Lecture 9:
Plasma Physics 1

APPH E6101x
Columbia University

Last Lecture

- Homework #4: Modeling Collisions and the Rosenbluth Potentials
- Force balance (*equilibrium*) in a magnetized plasma
    - Z-pinch
    - θ-pinch
    - Screw-pinch (straight tokamak)

Outline

- Grad-Shafranov Equation
- Conservation principles in magnetized plasma (“frozen-in” and conservation of particles/flux tubes)
- Alfvén Wave

MHD

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\mathbf{u}) = 0$$

$$\rho_{\mathrm{m}} \frac{\partial \mathbf{v}_{\mathrm{m}}}{\partial t} = \mathbf{j} \times \mathbf{B} - \nabla p + \rho_{\mathrm{m}} \mathbf{g}$$

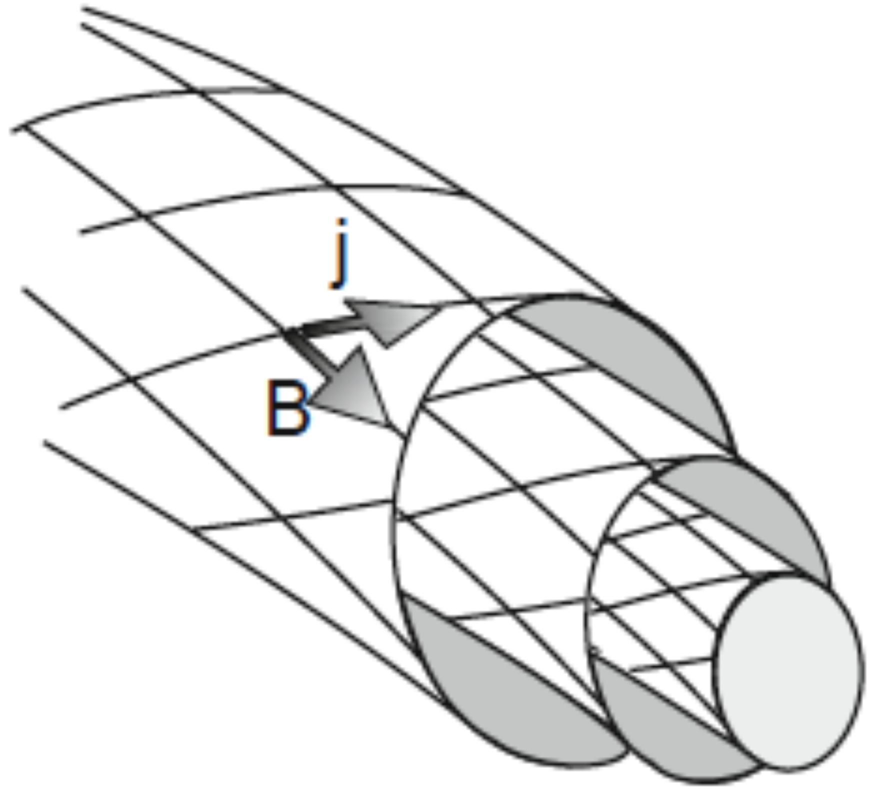
$$\mathbf{E} + \mathbf{v}_{\mathrm{m}} \times \mathbf{B} = \eta \mathbf{j} + \frac{1}{ne}(\mathbf{j} \times \mathbf{B} - \nabla p_{\mathrm{e}})$$

plus magnetostatics

Statics

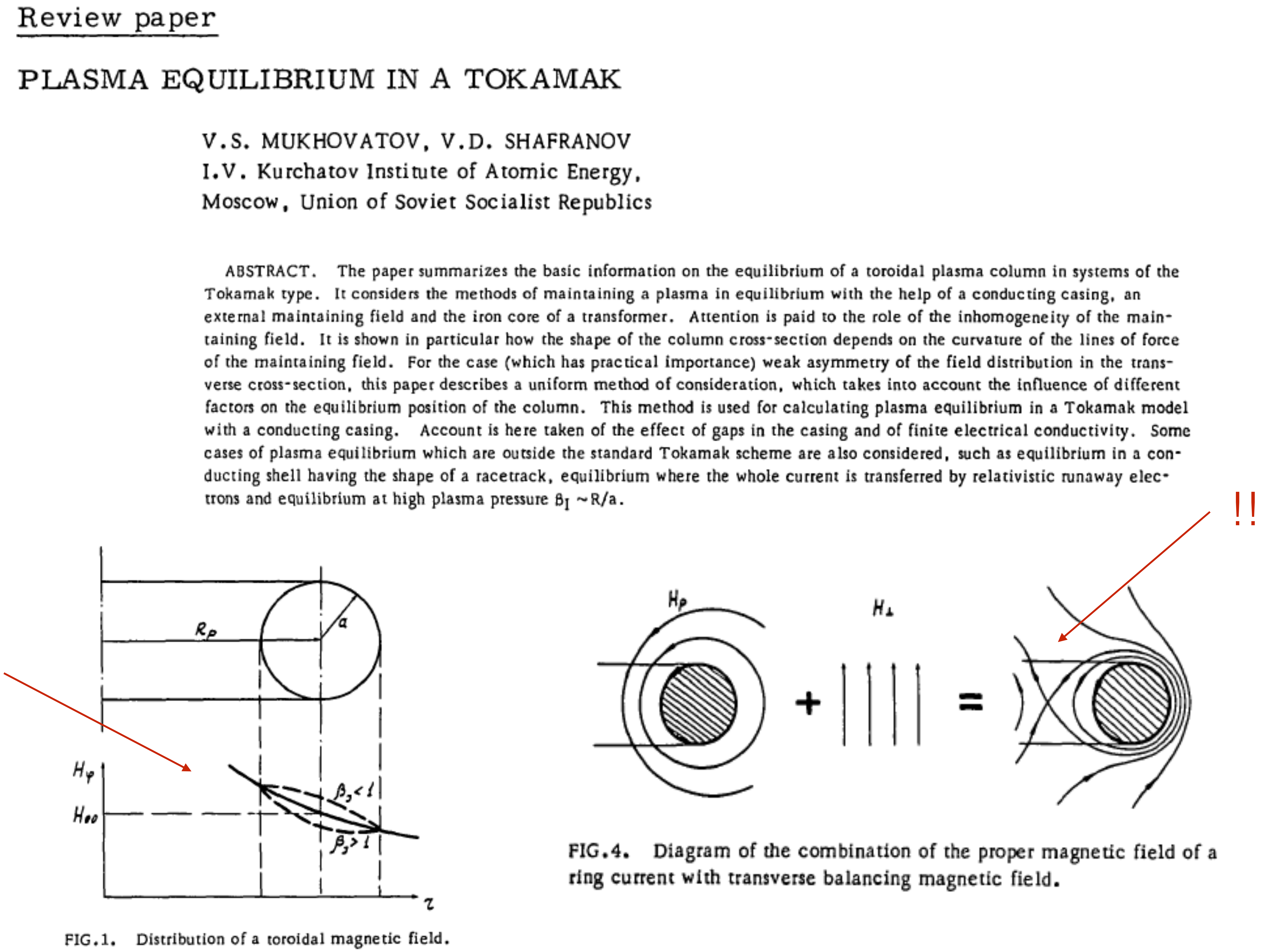
$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j}$$

$$0 = \mathbf{j} \times \mathbf{B} - \nabla p$$



$$\nabla_\perp p = -\frac{1}{2\mu_0} \nabla_\perp B^2 + \frac{B^2}{\mu_0} \overline{\kappa}$$

$$\boxed{\nabla_\perp \left(p + \frac{B^2}{2\mu_0} \right) = \frac{B^2}{\mu_0} \overline{\kappa}}$$



Review paper

PLASMA EQUILIBRIUM IN A TOKAMAK

V.S. MUKHOVATOV, V.D. SHAFRANOV
I.V. Kurchatov Institute of Atomic Energy,
Moscow, Union of Soviet Socialist Republics

ABSTRACT. The paper summarizes the basic information on the equilibrium of a toroidal plasma column in systems of the Tokamak type. It considers the methods of maintaining a plasma in equilibrium with the help of a conducting casing, an external maintaining field and the iron core of a transformer. Attention is paid to the role of the inhomogeneity of the maintaining field. It is shown in particular how the shape of the column cross-section depends on the curvature of the lines of force of the maintaining field. For the case (which has practical importance) weak asymmetry of the field distribution in the transverse cross-section, this paper describes a uniform method of consideration, which takes into account the influence of different factors on the equilibrium position of the column. This method is used for calculating plasma equilibrium in a Tokamak model with a conducting casing. Account is here taken of the effect of gaps in the casing and of finite electrical conductivity. Some cases of plasma equilibrium which are outside the standard Tokamak scheme are also considered, such as equilibrium in a conducting shell having the shape of a racetrack, equilibrium where the whole current is transferred by relativistic runaway electrons and equilibrium at high plasma pressure $\beta_I \sim R/a$.

FIG.1. Distribution of a toroidal magnetic field.

FIG.4. Diagram of the combination of the proper magnetic field of a ring current with transverse balancing magnetic field.

$\mu = 1 - \beta_p$

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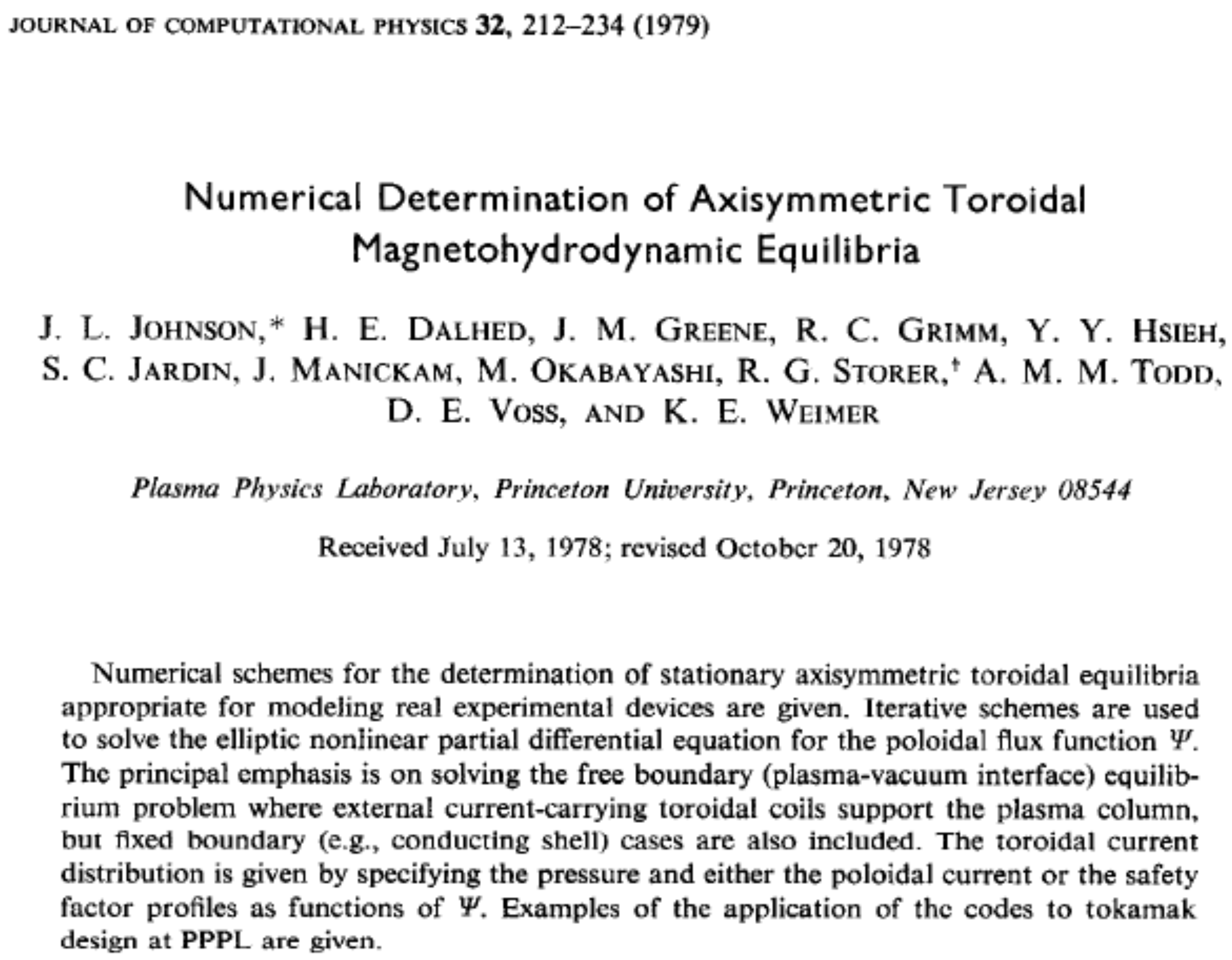

JOURNAL OF COMPUTATIONAL PHYSICS **32**, 212–234 (1979)

Numerical Determination of Axisymmetric Toroidal Magnetohydrodynamic Equilibria

J. L. JOHNSON,* H. E. DALHED, J. M. GREENE, R. C. GRIMM, Y. Y. HSIEH,
S. C. JARDIN, J. MANICKAM, M. OKABAYASHI, R. G. STORER,† A. M. M. TODD,
D. E. VOSS, AND K. E. WEIMER

Plasma Physics Laboratory, Princeton University, Princeton, New Jersey 08544

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Numerical schemes for the determination of stationary axisymmetric toroidal equilibria appropriate for modeling real experimental devices are given. Iterative schemes are used to solve the elliptic nonlinear partial differential equation for the poloidal flux function Ψ. The principal emphasis is on solving the free boundary (plasma-vacuum interface) equilibrium problem where external current-carrying toroidal coils support the plasma column, but fixed boundary (e.g., conducting shell) cases are also included. The toroidal current distribution is given by specifying the pressure and either the poloidal current or the safety factor profiles as functions of Ψ. Examples of the application of the codes to tokamak design at PPPL are given.

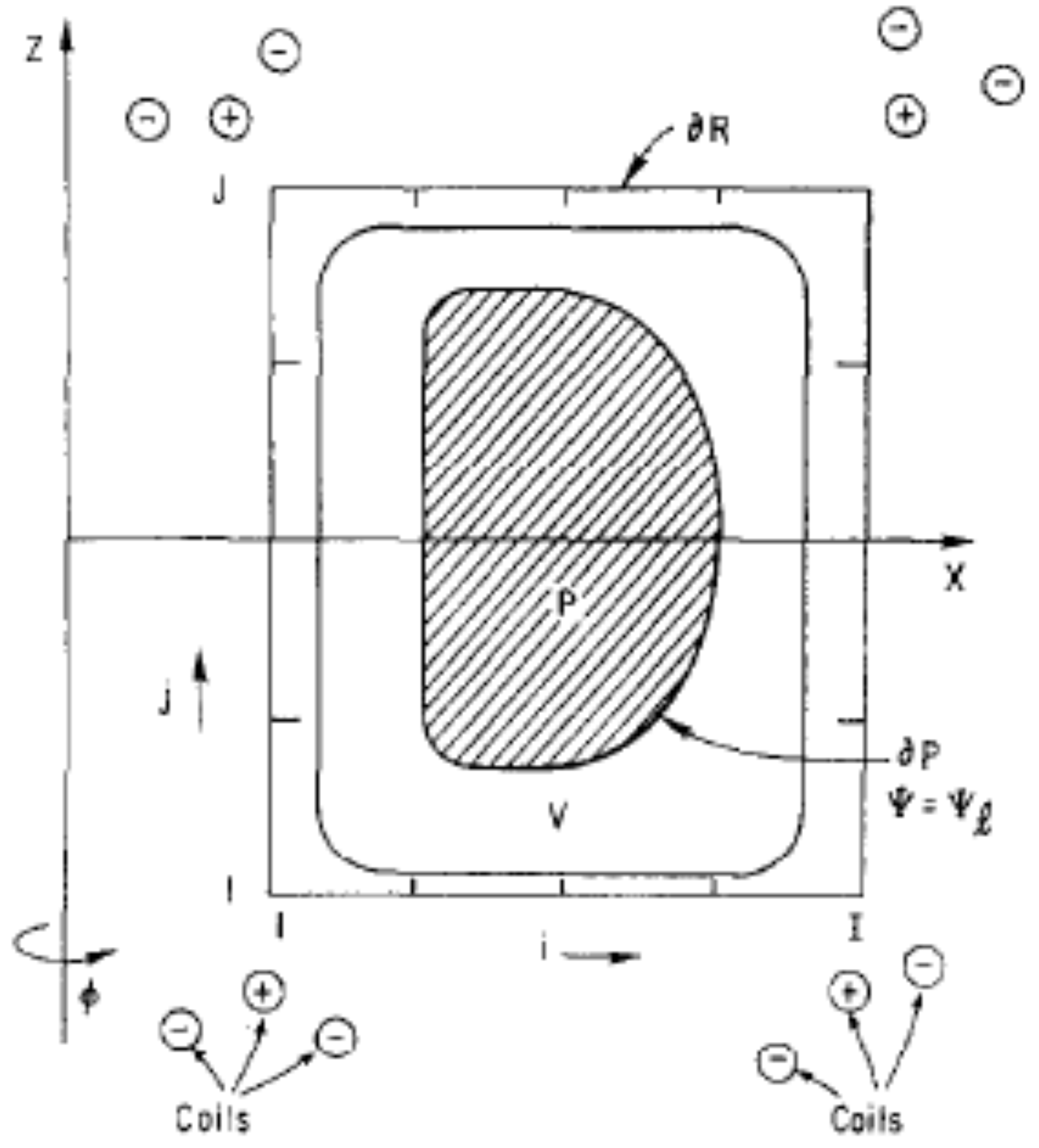


FIG. 1. Computational domain $\mathscr{R}$.

SOLUTION PROCEDURE WITH $p(\Psi)$ AND $g(\Psi)$ SPECIFIED

$$\nabla p = \mathbf{J} \times \mathbf{B},$$

$$\mathbf{J} = \nabla \times \mathbf{B},$$

$$\nabla \cdot \mathbf{B} = 0.$$

$$\mathbf{B} = \frac{1}{2\pi} \nabla\phi \times \nabla\Psi + RB_0 g \, \nabla\phi$$

Grad Shafranov Equation:

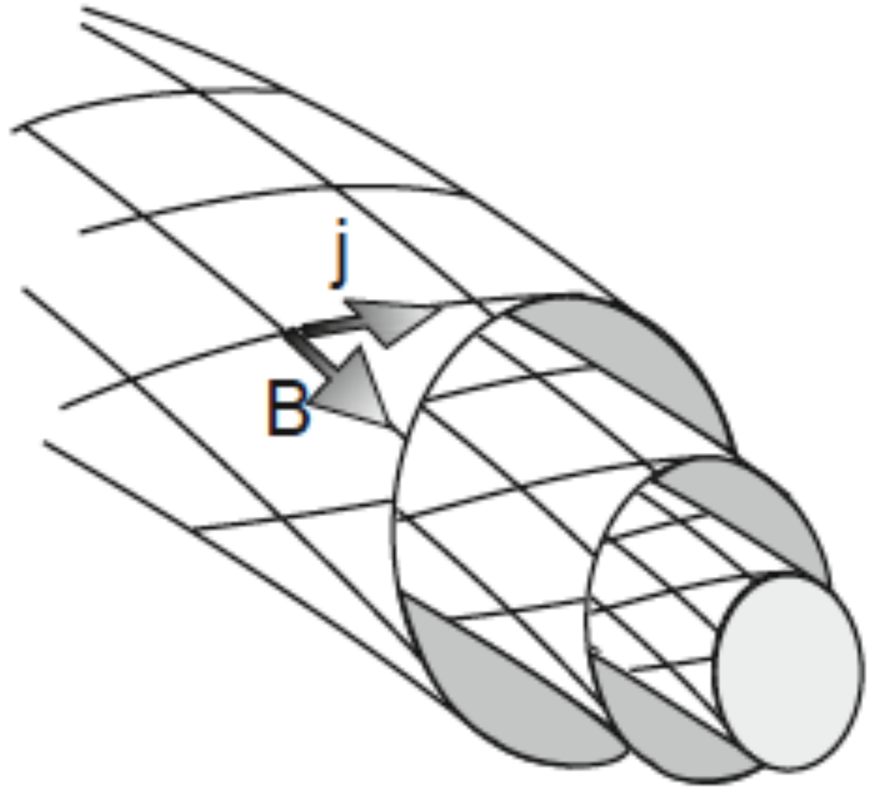
$$X \frac{\partial}{\partial X} \frac{1}{X} \frac{\partial \Psi}{\partial X} + \frac{\partial^2 \Psi}{\partial Z^2} = 2\pi X J_\phi ,$$

$$J_\phi = -2\pi \left(X \frac{dp}{d\Psi} + \frac{R^2 B_0^2}{2X} \frac{dg^2}{d\Psi} \right).$$

Grad-Shafranov Equation

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j}$$

$$0 = \mathbf{j} \times \mathbf{B} - \nabla p$$



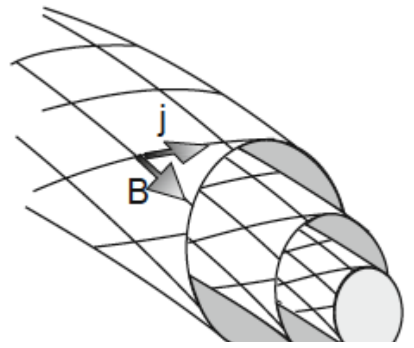
- $\bar{B} = \bar{\nabla}\varphi \times \bar{\nabla}\psi + G\bar{\nabla}\varphi$
- CYLINDRICAL SYMMETRY $\quad \bar{\nabla}\varphi = \frac{\hat{\varphi}}{R} \quad \bar{\nabla}\varphi \cdot \bar{\nabla}(\text{ANYTHING}) \approx 0$
- $P(\psi), G(\psi)$ $\quad G =$ TOROIDAL FLUX $\quad \bar{B}_\varphi = \frac{G}{R}\hat{\varphi}$
- $\bar{B}\cdot\bar{B} = (\nabla\varphi \times \nabla\psi)\cdot(\nabla\varphi \times \nabla\psi) + G^2|\nabla\varphi|^2$
  $= |\nabla\varphi|^2 \left(|\nabla\psi|^2 + G^2\right)$

Grad-Shafranov Equation

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j}$$

$$0 = \mathbf{j} \times \mathbf{B} - \nabla p$$

$$\mathbf{B} = \frac{1}{2\pi} \nabla\phi \times \nabla\Psi + R B_0 g \, \nabla\phi$$



- $$\mu_0 \bar{J} = \nabla \times \bar{B}$$

$$= \nabla \times (\nabla\varphi \times \nabla\psi) + \nabla \times (G \nabla \varphi)$$

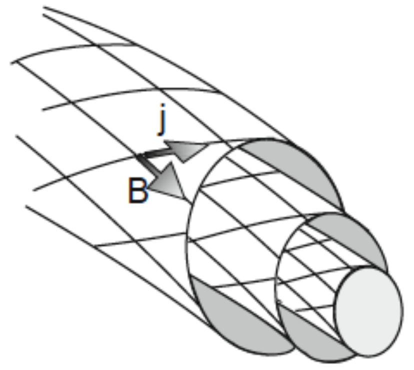
$$= \bar{\nabla}\varphi \, (\nabla \cdot \nabla\psi) - \bar{\nabla}\psi \, (\nabla \cdot \bar{\nabla}\varphi) + (\nabla\psi \cdot \bar{\nabla}) \bar{\nabla}\varphi$$

$$- (\nabla\varphi \cdot \bar{\nabla}) \bar{\nabla}\psi + G \, \nabla \times \nabla\varphi + \bar{\nabla} G \times \bar{\nabla}\varphi$$

$$= \underbrace{\bar{\nabla}\varphi \left[\nabla^2 \psi - \frac{1}{\partial r} \frac{\partial \psi}{\partial r} \right]}_{J_\varphi} + \underbrace{\frac{\partial G}{\partial \psi} \bar{\nabla}\psi \times \bar{\nabla}\varphi}_{\perp \ \text{POLOIDAL CURRENT WITHIN FLUX SURFACE}}$$

$$(10) \quad \nabla \times (\mathbf{A} \times \mathbf{B}) = \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A}) + (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B}$$

Grad-Shafranov Equation



$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j}$$

$$0 = \mathbf{j} \times \mathbf{B} - \nabla p$$

$$\mathbf{B} = \frac{1}{2\pi} \nabla\phi \times \nabla\Psi + RB_0 g \, \nabla\phi$$

$$\overline{\nabla p} = \bar{J} \times \bar{B}$$

$$\mu_0 \overline{\nabla\psi} \cdot \overline{\nabla p} = \overline{\nabla\psi} \; \mu_0 \cdot \left(\bar{J} \times \bar{B} \right)$$

$$\mu_0 |\nabla\psi|^2 \frac{\partial p}{\partial\psi} = \mu_0 \bar{J} \cdot \left(\bar{B} \times \nabla\psi \right)$$

$$= \mu_0 \bar{J} \cdot \Big[-\nabla\psi \times (\nabla\varphi \times \nabla\psi) + G \overline{\nabla\varphi} \times \nabla \Big]$$

$$= -\mu_0 J_\varphi |\nabla\varphi| |\nabla\psi|^2 + \mu_0 G \bar{J} \cdot \nabla\varphi \times \nabla\psi$$

$$= -\mu_0 J_\varphi |\nabla\varphi| |\nabla\psi|^2 + G \frac{\partial G}{\partial\psi} (\overline{\nabla\psi} \times \nabla\varphi) \cdot (\nabla\varphi \times \nabla\psi)$$

$$= -\mu_0 J_\varphi |\nabla\varphi| |\nabla\psi|^2 - G \frac{\partial G}{\partial\psi} |\nabla\varphi|^2 |\nabla\psi|^2$$

$$\boxed{\mu_0 J_\varphi |\nabla\varphi| = -\mu_0 \frac{\partial p}{\partial\psi} - |\nabla\varphi|^2 G \frac{\partial G}{\partial\psi}}$$

$$\left(\nabla\varphi = \frac{1}{R} \right)$$

$$\Delta^* \psi = -\mu_0 R^2 \frac{\partial p}{\partial\psi} - G \frac{\partial G}{\partial\psi}$$

$$\Delta^* = R \frac{\partial}{\partial R} \left(\frac{1}{R} \frac{\partial\psi}{\partial R} \right) + \frac{\partial^2\psi}{\partial z^2}$$

(1) $\mathbf{A} \cdot \mathbf{B} \times \mathbf{C} = \mathbf{A} \times \mathbf{B} \cdot \mathbf{C} = \mathbf{B} \cdot \mathbf{C} \times \mathbf{A} = \mathbf{B} \times \mathbf{C} \cdot \mathbf{A} = \mathbf{C} \cdot \mathbf{A} \times \mathbf{B} = \mathbf{C} \times \mathbf{A} \cdot \mathbf{B}$

(2) $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{C} \times \mathbf{B}) \times \mathbf{A} = (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C}$

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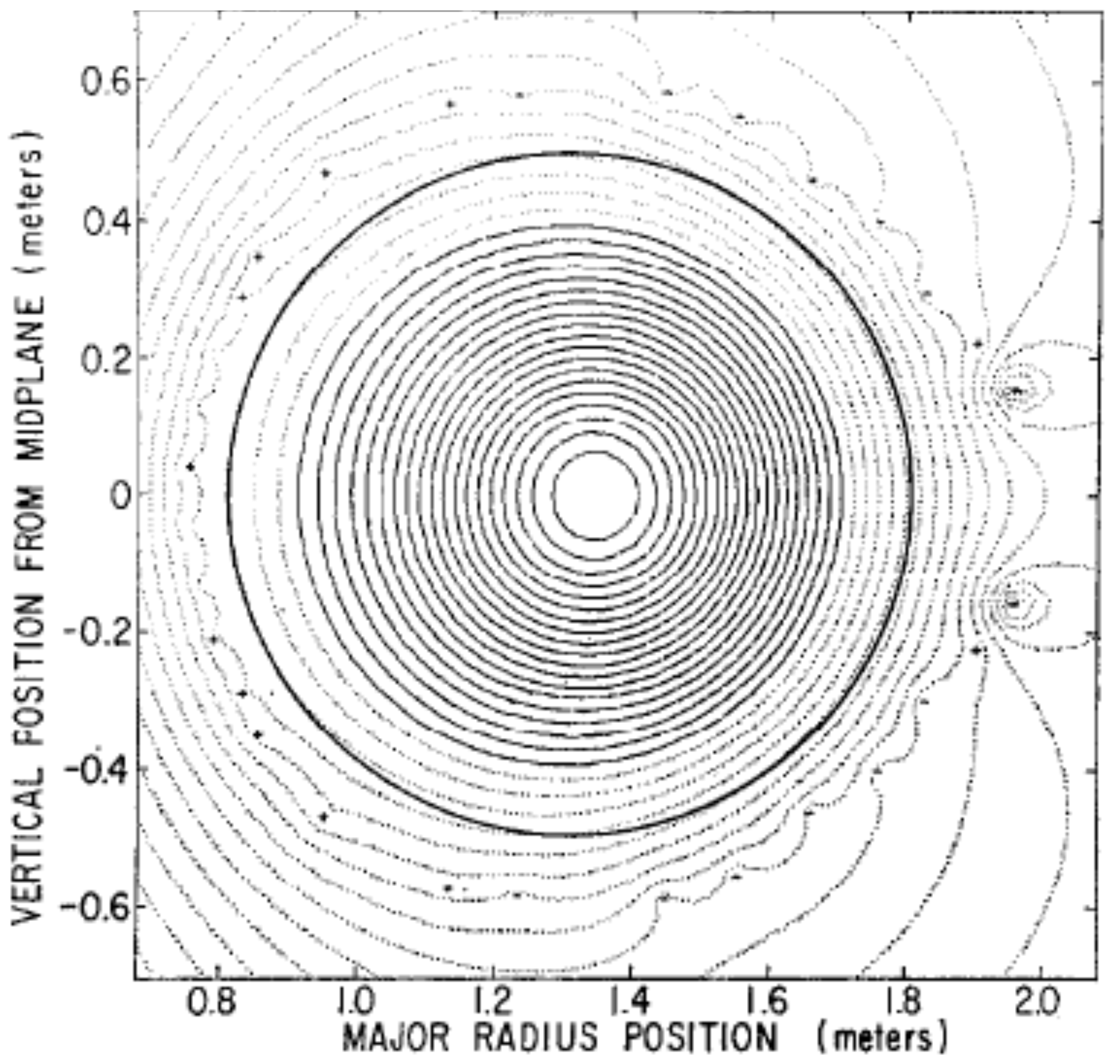
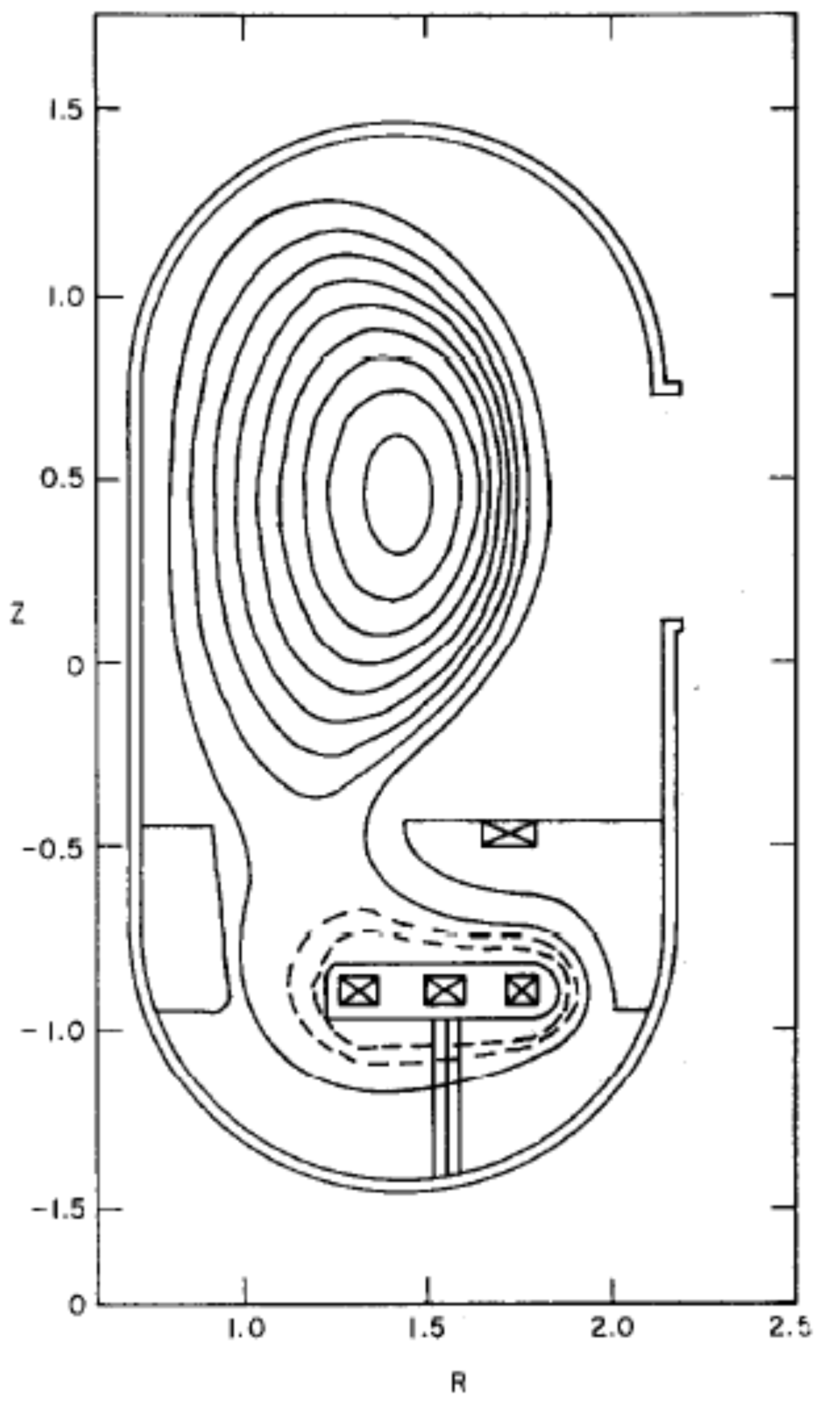
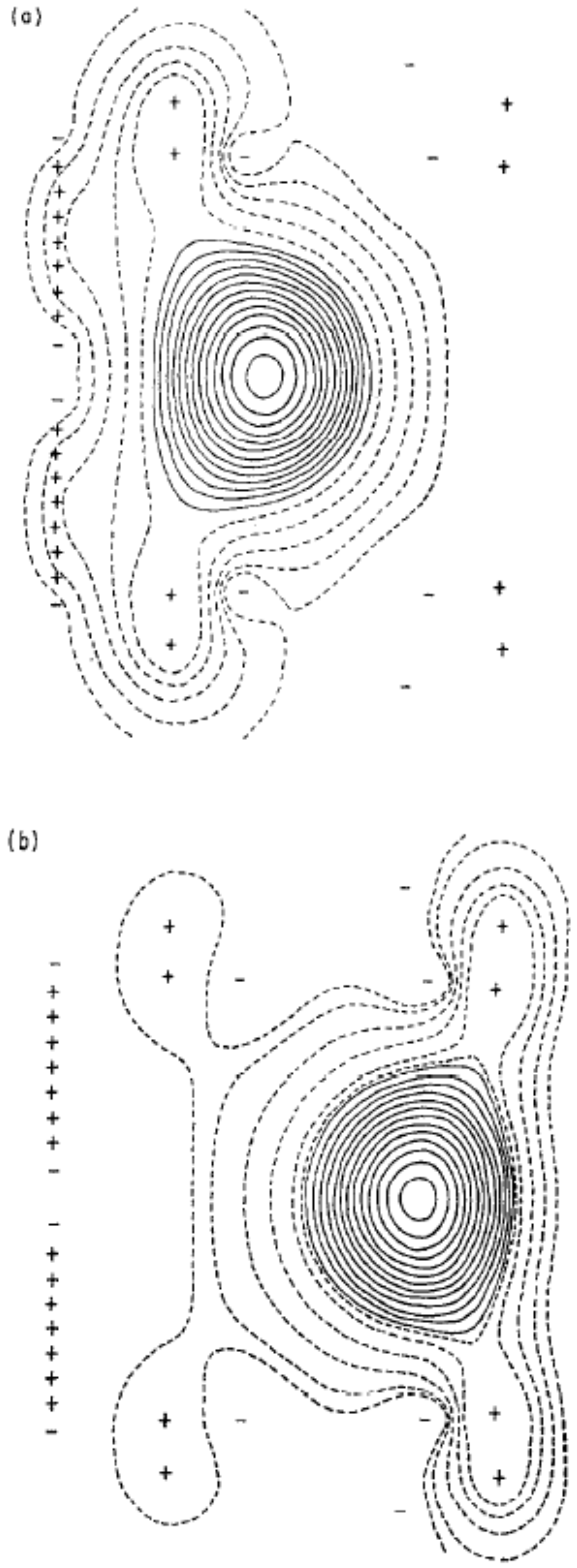


FIG. 4. A typical PLT equilibrium with $\beta_p = 0.23$ and $1.05 < q < 5.3$. The solid curve marks the position of the vacuum vessel. The pluses and minuses denote the poloidal field coils.





IG. 6. Typical PDX equilibria, showing that the plasma can be attac
s or (b) the outside divertor coils. Here $\alpha = \beta = 2$.

make it come true. The
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206/885-5446.
Garrison

aMack

The U.S. Department

looks like rings of lightning and lasts for 200 millionths of a second each time it appears. During every flash of the lightning ring, laser beams and magnetic sensors measure the dynamics of the ring at least once every millionth of a second, producing more than a megabyte of data.

The data is digitized, loaded into VAX computers, transferred to Macs, and then analyzed with TokaMack, an application Mauel wrote using Apple's Macintosh Programmer's Workshop (MPW) and MacApp. The analysis also involves four Cray com-

tohydrodynamic instabilities, called kinks, occurring in the plasma ring at very high pressure.

TokaMack is named after Tokamak, an earlier device developed by the late Russian physicist L. A. Artsimovich for magnetically confining ring lightning. Mauel offers the software as freeware to scientists doing similar research throughout the world. TokaMack requires a Mac II.

For further information, contact Michael Mauel at Columbia University, at 212/854-4455.
–Ann Garrison

Columbia professor Michael Mauel leads a team using Macs to test the use of magnetic force fields to confine hydrogen plasma.

81 Macworld News

- *Beyond HyperCard* Two
nies prepare more powerf
Card clones.
- *New MacWrite and Mac*
addition to new versions o
standbys, Claris introduce
CAD and the SmartForm S
- *And Now Presenting . . .*
and More II connect to si
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- *TokaMack* Columbia U
uses a Mac program in fus
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Plus, Jasmine's new BackF
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and more.

MHD

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\mathbf{u}) = 0$$

$$\rho_{\mathrm{m}} \frac{\partial \mathbf{v}_{\mathrm{m}}}{\partial t} = \mathbf{j} \times \mathbf{B} - \nabla p + \rho_{\mathrm{m}} \mathbf{g}$$

$$\mathbf{E} + \mathbf{v}_{\mathrm{m}} \times \mathbf{B} = \eta \mathbf{j} + \frac{1}{ne}(\mathbf{j} \times \mathbf{B} - \nabla p_{\mathrm{e}})$$

plus magnetostatics

“Frozen-in” Flux

The plasma moves along with the magnetic field
or
The plasma within flux tubes remains invariant

(Ohm’s Law & Faraday’s Law)

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v}_\mathrm{m} \times \mathbf{B})$$

$$\nabla \times (\mathbf{v}_\mathrm{m} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{v}_\mathrm{m} - (\mathbf{v}_\mathrm{m} \cdot \nabla)\mathbf{B} + \mathbf{v}_\mathrm{m} \underbrace{(\nabla \cdot \mathbf{B})}_{=0} - \mathbf{B}(\nabla \cdot \mathbf{v}_\mathrm{m})$$

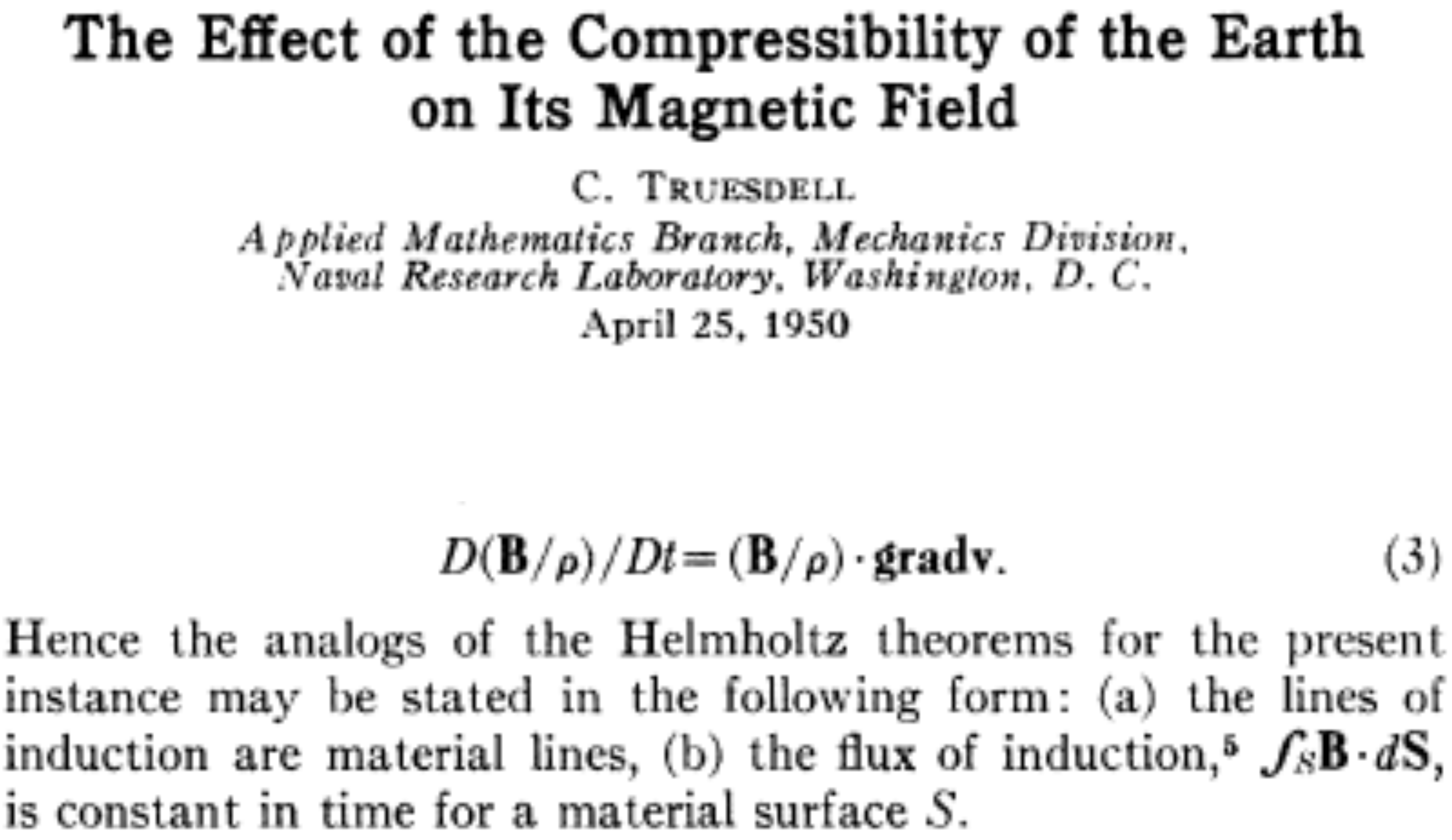
$$\nabla \cdot \mathbf{v}_\mathrm{m} = -\frac{1}{\rho_\mathrm{m}} \left(\frac{\partial \rho_\mathrm{m}}{\partial t} + (\mathbf{v}_\mathrm{m} \cdot \nabla)\rho_\mathrm{m} \right) = -\frac{1}{\rho_\mathrm{m}} \frac{\mathrm{d}\rho_\mathrm{m}}{\mathrm{d}t}$$

(10) $\nabla \times (\mathbf{A} \times \mathbf{B}) = \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A}) + (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B}$

$$\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}t} = (\mathbf{B} \cdot \nabla)\mathbf{v}_\mathrm{m} + \frac{\mathbf{B}}{\rho_\mathrm{m}} \frac{\mathrm{d}\rho_\mathrm{m}}{\mathrm{d}t}$$

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathbf{B}}{\rho_\mathrm{m}}\right) = \left(\frac{\mathbf{B}}{\rho_\mathrm{m}} \cdot \nabla\right)\mathbf{v}_\mathrm{m}}$$

“Frozen-in” Flux



The Effect of the Compressibility of the Earth on Its Magnetic Field

C. TRUESDELL

Applied Mathematics Branch, Mechanics Division,
Naval Research Laboratory, Washington, D. C.
April 25, 1950

$$D(\mathbf{B}/\rho)/Dt=(\mathbf{B}/\rho)\cdot\mathbf{grad}\,\mathbf{v}. \qquad (3)$$

Hence the analogs of the Helmholtz theorems for the present instance may be stated in the following form: (a) the lines of induction are material lines, (b) the flux of induction,[5] $\int_S \mathbf{B}\cdot d\mathbf{S}$, is constant in time for a material surface S.

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathbf{B}}{\rho_\mathrm{m}}\right)=\left(\frac{\mathbf{B}}{\rho_\mathrm{m}}\cdot\nabla\right)\mathbf{v}_\mathrm{m}$$

"Frozen-in" Flux

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v}_{\mathrm{m}} \times \mathbf{B})$$

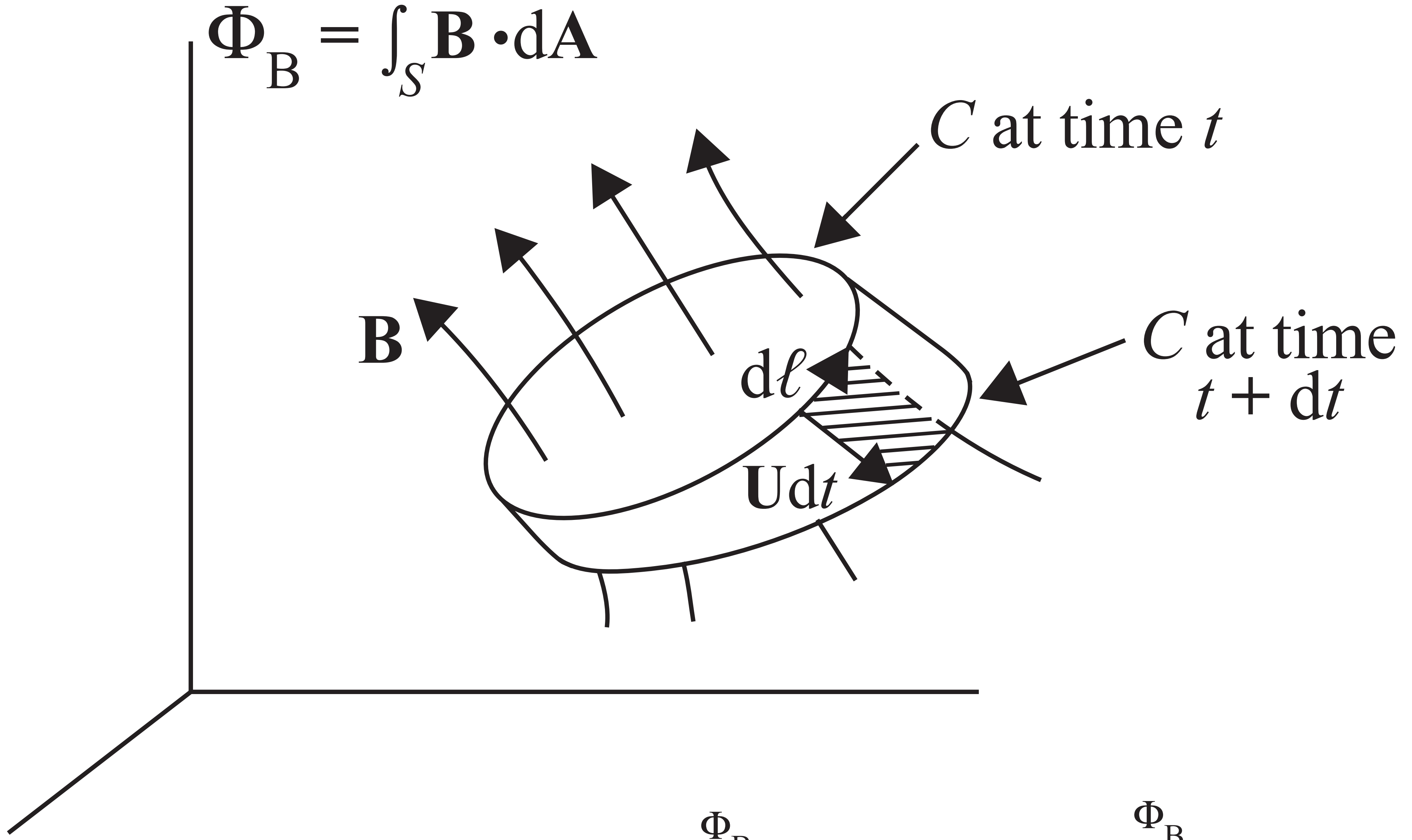
$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathbf{B}}{\rho_{\mathrm{m}}}\right) = \left(\frac{\mathbf{B}}{\rho_{\mathrm{m}}} \cdot \nabla\right)\mathbf{v}_{\mathrm{m}}}$$

$$(34) \quad \int_S d\mathbf{S} \cdot \nabla \times \mathbf{A} = \oint_C d\mathbf{l} \cdot \mathbf{A}$$

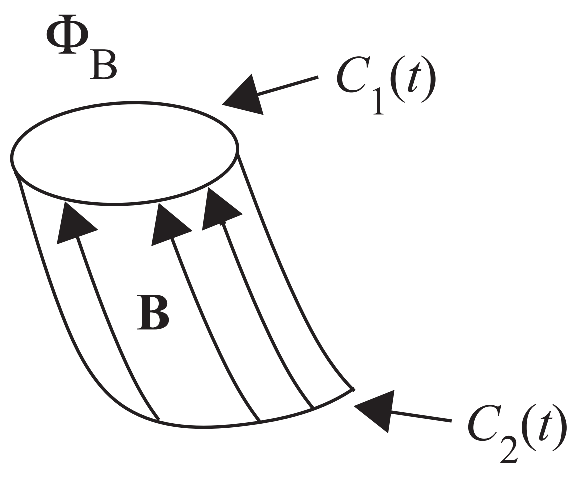
$$\frac{\mathrm{d}\Phi_{\mathrm{B}}}{\mathrm{d}t} = \int_S \frac{\partial \mathbf{B}}{\partial t} \cdot \mathrm{d}\mathbf{A} + \int_C \mathrm{d}\boldsymbol{\ell} \cdot (\mathbf{B} \times \mathbf{U}).$$

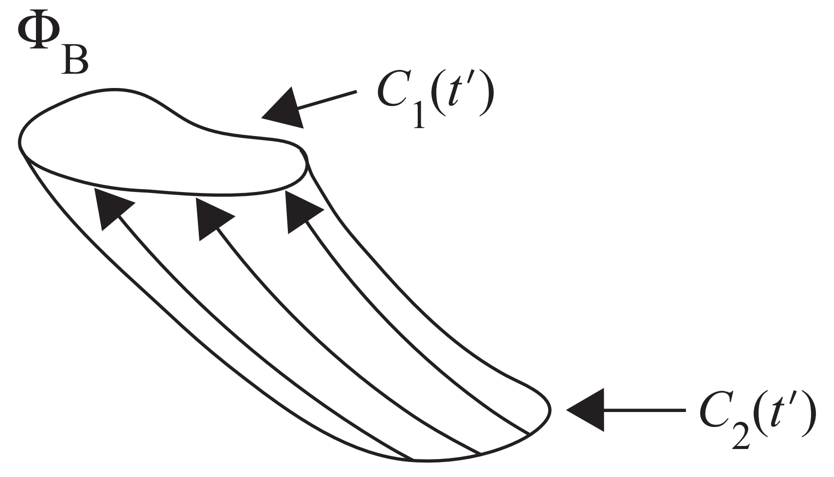
$$\frac{\mathrm{d}\Phi_{\mathrm{B}}}{\mathrm{d}t} = \int_S \frac{\partial \mathbf{B}}{\partial t} \cdot \mathrm{d}\mathbf{A} + \int_C \mathbf{B} \cdot (\mathbf{U} \times \mathrm{d}\boldsymbol{\ell}).$$

$$\frac{\mathrm{d}\Phi_{\mathrm{B}}}{\mathrm{d}t} = \int_S \left[\frac{\partial \mathbf{B}}{\partial t} - \boldsymbol{\nabla} \times (\mathbf{U} \times \mathbf{B})\right] \cdot \mathrm{d}\mathbf{A} = 0,$$



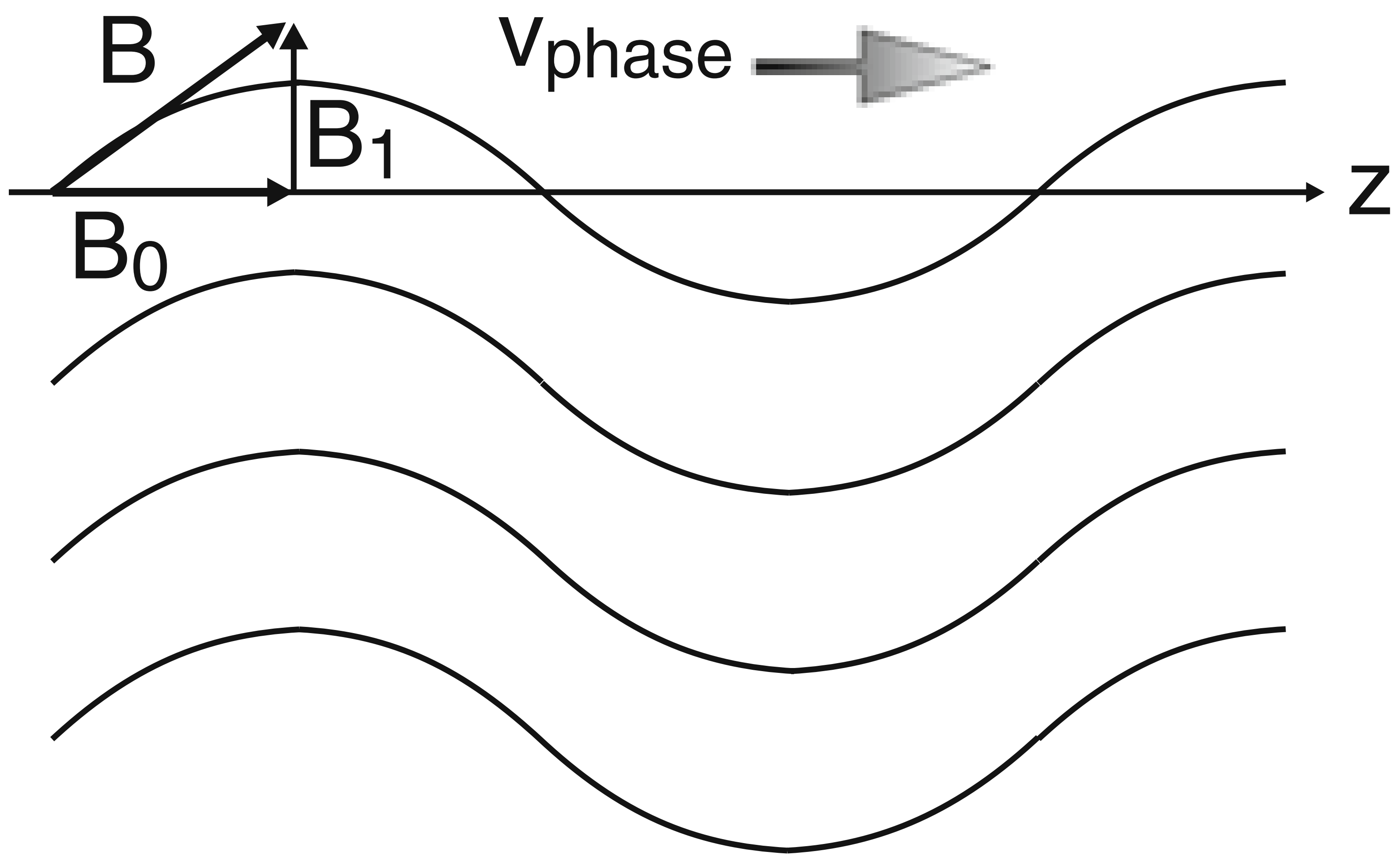
$$\Phi_{\mathrm{B}} = \int_S \mathbf{B} \cdot \mathrm{d}\mathbf{A}$$



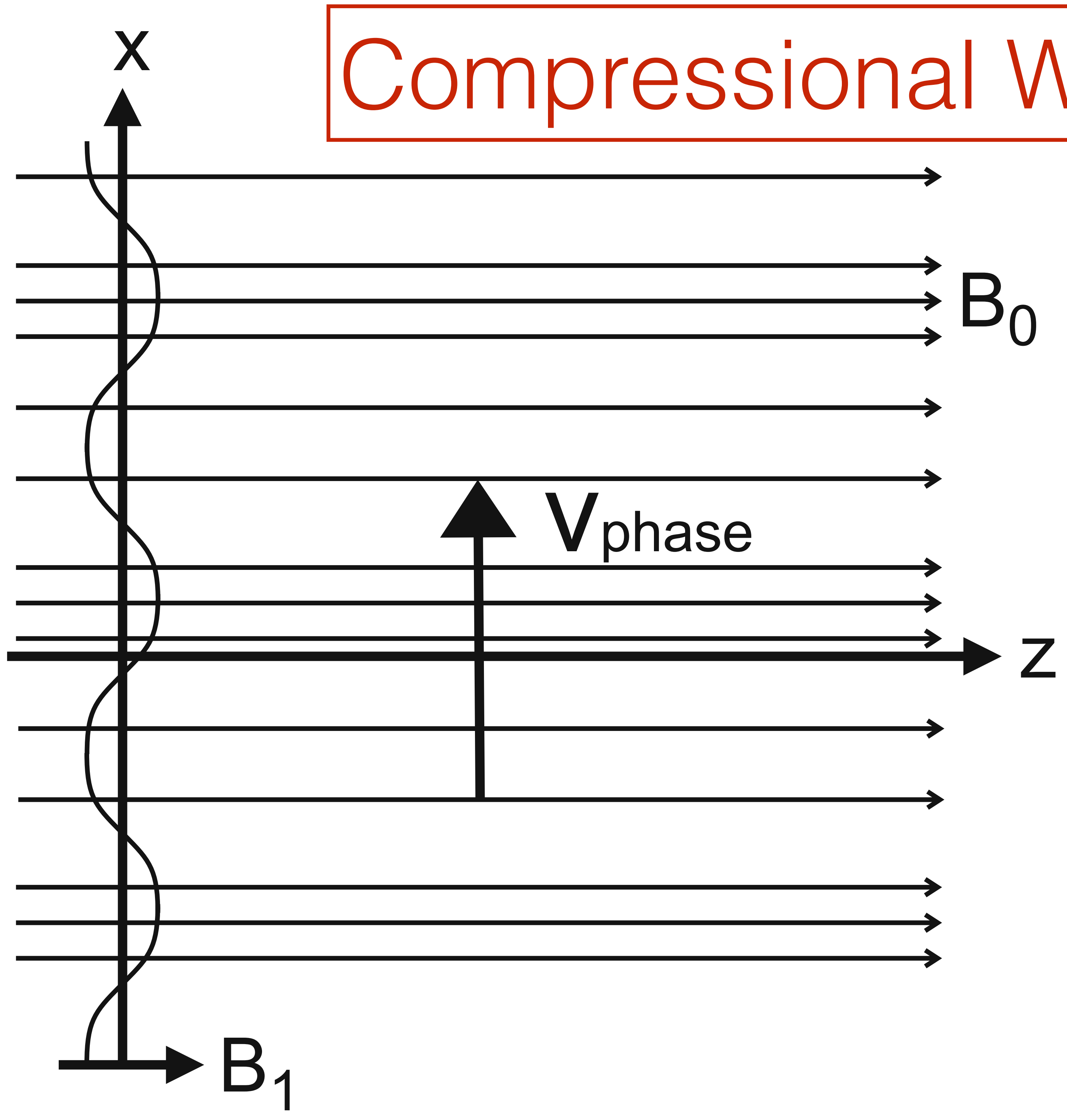


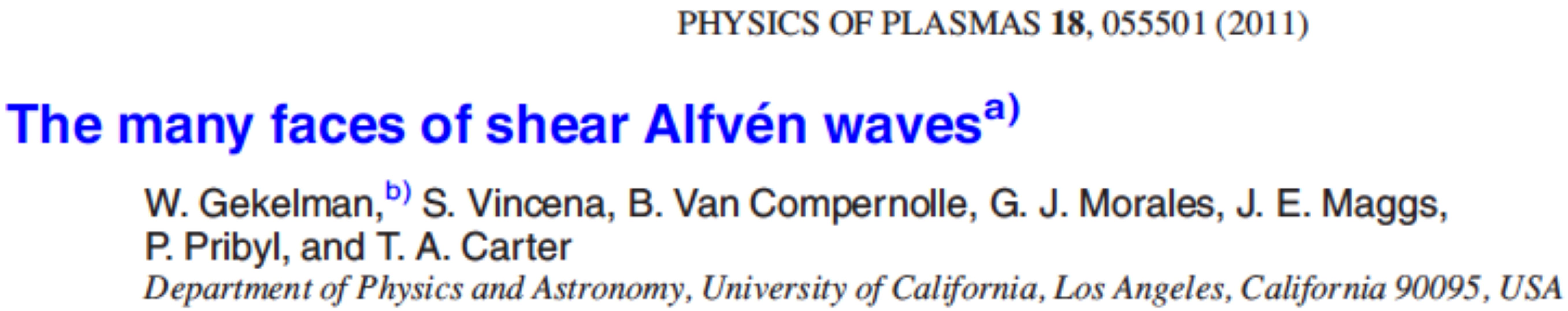
Alfvén Waves: Two Types

Shear Wave



Compressional Wave





PHYSICS OF PLASMAS **18**, 055501 (2011)

The many faces of shear Alfvén waves[a)]

W. Gekelman,[b)] S. Vincena, B. Van Compernolle, G. J. Morales, J. E. Maggs, P. Pribyl, and T. A. Carter
Department of Physics and Astronomy, University of California, Los Angeles, California 90095, USA

https://doi.org/10.1063/1.3592210

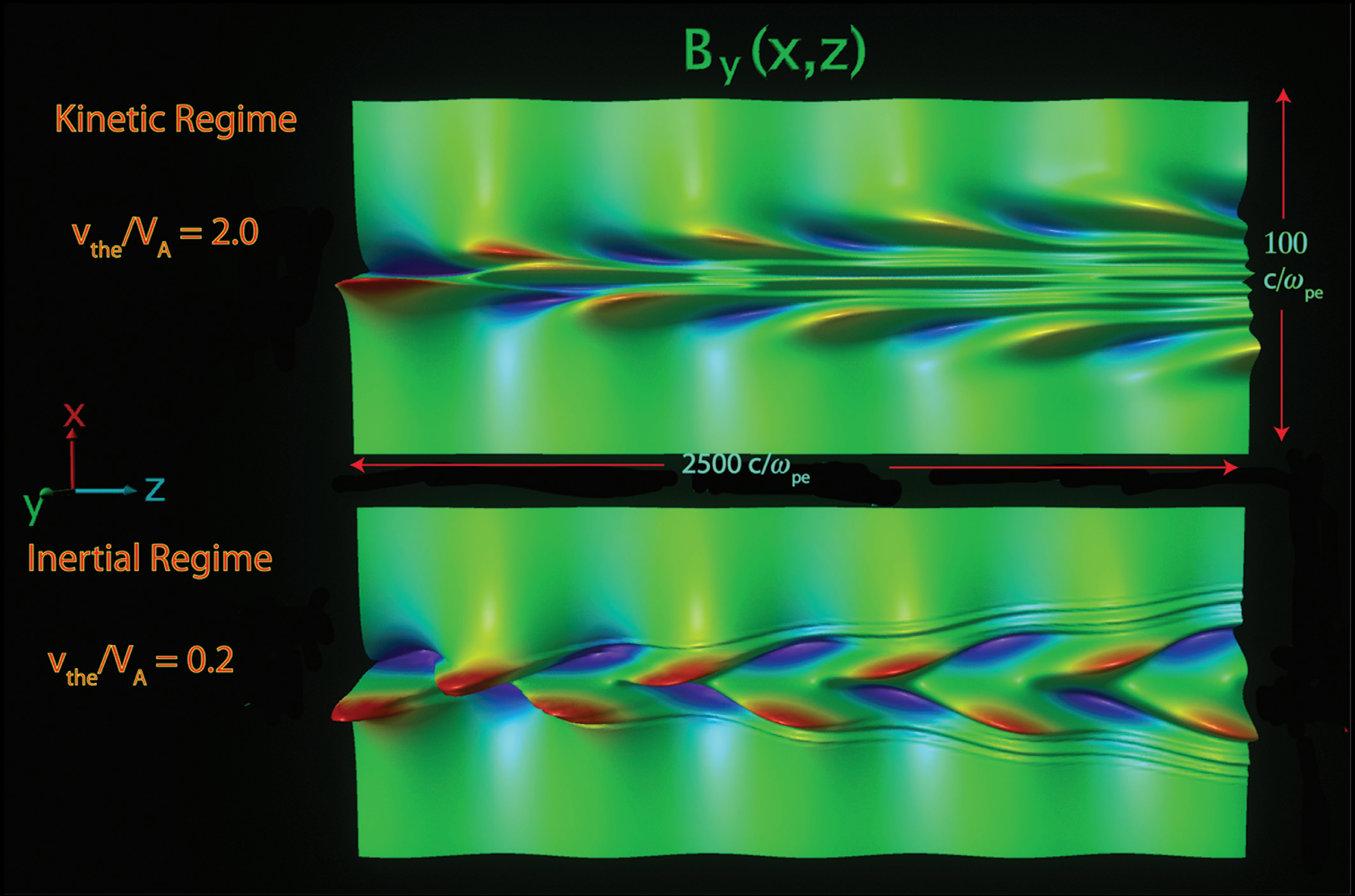


FIG. 1. (Color) Theoretical patterns of one component, B of the Alfvén wave in the kinetic and inertial regimes. The waves propagate from left to right

JOURNAL OF GEOPHYSICAL RESEARCH, SPACE PHYSICS VOL. 74, NO. 3, FEBRUARY 1, 1969

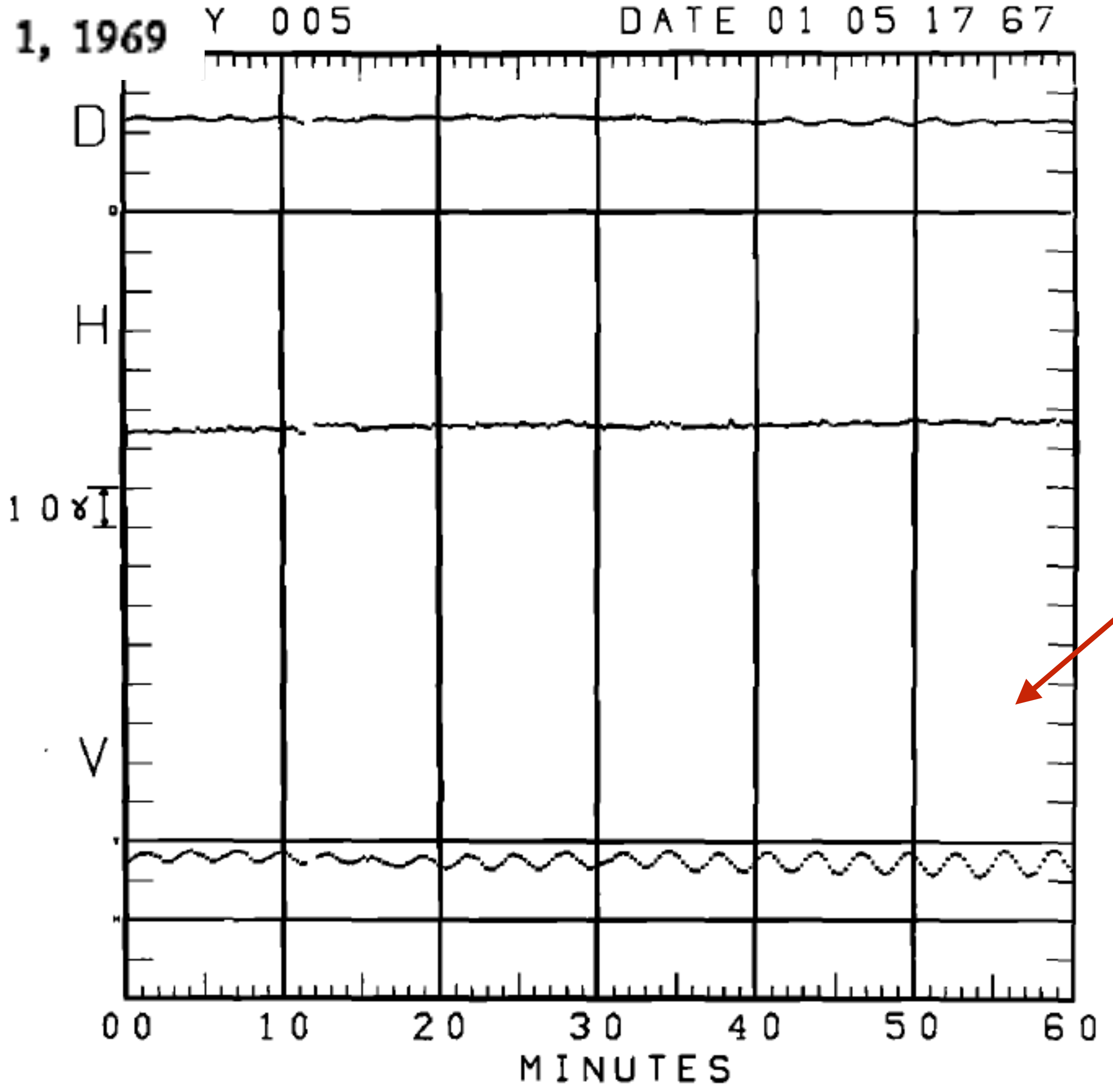


Fig. 2. An example of the transverse oscillations observed at ATS 1. The magnetogram covers one hour; each point is a 15-sec average of the data. The starting time (UT) for the

hour, year). The three horizontal

!!

Standing Alfvén Waves in the Magnetosphere

W. D. CUMMINGS AND R. J. O'SULLIVAN

Department of Planetary and Space Science
University of California, Los Angeles, California 90024

P. J. COLEMAN, JR.

Department of Planetary and Space Science and
Institute of Geophysics and Planetary Physics
University of California, Los Angeles, California 90024

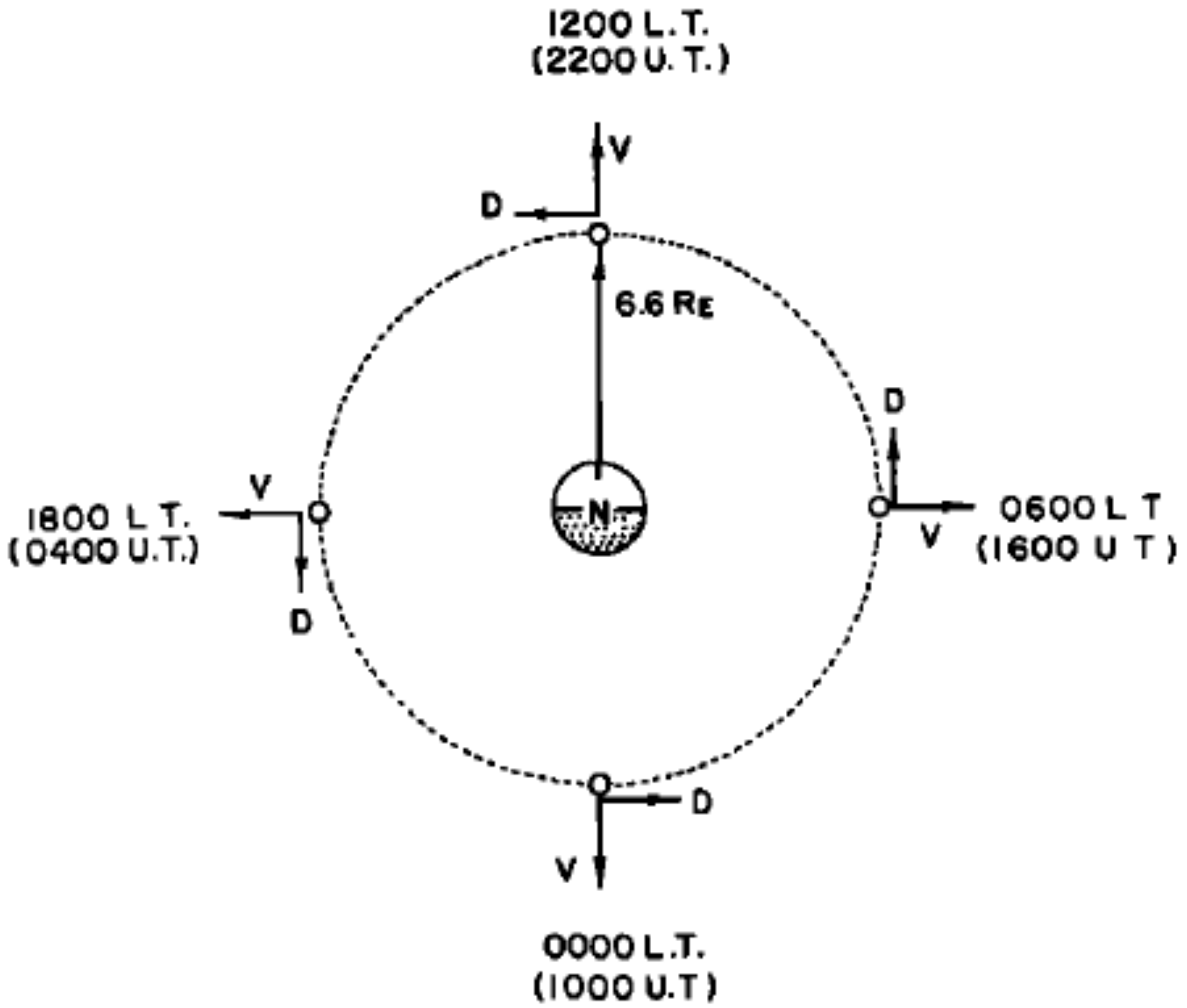


Fig. 1. The *DHV* coordinate system has its origin at the satellite and corotates with it at 150°W longitude in the geographic equatorial plane. *D* is positive eastward, *V* is positive outward, and *H* is positive northward, i.e., perpendicular to the equatorial plane.

Despite the above reservations, which have not been examined in detail, there is reasonable agreement between theory and experiment if we interpret the oscillations observed at ATS 1 as the second harmonic of an MHD standing wave resonance. According to this interpretation, the oscillations with $T \simeq T_1$ represent the standing wave when the plasmapause is beyond the geosynchronous orbit. In agreement with previous observations: (1) the number density in the plasmasphere is $\simeq 100/\mathrm{cm}^3$; (2) the distribution of plasma along the field line is roughly independent of geocentric distance; and (3) the equatorial geocentric distance to the plasmapause is greater than 6.6 R_E only for very quiet geomagnetic conditions. The oscillations with

Low-Frequency MHD Dynamics

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j}$$

$$\rho_\mathrm{m} \frac{\partial \mathbf{v}_\mathrm{m}}{\partial t} = \mathbf{j} \times \mathbf{B}$$

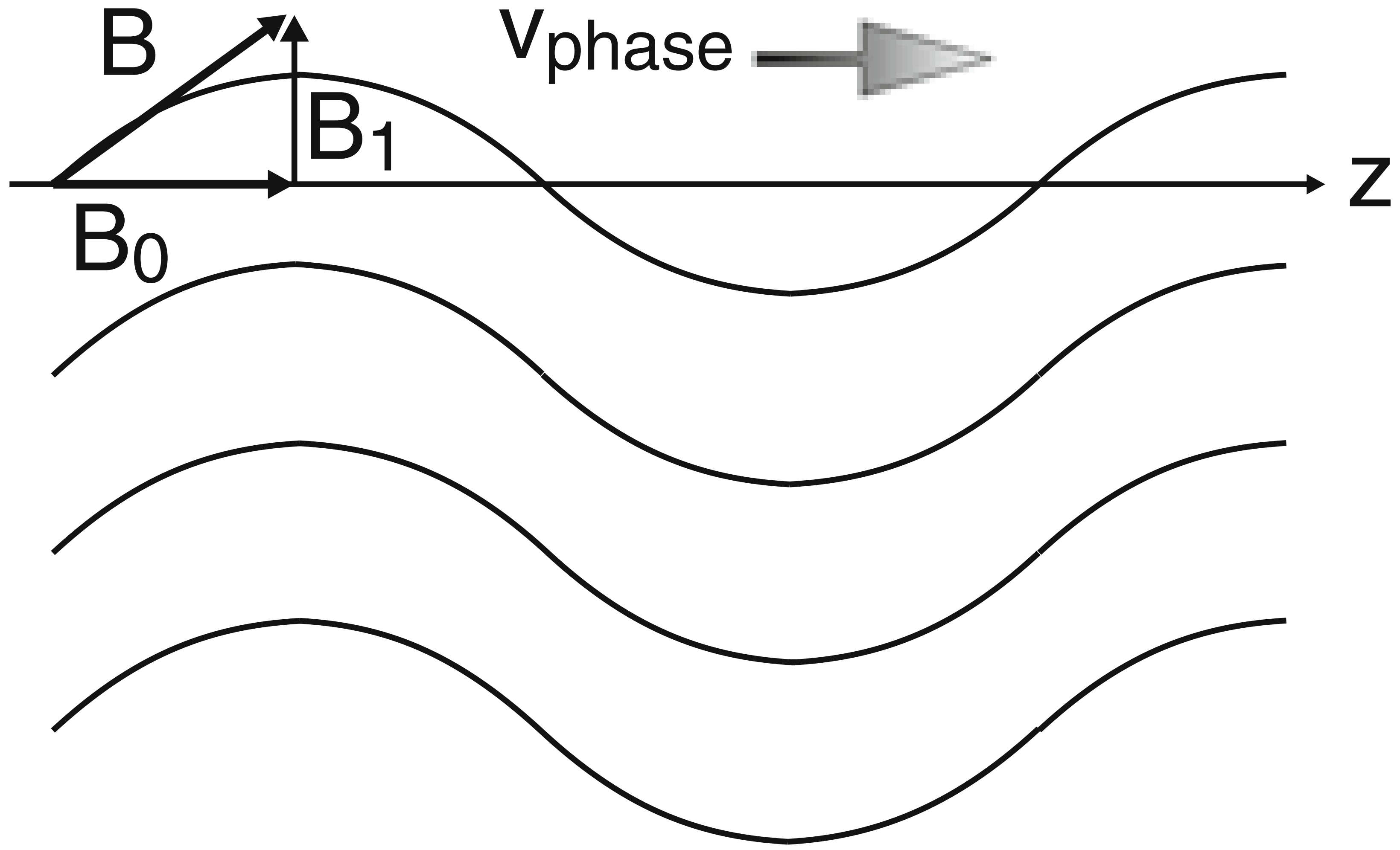
$$\frac{\partial \mathbf{B}}{\partial t} = (\mathbf{B} \cdot \nabla)\mathbf{v}_\mathrm{m} - (\mathbf{v}_\mathrm{m} \cdot \nabla)\mathbf{B} - \mathbf{B}(\nabla \cdot \mathbf{v}_\mathrm{m})$$

For shear waves: $\nabla \cdot \mathrm{v_m} = 0$

Linearize: Waves

$$\mathbf{B} = \mathbf{B}_0 + \mathbf{B}_1$$

$$\mathbf{v}_\mathrm{m} = \mathbf{v}_0 + \mathbf{v}_1 \,.$$



$$\rho_\mathrm{m} \frac{\partial \mathbf{v}_1}{\partial t} = \frac{1}{\mu_0} (\nabla \times \mathbf{B}_1) \times \mathbf{B}_0$$

$$\frac{\partial \mathbf{B}_1}{\partial t} = (\mathbf{B}_0 \cdot \nabla) \mathbf{v}_1 \,.$$

For shear waves, when $\nabla \cdot v_m = 0$

Linear Shear Waves

$$\rho \frac{\partial \tilde{\bar{V}}}{\partial t} = \frac{1}{\mu_0}\left(\nabla \times \tilde{\bar{B}}\right) \times \bar{B}_0$$

$$\frac{\partial \tilde{B}_1}{\partial t} = (B_0 \cdot \nabla)\, \tilde{\bar{V}}$$

“Phasors” (!!)

$$\frac{\partial}{\partial t} \longrightarrow -i\omega \qquad e^{-j\omega t} \sim (\tilde{v}, \tilde{B})$$

$$\nabla \to i\bar{k} \qquad e^{+j\bar{k}\cdot\bar{x}} \sim (\tilde{v}, \tilde{B})$$

$$-\rho\,\omega\,\bar{V} = \frac{1}{\mu_0}\left(\bar{k} \times \tilde{B}\right) \times \bar{B}_0$$

$$-\omega\,\tilde{B} = \left(\bar{B}_0 \cdot \bar{k}\right) \tilde{V}$$

Linear Shear Plane Waves

$$(\bar{k} \times \tilde{B}) \times \bar{B}_0 = \tilde{B}(\bar{k} \cdot \bar{B}_0) - \bar{k}(\bar{B}_0 \cdot \tilde{B})$$

$$\rho\,\omega^2 \tilde{B} = (\bar{B}_0 \cdot \bar{k}) \frac{1}{\mu_0} \left[\tilde{B}(k \cdot B_0) - \bar{k}(\bar{B}_0 \cdot \tilde{B}) \right]$$

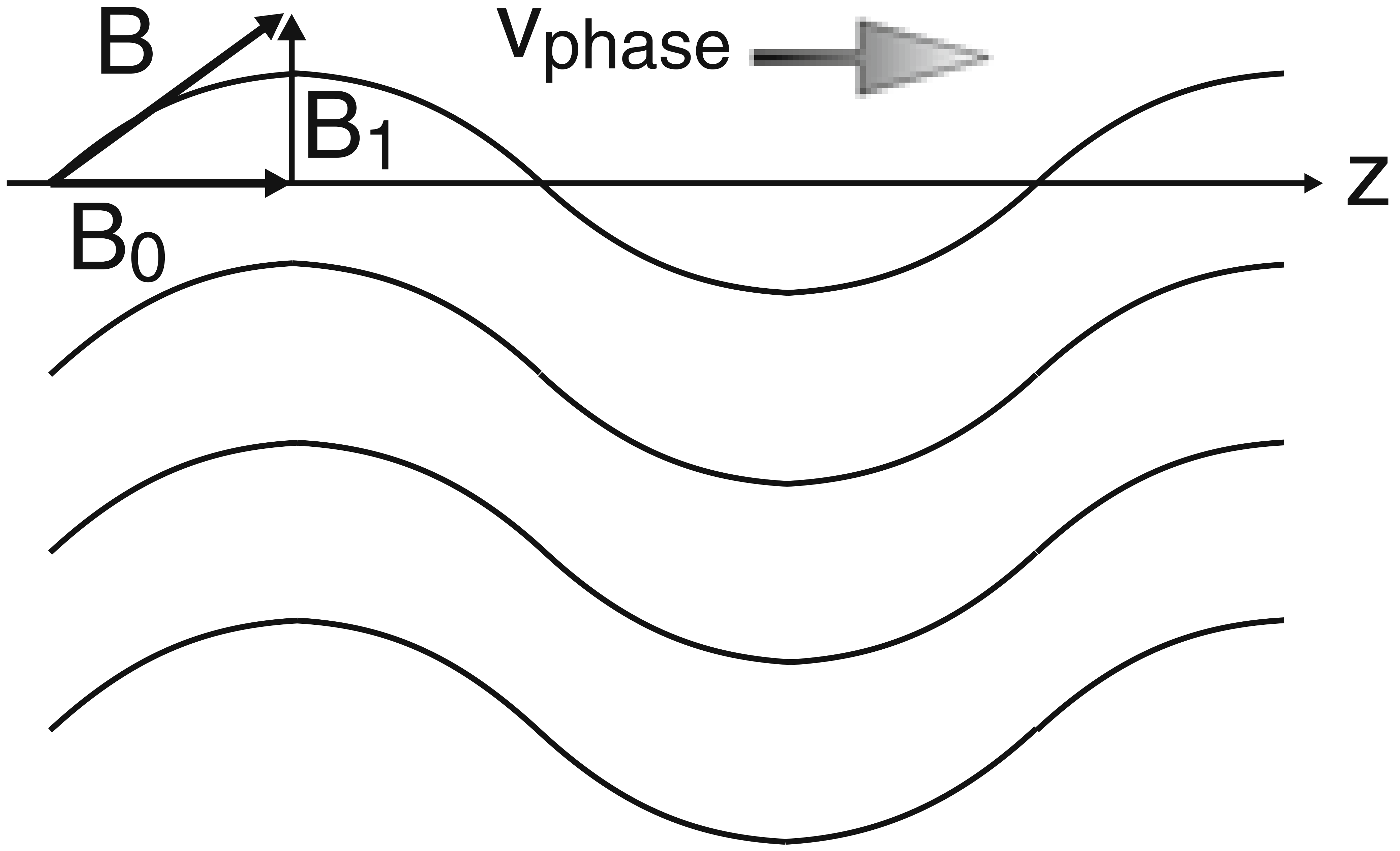
$$V_A^2 = \frac{B_0^2}{\rho \mu_0}$$

$$\boxed{\frac{\omega^2}{V_A^2} \tilde{B} = k_\parallel^2 \tilde{B} - \bar{k}\, k_\parallel (\hat{b} \cdot \tilde{B})}$$

Shear Alfvén Waves

$$\boxed{\tilde{\bar{V}} = -\left(\frac{\omega}{k_\parallel}\right)\frac{\tilde{\bar{B}}}{B_0}}$$

$$V_A^2 = \frac{B_0^2}{\rho\mu_0}$$

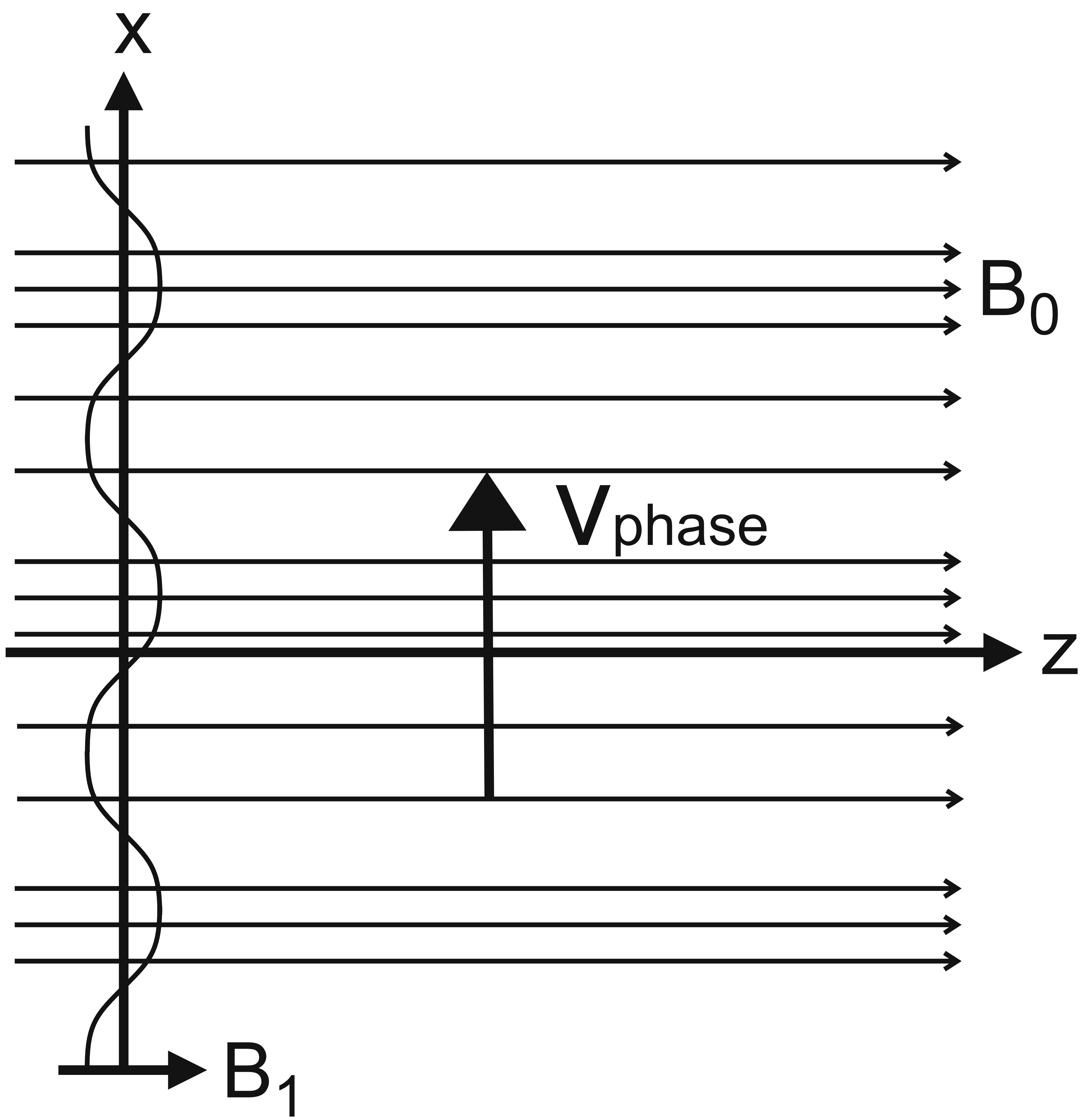


Compressional Alfvén Waves

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j}$$

$$\rho_{\mathrm{m}} \frac{\partial \mathbf{v}_{\mathrm{m}}}{\partial t} = \mathbf{j} \times \mathbf{B}$$

$$\frac{\partial \mathbf{B}}{\partial t} = \underbrace{(\mathbf{B} \cdot \nabla)\mathbf{v}_{\mathrm{m}} - (\mathbf{v}_{\mathrm{m}} \cdot \nabla)\mathbf{B}}_{= 0} - \underbrace{\mathbf{B}(\nabla \cdot \mathbf{v}_{\mathrm{m}})}_{\neq 0}$$



(what about plasma pressure?)

Next Lecture

- Chapter 6: “Plasma Waves”
- Quiz #1 Review
- “More Practice Problems”

More Practice Problems

AP 6101
Practice for Quiz #1
From Fitzpatrick, and Gurnett and Bhattacharjee

Dielectric of a Magnetized Plasma

2. A quasi-neutral slab of cold (i.e., $\lambda_D \to 0$) plasma whose bounding surfaces are normal to the x-axis consists of electrons of mass m_e, charge $-e$, and mean number density n_e, as well as ions of mass m_i, charge e, and mean number density n_e. The slab is fully magnetized by a uniform y-directed magnetic field of magnitude B. The slab is then subject to an externally generated, uniform, x-directed electric field that is gradually ramped up to a final magnitude E_0. Show that, as a consequence of ion polarization drift, the final magnitude of the electric field inside the plasma is

$$E_1 \simeq \frac{E_0}{\epsilon},$$

where

$$\epsilon = 1 + \frac{c^2}{V_A^2},$$

and $V_A = B/\sqrt{\mu_0\, n_e\, m_i}$ is the so-called Alfvén velocity.

Thermal Equilibrium

5. A uniform isothermal quasi-neutral plasma with singly-charged ions is placed in a relatively weak gravitational field of acceleration $\mathbf{g} = -g\,\mathbf{e}_z$. Assuming, first, that both species are distributed according to the Maxwell-Boltzmann statistics; second, that the perturbed electrostatic potential is a function of z only; and, third, that the electric field is zero at $z = 0$ (and well behaved as $z \to \infty$), demonstrate that the electric field in the region $z > 0$ takes the form $\mathbf{E} = E_z\,\mathbf{e}_z$, where

$$E_z(z) = E_0\left[1 - \exp\left(\frac{\sqrt{2}\,z}{\lambda_D}\right)\right],$$

and

$$E_0 = \frac{m_i\,g}{2\,e}.$$

Here, λ_D is the Debye length, e the magnitude of the electron charge, and m_i the ion mass.

Adiabatic Invariants

6. A particle of charge e, mass m, and energy $\mathcal{E}$, is trapped in a one-dimensional magnetic well of the form

$$B(x,t) = B_0\,(1 + k^2\,x^2),$$

where B_0 is constant, and $k(t)$ is a very slowly increasing function of time. Suppose that the particle's mirror points lie at $x = \pm x_m(t)$, and that its bounce time is $\tau_b(t)$. Demonstrate that, as a consequence of the conservation of the first and second adiabatic invariants,

$$x_m(t) = x_m(0)\left[\frac{k(0)}{k(t)}\right]^{1/2},$$

$$\tau_b(t) = \tau_b(0)\left[\frac{k(0)}{k(t)}\right],$$

$$\mathcal{E}(t) = \mathcal{E}_{0\,\perp} + \left[\frac{k(t)}{k(0)}\right]\mathcal{E}_{0\,\parallel}.$$

Here, $\mathcal{E}_{0\,\perp}$ is the perpendicular energy [i.e., $(1/2)\,m\,v_\perp^2$], and $\mathcal{E}_{0\,\parallel}$ is the parallel energy [i.e., $(1/2)\,m\,v_\parallel^2$], both evaluated at $x = 0$ and $t = 0$. Assume that the particle's gyroradius is relatively small, and that the electric field-strength is negligible.

Drift Velocity w Collisions

8. Consider a spatially uniform, unmagnetized plasma in which both species have zero mean flow velocity. Let n_e and T_e be the electron number density and temperature, respectively. Let $\mathbf{E}$ be the ambient electric field. The electron distribution function f_e satisfies the simplified kinetic equation

$$-\frac{e}{m_e}\,\mathbf{E}\cdot\nabla_v f_e = C_e.$$

We can crudely approximate the electron collision operator as

$$C_e = -\nu_e\,(f_e - f_0)$$

where ν_e is the effective electron-ion collision frequency, and

$$f_0 = \frac{n_e}{\pi^{3/2}\,v_{t\,e}^3}\,\exp\!\left(-\frac{v^2}{v_{t\,e}^2}\right).$$

Here, $v_{t\,e} = \sqrt{2\,T_e/m_e}$. Suppose that $E \ll m_e\,\nu_e\,v_{t\,e}/e$. Demonstrate that it is a good approximation to write

$$f_e = f_0 + \frac{e}{m_e\,\nu_e}\,\mathbf{E}\cdot\nabla_v f_0.$$

Hence, show that

$$\mathbf{j} = \sigma\,\mathbf{E},$$

where

$$\sigma = \frac{e^2\,n_e}{m_e\,\nu_e}.$$

Static MHD Equilibrium

7.6. For a force-balanced MHD equilibrium in a cylindrical geometry with $\mathbf{B} = [0, B_\phi(\rho), B_z(\rho)]$ the radial component of the pressure balance condition $\mathbf{J} \times \mathbf{B} = \boldsymbol{\nabla} P$ can be written

$$\frac{\partial}{\partial \rho}\left(P + \frac{B_\phi^2}{2\mu_0} + \frac{B_z^2}{2\mu_0}\right) = [(\mathbf{B} \cdot \boldsymbol{\nabla})\mathbf{B}]_\rho.$$

Show that $[(\mathbf{B} \cdot \boldsymbol{\nabla})\mathbf{B}]_\rho = -B_\phi^2/\rho$.
Hint: Use the identity $\boldsymbol{\nabla}(\mathbf{F} \cdot \mathbf{G}) = (\mathbf{F} \cdot \boldsymbol{\nabla})\mathbf{G} + (\mathbf{G} \cdot \boldsymbol{\nabla})\mathbf{F} + \mathbf{F} \times (\boldsymbol{\nabla} \times \mathbf{G}) + \mathbf{G} \times (\boldsymbol{\nabla} \times \mathbf{G})$.

Alfvén Waves with Collisions/Viscosity

1. We can add viscous effects to the MHD momentum equation by including a term $\mu\,\nabla^2\mathbf{V}$, where μ is the dynamic viscosity, so that

   $$\rho\,\frac{d\mathbf{V}}{dt} = \mathbf{j}\times\mathbf{b} - \nabla p + \mu\,\nabla^2\mathbf{V}.$$

   Likewise, we can add finite conductivity effects to the Ohm's law by including the term $(1/\mu_0\,\sigma)\,\nabla^2\mathbf{B}$, to give

   $$\frac{\partial\mathbf{B}}{\partial t} = \nabla\times(\mathbf{V}\times\mathbf{B}) + \frac{1}{\mu_0\,\sigma}\,\nabla^2\mathbf{B},$$

   Show that the modified dispersion relation for Alfvén waves can be obtained from the standard one by multiplying both ω^2 and V_S^2 by a factor

   $$[1 + \mathrm{i}\,k^2/(\mu_0\,\sigma\,\omega)],$$

   and ω^2 by an additional factor

   $$[1 + \mathrm{i}\,\mu\,k^2/(\rho_0\,\omega)].$$

   If the finite conductivity and viscous corrections are small (i.e., $\sigma\to\infty$ and $\mu\to 0$), show that, for parallel ($\theta = 0$) propagation, the dispersion relation for the shear-Alfvén wave reduces to

   $$k \simeq \frac{\omega}{V_A} + \mathrm{i}\,\frac{\omega^2}{2\,V_A^3}\left(\frac{1}{\mu_0\,\sigma} + \frac{\mu}{\rho_0}\right).$$