omega_b + k v_0$

$$\tilde{v}_p \simeq -j \frac{q}{m_0} \frac{\tilde{E}}{k v_0} \qquad \tilde{n}_p = -j \frac{q n_{p0}}{m_0} \frac{k \cdot \tilde{E}}{(k v_0)^2}$$

BOTH ARE SMALL BECAUSE $\omega \gg \omega_{p0}$

$$\tilde{v}_B = -j \frac{q}{m_0} \frac{\tilde{E}}{\omega_b} \qquad \tilde{n}_B = -j \frac{q n_{b0}}{m_0} \frac{k \cdot \tilde{E}}{\omega_b^2}$$

$$\therefore \text{(WAVE ENERGY)} = \frac{\epsilon_0}{2} |\tilde{E}|^2 + \frac{\epsilon_0}{2} |\tilde{E}|^2 \frac{\omega_b^2}{\omega_b^2} + m_0 V_0 k \frac{|\tilde{E}|^2}{\omega_b^2} + \cdots$$

*** → SO FAST BEAM MODE $\gg 2 \times \left(\frac{\epsilon_0}{2} |\tilde{E}|^2 \right)$

Energetics of "Slow" Beam Mode

WAVE #3 "SLOW" BEAM MODE $\omega \simeq k v_0 - \omega_B$

$\tilde{v}_p$ ~ SMALL

$$\tilde{v}_B = -\frac{q}{m_0}\frac{\tilde{E}}{(-\omega_B)} \qquad \tilde{n}_B = -j\,\frac{q\, n_{b0}}{m_0}\,\frac{k\cdot\tilde{E}}{(\omega_B^2)}$$

SEE!

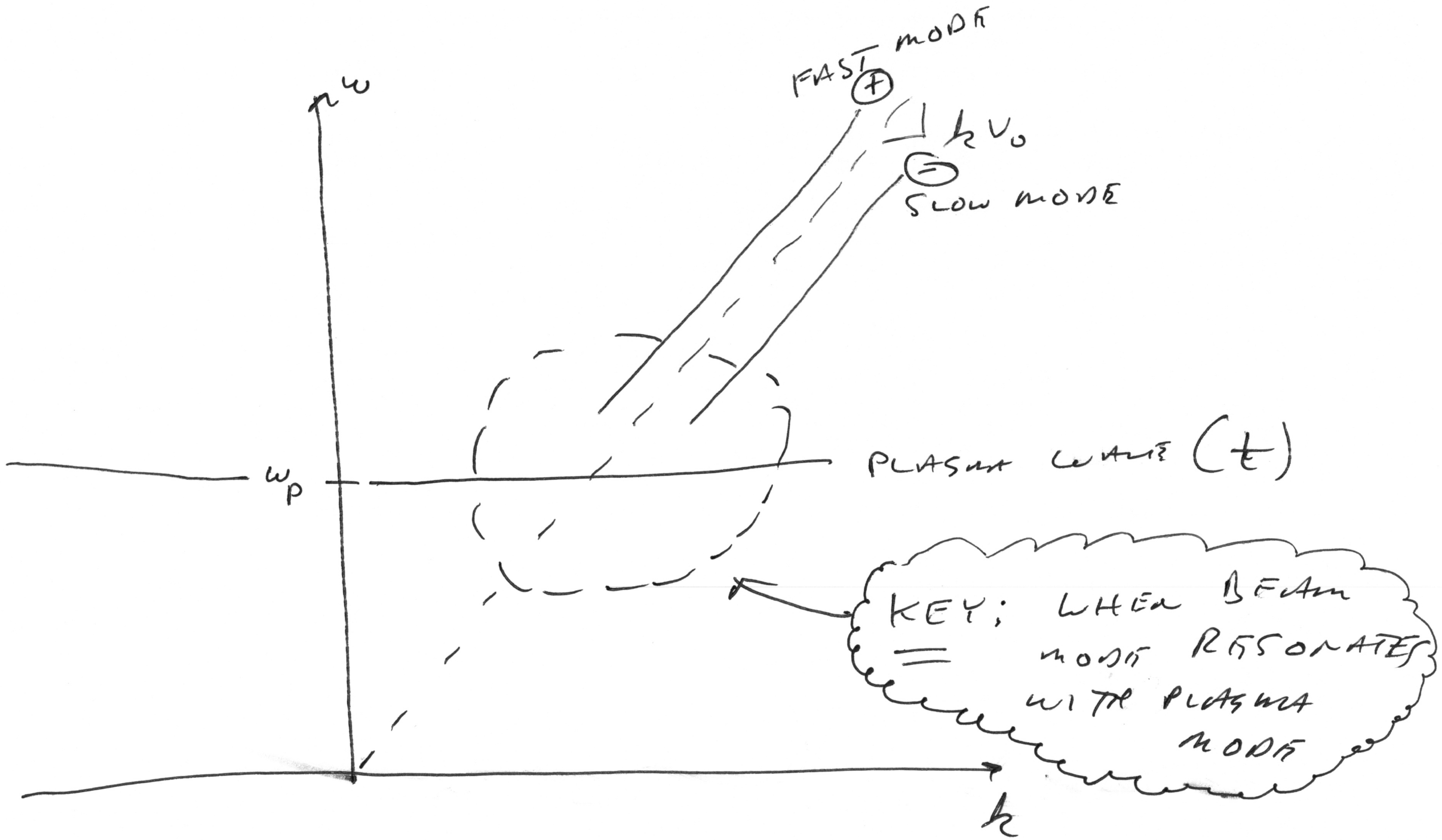
$$\therefore\ (\text{WAVE ENERGY}) \simeq \frac{\epsilon_0}{2}|\tilde{E}|^2 + \frac{\epsilon_0}{2}\frac{\omega_b^2}{\omega_B^2}|\tilde{E}|^2 - m_0 v_0 k \frac{|\tilde{E}|^2}{\omega_B^3}$$

SEE!

WITH $\frac{k_0 v_B}{\omega_B} \gg 1$, THE "WAVE ENERGY" IS {NEGATIVE}! ***

(WHAT DOES NEGATIVE ENERGY MEAN?)

Dispersion Diagram of E-Beam Plasma System



INSTABILITY OCCURS WHEN

POSITIVE ENERGY PLASMA MODE ≃ NEGATIVE ENERGY BEAM MODE ***

Dispersion Diagram of E-Beam Plasma System

RESONANT INTERACTION BETWEEN PLASMA + BEAM:

$$D(\omega, k) = 1 - \frac{\omega_p^2}{\omega^2} - \frac{\omega_B^2}{(\omega - kv_0)^2}$$

DEFINE $\omega \simeq \omega_p + \Delta\omega$

$$D(\Delta\omega, k) = 1 - \frac{\omega_p^2}{(\omega_p + \Delta\omega)^2} - \frac{\omega_B^2}{(\omega_p - kv_0 + \Delta\omega)^2}$$

$$\simeq 1 - \frac{1}{\left(1 + \frac{\Delta\omega}{\omega_p}\right)^2} - \frac{\omega_B^2}{(\Delta\omega)^2} \quad (\text{with } \omega_p - kv_0 \simeq 0)$$

$$\simeq + \frac{2\Delta\omega}{\omega_p} - \frac{\omega_B^2}{(\Delta\omega)^2}$$

OR

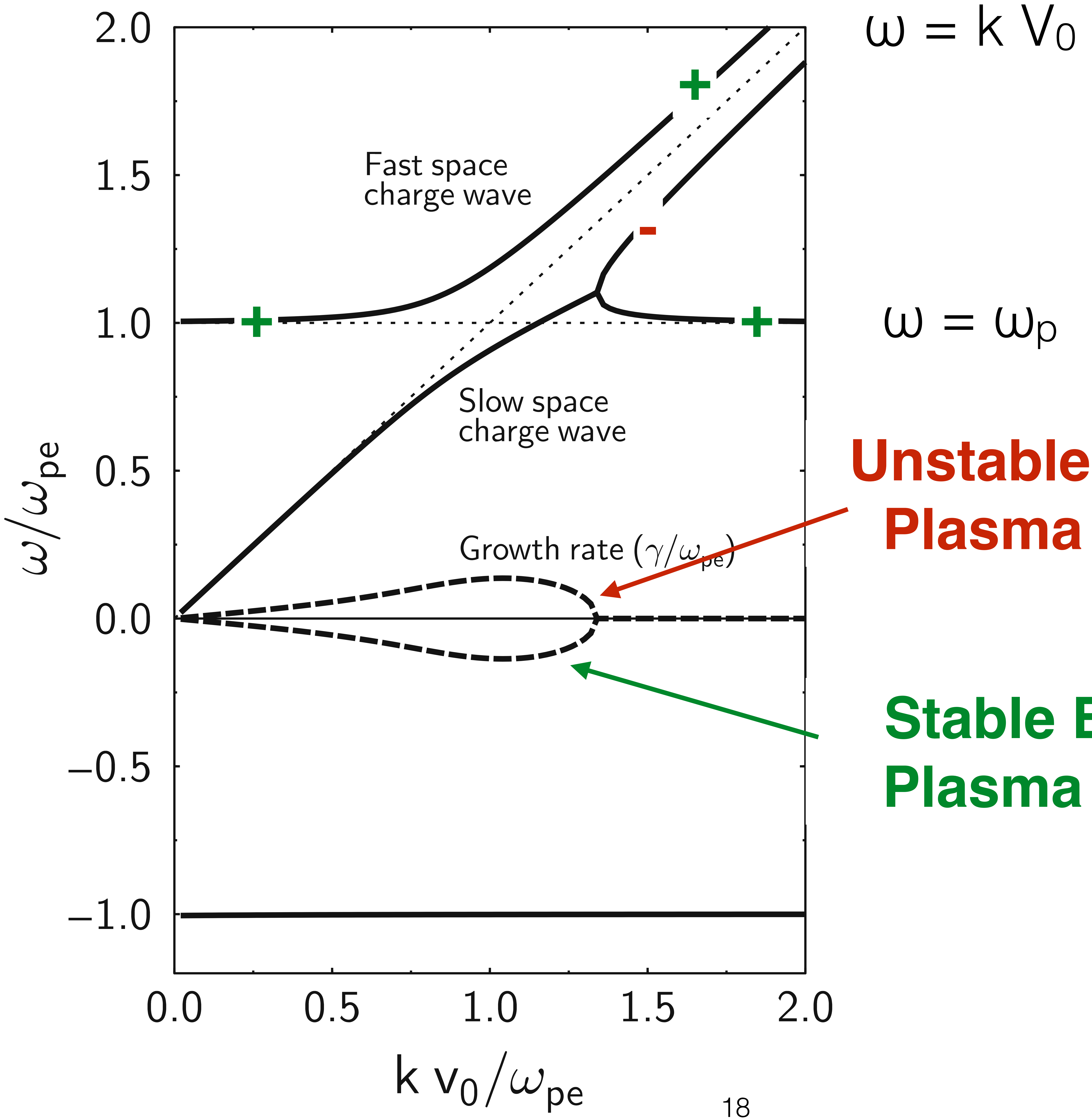
$$(\Delta\omega)^3 = 2\,\omega_B^2\,\omega_p$$

$$\therefore \Delta\omega = (1)^{1/3}\sqrt[3]{2\omega_B^2\omega_p}$$

$$1^{1/3} = \begin{cases} 1 \\ e^{i120^\circ} \\ e^{-i120^\circ} \end{cases}$$

$$= -\frac{1}{2} \pm \frac{i\sqrt{3}}{2} \quad !!!$$

← ***



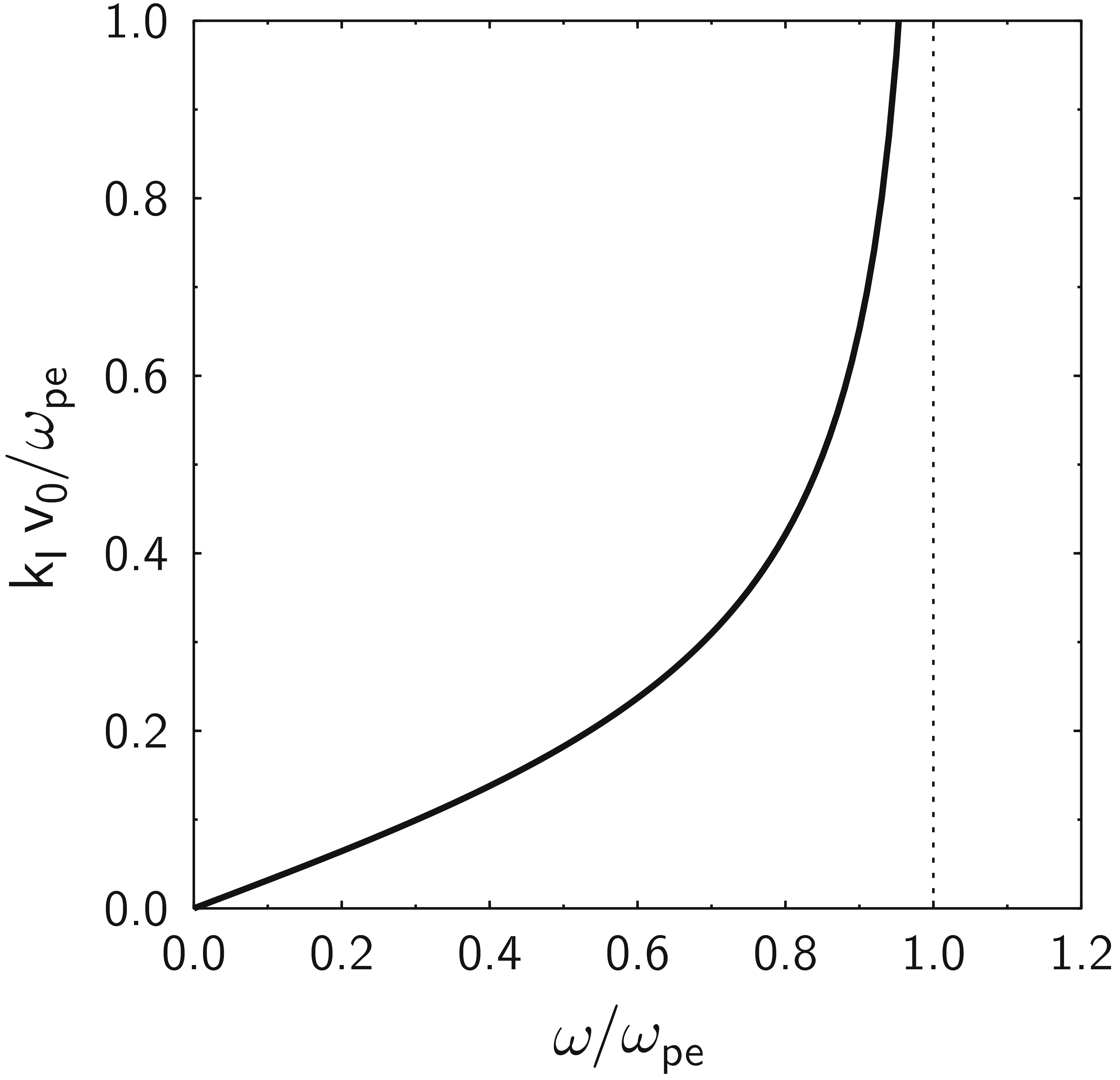
Unstable Beam-Plasma Mode

Stable Beam-Plasma Mode

8.1.5 Temporal or Spatial Growth

$$k v_0 = \omega + \mathrm{i} \left(\frac{\alpha_\mathrm{b} \omega_\mathrm{pe}^2 \omega^2}{\omega_\mathrm{pe}^2 - \omega^2} \right)^{1/2}$$

$$k_\mathrm{I} = \frac{\omega_\mathrm{pe}}{v_0} \frac{\alpha_\mathrm{b}^{1/2} \omega}{(\omega_\mathrm{pe}^2 - \omega^2)^{1/2}}$$



These waves are convectively unstable: growth as wave propagates at V_0

Summary

- E-Beam Plasma is unstable when $kV_0 \sim \omega_p$
- "Slow" beam mode is a "negative energy" wave
- Electric field energy increases as unstable wave grows
- Instabilities result from a source of energy and when a wave, or displacement, releases the source of energy
- NEXT LECTURE: ideal "macroinstability", flute-interchange mode, and kink mode

ANNALS OF PHYSICS: 1, 120–140 (1957)

Stability of Plasmas Confined by Magnetic Fields[1]

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In this paper, we examine the question of the stability of plasmas confined by magnetic fields. Whereas previous studies of this problem have started from the magnetohydrodynamic equations, we pay closer attention to the motions of individual particles. Our results are similar to, but more general than, those which follow from the magnetohydrodynamic equations.

that the internal energy of the plasma per unit mass is

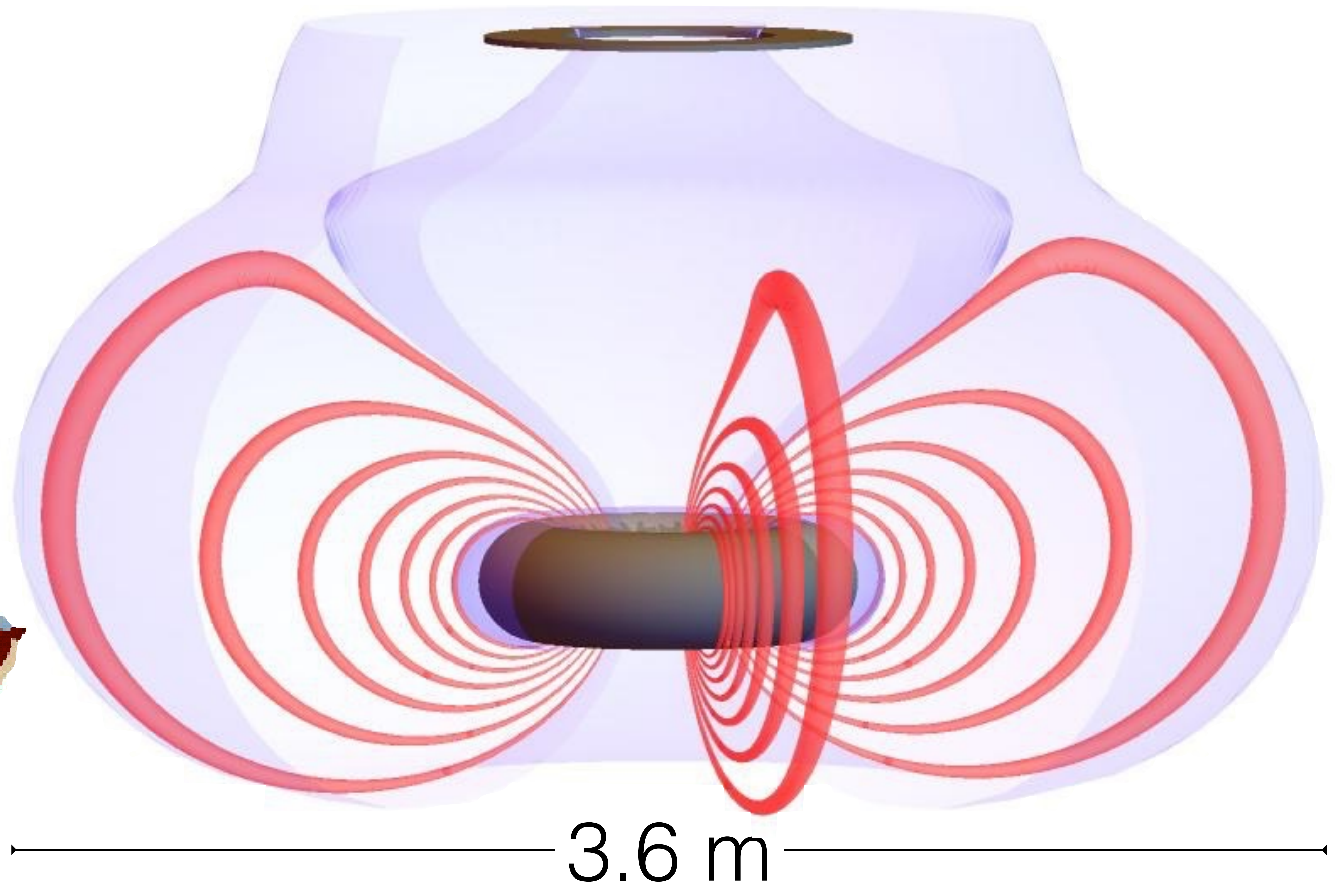
$$E_p = \frac{pv}{\gamma - 1} \tag{22}$$

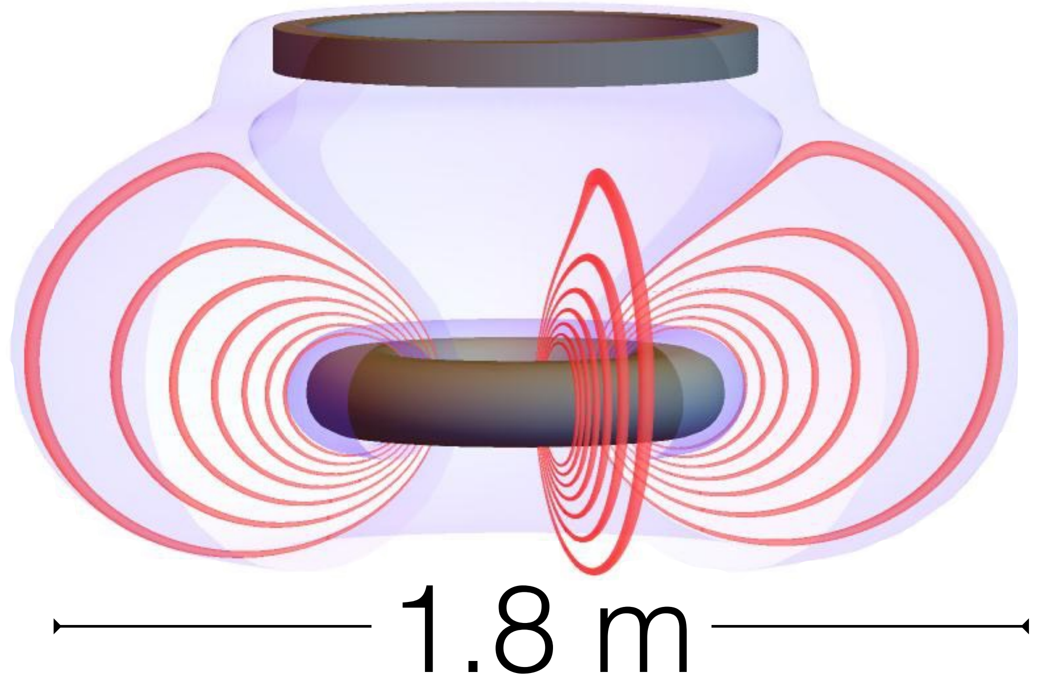
where p is the pressure and v the specific volume. In any adiabatic motion,

$$p \sim v^{-\gamma} \tag{23}$$

Laboratory Magnetospheres: Designed for Maximum Flux Tube Expansion

$$V = \int \frac{dl}{B} \propto L^4$$





Levitated Dipole Experiment (LDX)

Flux Tube Expansion:

$\delta V(out)/\delta V(in) = 100$

Ring Trap 1 (RT-1)

Flux Tube Expansion:

$\delta V(out)/\delta V(in) = 40$