Lecture 8:
Plasma Physics 1

APPH E6101x
Columbia University

Last Lecture (online)

- Moments of the distribution function
- Fluid equations (“two fluid”)
- The “closure problem”
- MHD equations (“single fluid”)

Outline

- Homework #4: Modeling Collisions and the Rosenbluth Potentials
- Force balance (*equilibrium*) in a magnetized plasma
  - Z-pinch
  - θ-pinch
  - Screw-pinch (straight tokamak)
  - Grad-Shafranov Equation
- Conservation principles in magnetized plasma ("frozen-in" and conservation of particles/flux tubes)

Modeling Collisions

$$\frac{\partial f_s}{\partial t} + \mathbf{v} \cdot \frac{\partial f_s}{\partial \mathbf{r}} + \frac{e_s}{m_s}(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \frac{\partial f_s}{\partial \mathbf{v}} = C_s(f),$$

- “Weakly ionized plasma” collisions with neutrals
- Fully ionized plasma: Coulomb collisions

“Simple” Fixed-σ Model for Weakly Ionized Plasma

$$C_s(f) \approx -\frac{1}{\tau_{col}}(f - f_0)$$

PERTURBED DISTRIBUTION

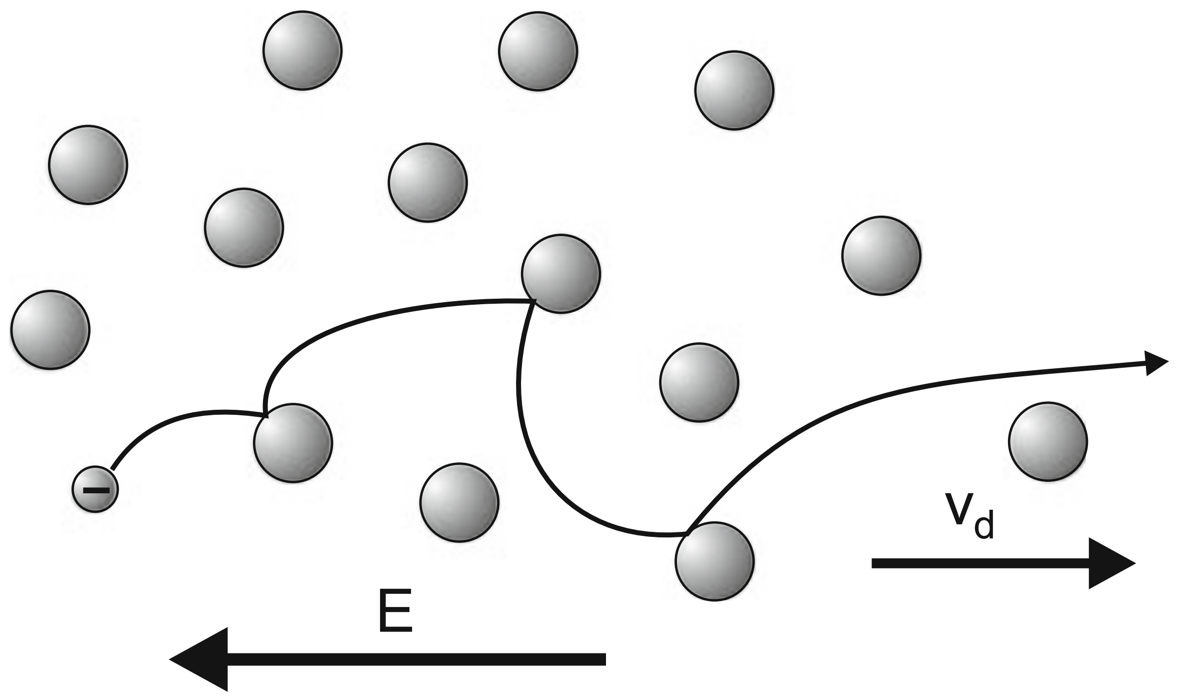
MAXWELLIAN “EQUILIBRIUM”

$$\tau_{col} \sim \frac{1}{\sigma V_{TH}\, n_0}$$

NEUTRAL DENSITY

THERMAL VELOCITY

CROSS SECTION $\approx 5 \times 10^{-15}\ cm^2$



$$\iiint d^3v\, C_s(f) = 0$$

$$\iiint d^3v\, \bar{v}\, C_s(f) = -\bar{V} n_0 / \tau_{col} \qquad (\text{IF } \langle f_0 \bar{v} \rangle = 0)$$

COLLISIONAL DRAG

THEN

$$\boxed{n \frac{\partial \bar{V}}{\partial t} = -\frac{e}{m} n \bar{E} - \frac{e}{m} n \bar{V} \times \bar{B} - \bar{V} n_0 / \tau_{col}}$$

FOR e^-, e^+

Modeling Collisions in Fully Ionized Plasma

Rutherford scattering cross-section,

$$\frac{d\sigma}{d\Omega} = \frac{1}{4}\left(\frac{e_s\, e_{s'}}{4\pi\,\epsilon_0\,\mu_{ss'}\,u_{ss'}^2}\right)^2 \frac{1}{\sin^4(\chi/2)} \tag{3.70}$$

(Rutherford 1911). It is immediately apparent, from the previous formula, that two-particle Coulomb collisions are dominated by small-angle (i.e., small χ) scattering events.

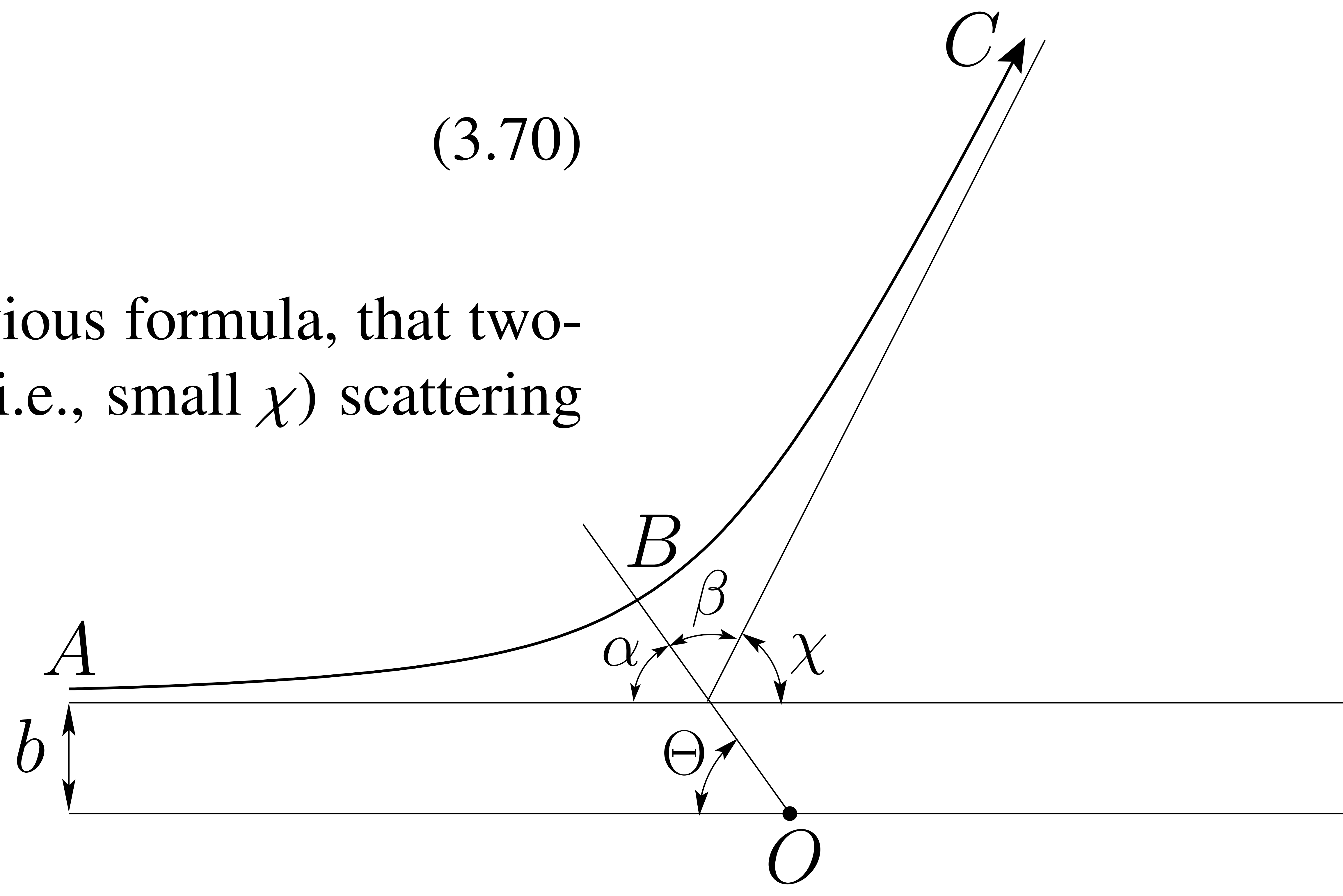


Figure 3.1 A two-body Coulomb collision.

Modeling Collisions in Fully Ionized Plasma

CURRENTS DRIVEN BY ELECTRON CYCLOTRON WAVES

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ABSTRACT. Certain aspects of the generation of steady-state currents by electron cyclotron waves are explored. A numerical solution of the Fokker-Planck equation is used to verify the theory of Fisch and Boozer and to extend their results into the non-linear regime. Relativistic effects on the current generated are discussed. Applications to steady-state tokamak reactors are considered.

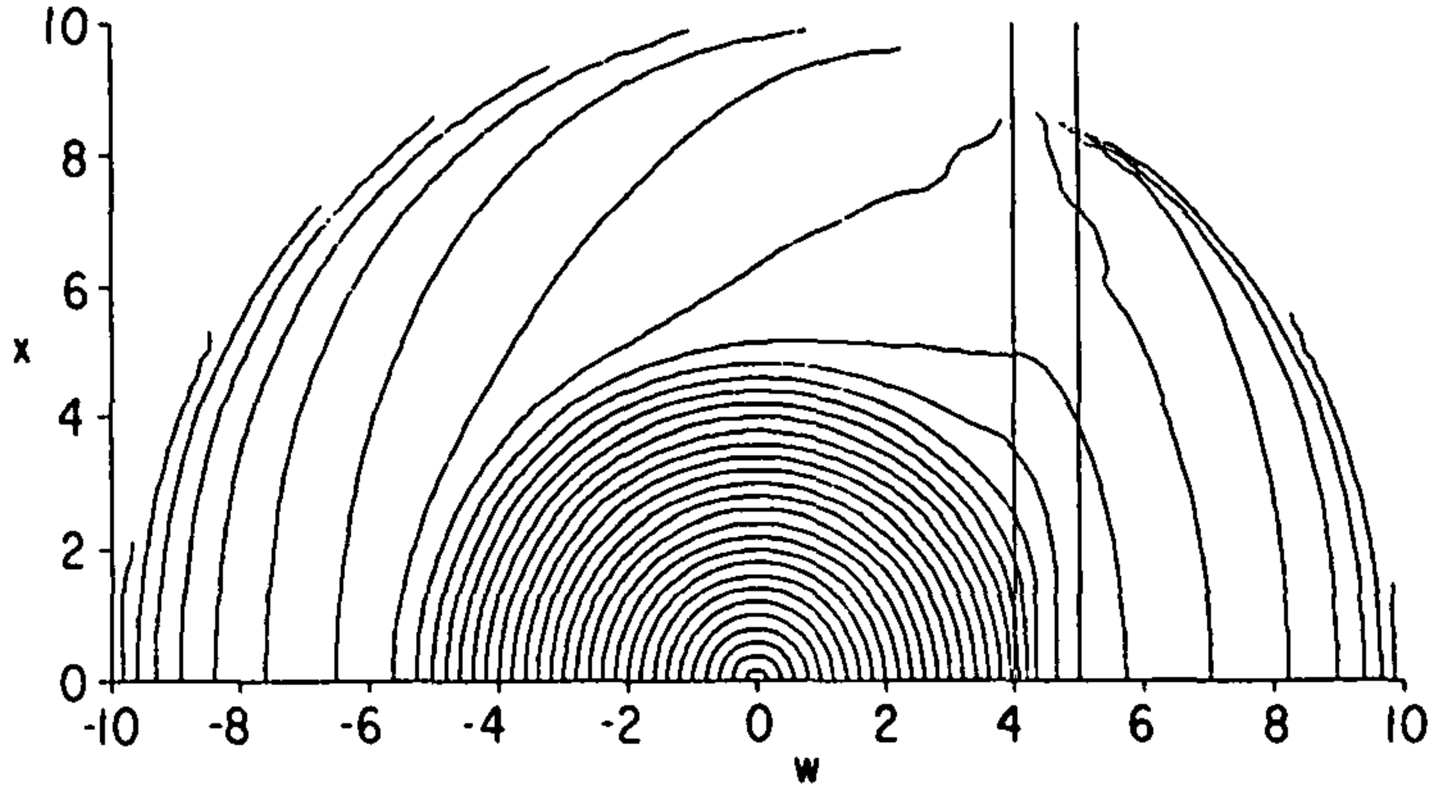

FIG.5. Steady-state distribution for D = 0.25, $w_1 = 4$, *and* $w_2 = 5$ *(cyclotron damping).*

https://doi.org/10.1088/0029-5515/21/12/004

Electron-cyclotron heating in a pulsed mirror experiment

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(Received 5 December 1983; accepted 6 August 1984)

Experimental measurements of electron-cyclotron resonance heating (ECRH) of a highly ionized plasma in mirror geometry is compared to a two-dimensional, time-dependent, Fokker–Planck simulation. Measurements of the absorption strength of the electrons and of the energy confinement of the ions helped to specify the parameters of the code. The electron energy distribution is measured with an end-loss analyzer and a target x-ray detector. These characterize a non-Maxwellian distribution consisting of "passing" (10 eV $< T_{e,p} <$ 30 eV), "warm" (50 eV $< T_{e,w} <$ 300 eV), and "hot" (1.2 keV $< T_{e,h} <$ 4.0 keV) electron populations. The temperature and fractional densities of the warm and hot populations depend on the absorbed power and total density. A similar distribution is calculated with the simulation program that reproduces the end-loss and x-ray signals. Both the experimental measurements and the simulation are described.

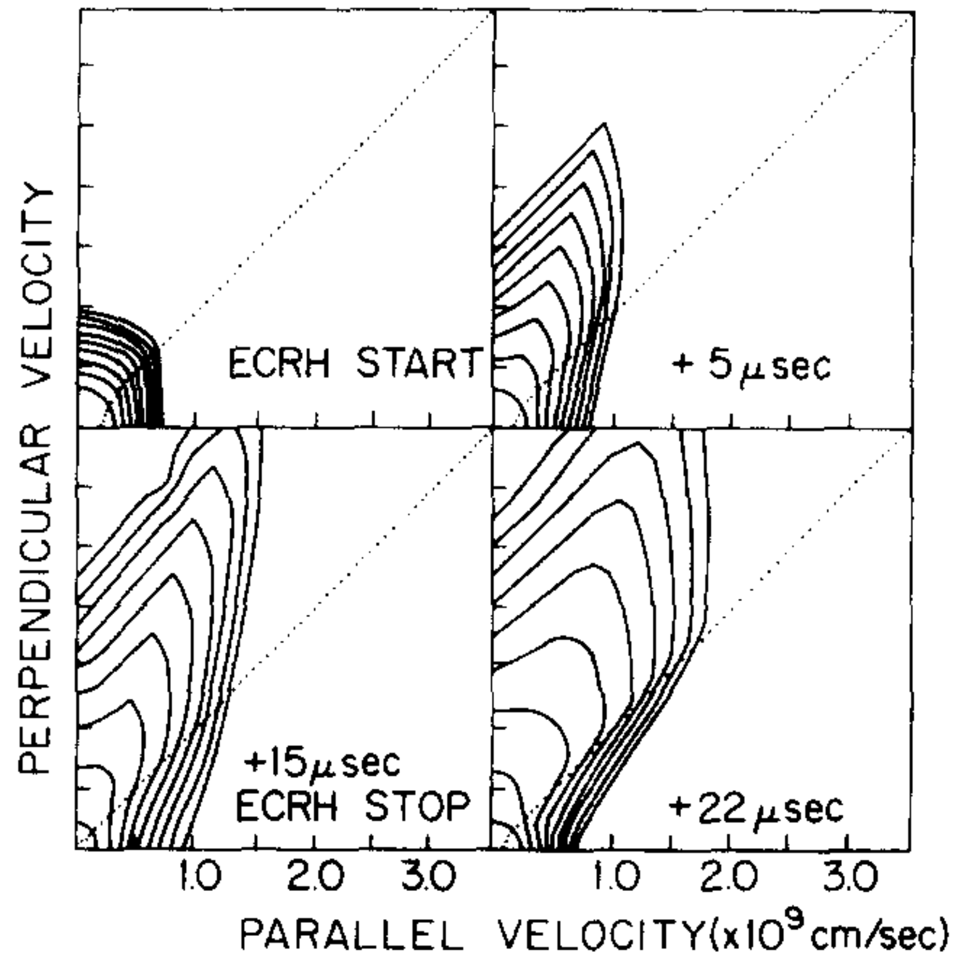

FIG. 9. An example of the development of the electron velocity distribution during and after ECRH. Shown are four times: (a) at the start of the ECRH, (b) after 5.0 μsec, (c) at the end of the 15 μsec rf pulse, and (d) 5.0 μsec after the ECRH was turned off.

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Modeling Collisions in Fully Ionized Plasma

$$C_{ss'} = \iiint u_{ss'} \frac{d\sigma}{d\Omega} \left(f'_s f'_{s'} - f_s f_{s'}\right) d^3\mathbf{v}_{s'}\, d\Omega,$$

$$u_{ss'} = |\mathbf{v}_s - \mathbf{v}_{s'}|$$

$$\mathbf{v}'_s = \mathbf{v}_s + \frac{\mu_{ss'}}{m_s} \mathbf{g}_{ss'}, \qquad \mathbf{g}_{ss'} = \mathbf{u}'_{ss'} - \mathbf{u}_{ss'}$$

$$f_s(\mathbf{v}'_s) \simeq f_s(\mathbf{v}_s) + \frac{\mu_{ss'}}{m_s} \mathbf{g}_{ss'} \cdot \frac{\partial f_s(\mathbf{v}_s)}{\partial \mathbf{v}_s} + \frac{1}{2} \frac{\mu^2_{ss'}}{m^2_s} \mathbf{g}_{ss'} \mathbf{g}_{ss'} : \frac{\partial^2 f_s(\mathbf{v}_s)}{\partial \mathbf{v}_s \partial \mathbf{v}_s}.$$

Modeling Collisions in Fully Ionized Plasma

$$C_{ss'} = \iiint u_{ss'} \frac{d\sigma}{d\Omega} (f'_s f'_{s'} - f_s f_{s'}) \, d^3\mathbf{v}_{s'} \, d\Omega,$$

$$\mathbf{g}_{ss'} = \mathbf{u}'_{ss'} - \mathbf{u}_{ss'}$$

$$f'_s f'_{s'} - f_s f_{s'} \simeq \mu_{ss'} \mathbf{g}_{ss'} \cdot \left(\frac{\partial f_s}{\partial \mathbf{v}_s} \frac{f_{s'}}{m_s} - \frac{f_s}{m_{s'}} \frac{\partial f_{s'}}{\partial \mathbf{v}_{s'}} \right)$$

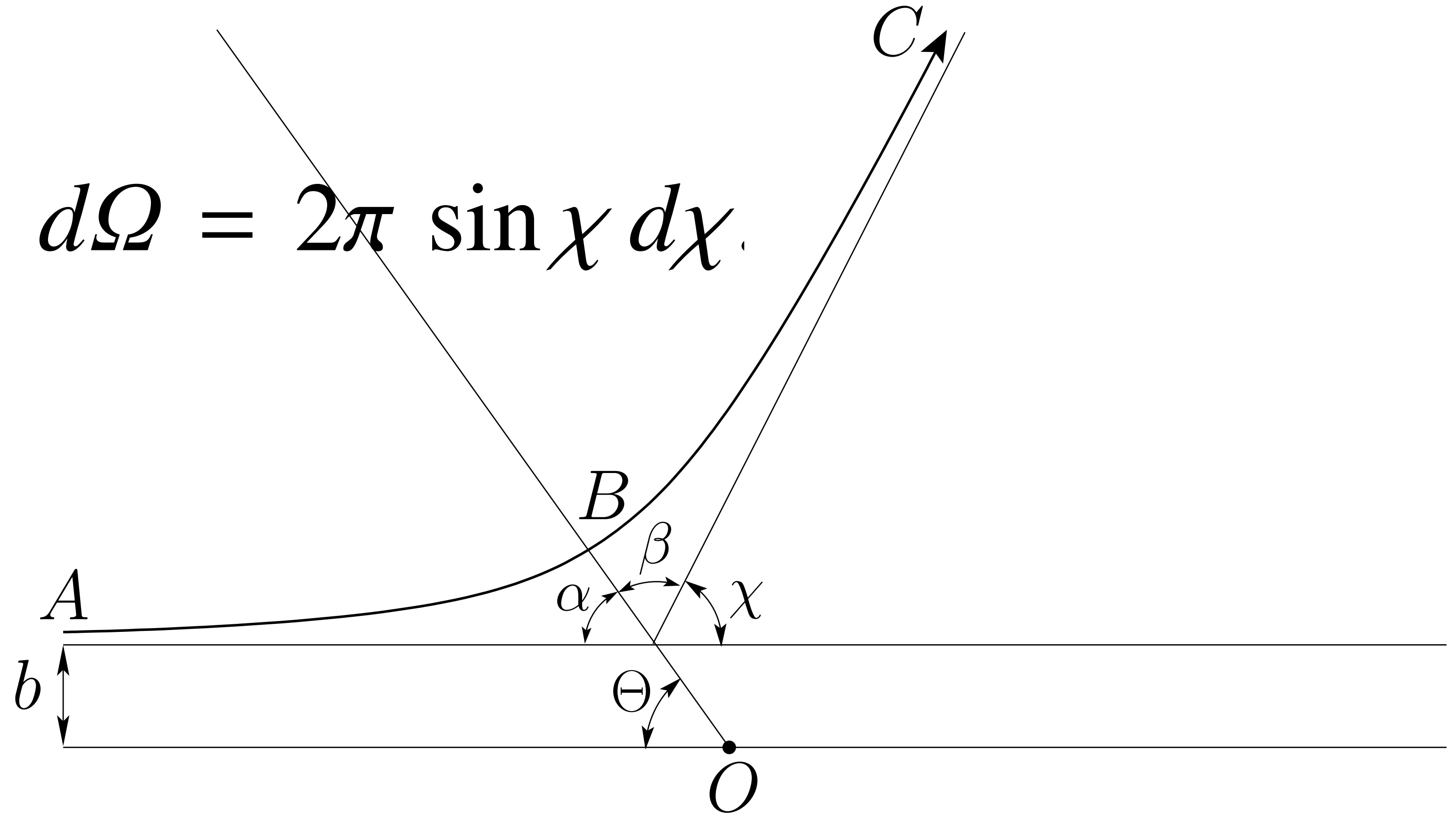
$$+ \frac{1}{2} \mu_{ss'}^2 \mathbf{g}_{ss'} \mathbf{g}_{ss'} : \left(\frac{\partial^2 f_s}{\partial \mathbf{v}_s \partial \mathbf{v}_s} \frac{f_{s'}}{m_s^2} + \frac{f_s}{m_{s'}^2} \frac{\partial^2 f_{s'}}{\partial \mathbf{v}_{s'} \partial \mathbf{v}_{s'}} - \frac{2}{m_s m_{s'}} \frac{\partial f_s}{\partial \mathbf{v}_s} \frac{\partial f_{s'}}{\partial \mathbf{v}_{s'}} \right)$$

Modeling Collisions in Fully Ionized Plasma

$$C_{ss'} \simeq \frac{1}{4}\left(\frac{e_s\, e_{s'}}{4\pi\,\epsilon_0\,\mu_{ss'}}\right)^2$$
$$\times \iint \left[\mu_{ss'}\,\mathbf{g}_{ss'}\cdot\mathbf{J}_{ss'} + \frac{1}{2}\,\mu_{ss'}^2\,\mathbf{g}_{ss'}\mathbf{g}_{ss'} : \left(\frac{1}{m_s}\,\frac{\partial}{\partial\mathbf{v}_s} - \frac{1}{m_{s'}}\,\frac{\partial}{\partial\mathbf{v}_{s'}}\right)\mathbf{J}_{ss'}\right]\frac{d^3\mathbf{v}_{s'}\,d\Omega}{u_{ss'}^3\,\sin^4(\chi/2)}, \tag{3.78}$$

where

$$d\Omega = 2\pi\,\sin\chi\,d\chi.$$



$$\mathbf{J}_{ss'} = \frac{\partial f_s}{\partial\mathbf{v}_s}\,\frac{f_{s'}}{m_s} - \frac{f_s}{m_{s'}}\,\frac{\partial f_{s'}}{\partial\mathbf{v}_{s'}}. \tag{3.79}$$

$$\mathbf{g}_{ss'} = \mathbf{u}'_{ss'} - \mathbf{u}_{ss'}$$

Modeling Collisions in Fully Ionized Plasma

Landau Collision Operator

Collisions create a "flux" in velocity-space

$$C_{ss'} = -\frac{1}{m_s}\frac{\partial}{\partial \mathbf{v}_s}\cdot \mathbf{A}_{ss'},$$

$$\mathbf{A}_{ss'} = \mathbf{B}_{ss'}\, f_s - \mathbf{D}_{ss'}\cdot\frac{\partial f_s}{\partial \mathbf{v}_s},$$

"Drag" "Diffusion"

$$\mathbf{B}_{ss'} = \frac{\gamma_{ss'}}{m_{s'}}\int \mathbf{w}_{ss'}\cdot\frac{\partial f_{s'}}{\partial \mathbf{v}_{s'}}\, d^3\mathbf{v}_{s'},$$

$$\mathbf{D}_{ss'} = \frac{\gamma_{ss'}}{m_s}\int \mathbf{w}_{ss'}\, f_{s'}\, d^3\mathbf{v}_{s'}.$$

Rosenbluth Potentials

$$\mathbf{B}_{ss'} = \frac{2\,\gamma_{ss'}}{m_{s'}} \frac{\partial H_{s'}}{\partial \mathbf{v}_s}, \quad G_{s'}(\mathbf{v}_s) = \int u_{ss'}\, f_{s'}\, d^3\mathbf{v}_{s'},$$

$$\mathbf{D}_{ss'} = \frac{\gamma_{ss'}}{m_s} \frac{\partial^2 G_{s'}}{\partial \mathbf{v}_s \partial \mathbf{v}_s}. \quad H_{s'}(\mathbf{v}_s) = \int u_{ss'}^{-1}\, f_{s'}\, d^3\mathbf{v}_{s'}.$$

$$\boxed{\begin{aligned} \nabla_v^2 H_{s'} &= -4\pi\, f_{s'}(\mathbf{v}), \\ \nabla_v^2 G_{s'} &= 2\, H_{s'}(\mathbf{v}), \end{aligned}}$$

Rosenbluth Potentials

APPENDIX: NUMERICAL TECHNIQUES

Here we describe with more detail the numerical technique used to solve the two-dimensional partial equation [Eq. (32)] given by

$$\frac{\partial F}{\partial t} = S + \frac{\Delta F}{\tau_B} - \frac{1}{v^2}\frac{\partial}{\partial v} v^2 \Gamma_v - \frac{1}{v \sin\theta}\frac{\partial}{\partial \theta}\sin\theta \Gamma_\theta . \tag{A1}$$

The rf currents are given by Eqs. (17), (18), (29), and (30), or

$$\Gamma_v = |\cos\theta| (\Gamma_E / L),$$
$$\Gamma_\theta = \frac{B}{\sin\theta}\left(\frac{1}{R_{\rm res}} - \sin^2\theta\right)\frac{\Gamma_E}{L}. \tag{A2}$$

For the collisional currents, the form used by Cutter *et al.*[20] is used with

$$\Gamma_v = \Gamma_{e\alpha}\left(1 - \frac{M_{e\alpha}}{m_e}\right) F \frac{\partial H}{\partial v} - \frac{\Gamma_{e\alpha}}{2} \times \left[\frac{\partial^2 G}{\partial v^2}\frac{\partial F}{\partial v} + \frac{1}{v^2}\left(\frac{\partial^2 G}{\partial v\, \partial\theta} - \frac{\partial G}{\partial v\, \partial \theta}\right)\frac{\partial F}{\partial \theta}\right], \tag{A3}$$

$$\Gamma_\theta = \Gamma_{e\alpha}\left(1 - \frac{M_{e\alpha}}{m_e}\right) F \frac{1}{v}\frac{\partial H}{\partial \theta} - \frac{\Gamma_{e\alpha}}{2} \times \left[\frac{1}{v^2}\left(\frac{1}{v}\frac{\partial^2 G}{\partial \theta^2} + \frac{\partial G}{\partial v}\right)\frac{\partial F}{\partial \theta} - \frac{1}{v}\frac{\partial^2 G}{\partial v \partial \theta}\frac{\partial F}{\partial v}\right],$$

where $\Gamma_{e\alpha} = 4\pi e^2 e_\alpha^2 \lambda_{e\alpha} / m_e^2$. Here H and G are the Rosenbluth potentials which are approximated by a truncated expansion of spherical harmonics.[20] The first seven, even poly-

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MHD

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\mathbf{u}) = 0$$

$$\rho_{\mathrm{m}} \frac{\partial \mathbf{v}_{\mathrm{m}}}{\partial t} = \mathbf{j} \times \mathbf{B} - \nabla p + \rho_{\mathrm{m}} \mathbf{g}$$

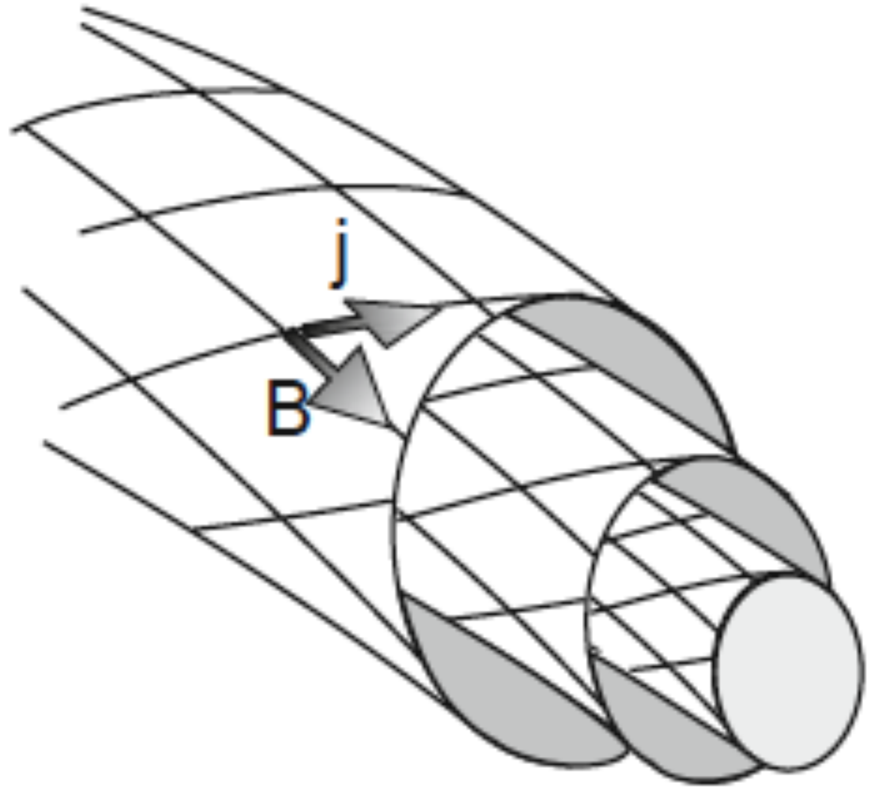
$$\mathbf{E} + \mathbf{v}_{\mathrm{m}} \times \mathbf{B} = \eta \mathbf{j} + \frac{1}{ne} (\mathbf{j} \times \mathbf{B} - \nabla p_{\mathrm{e}})$$

plus magnetostatics

Statics

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j}$$

$$0 = \mathbf{j} \times \mathbf{B} - \nabla p$$



(12) $\nabla(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A}) + (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A}$

$$\nabla p = \bar{J} \times \bar{B}$$

$$= \frac{1}{\mu_0}\left(-\bar{B} \times (\nabla \times B)\right)$$

$$= \frac{1}{\mu_0}\left(-\nabla(B^2/2) + (\bar{B}\cdot\nabla)\bar{B}\right)$$

BUT $(\bar{B}\cdot\bar{\nabla})\bar{B} = \bar{B}\cdot\nabla(\hat{b}B) = B^2\bar{\kappa} + \hat{b}\,(\bar{B}\cdot\nabla)(B)$

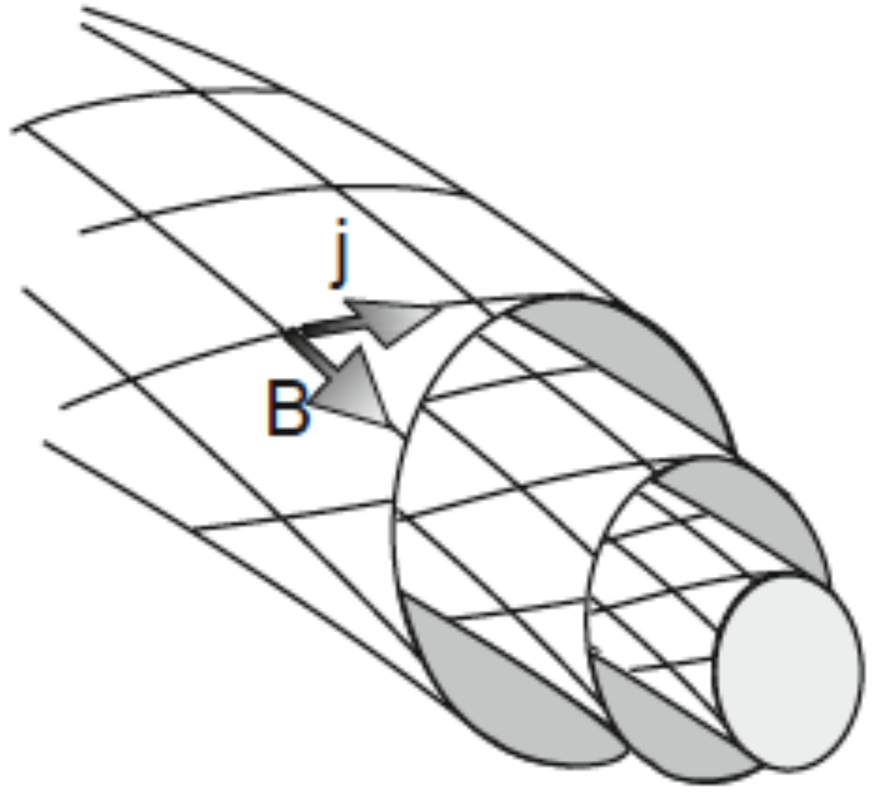
$$\bar{\kappa} = \hat{b}\cdot\nabla\hat{b}$$

CURVATURE

Statics

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j}$$

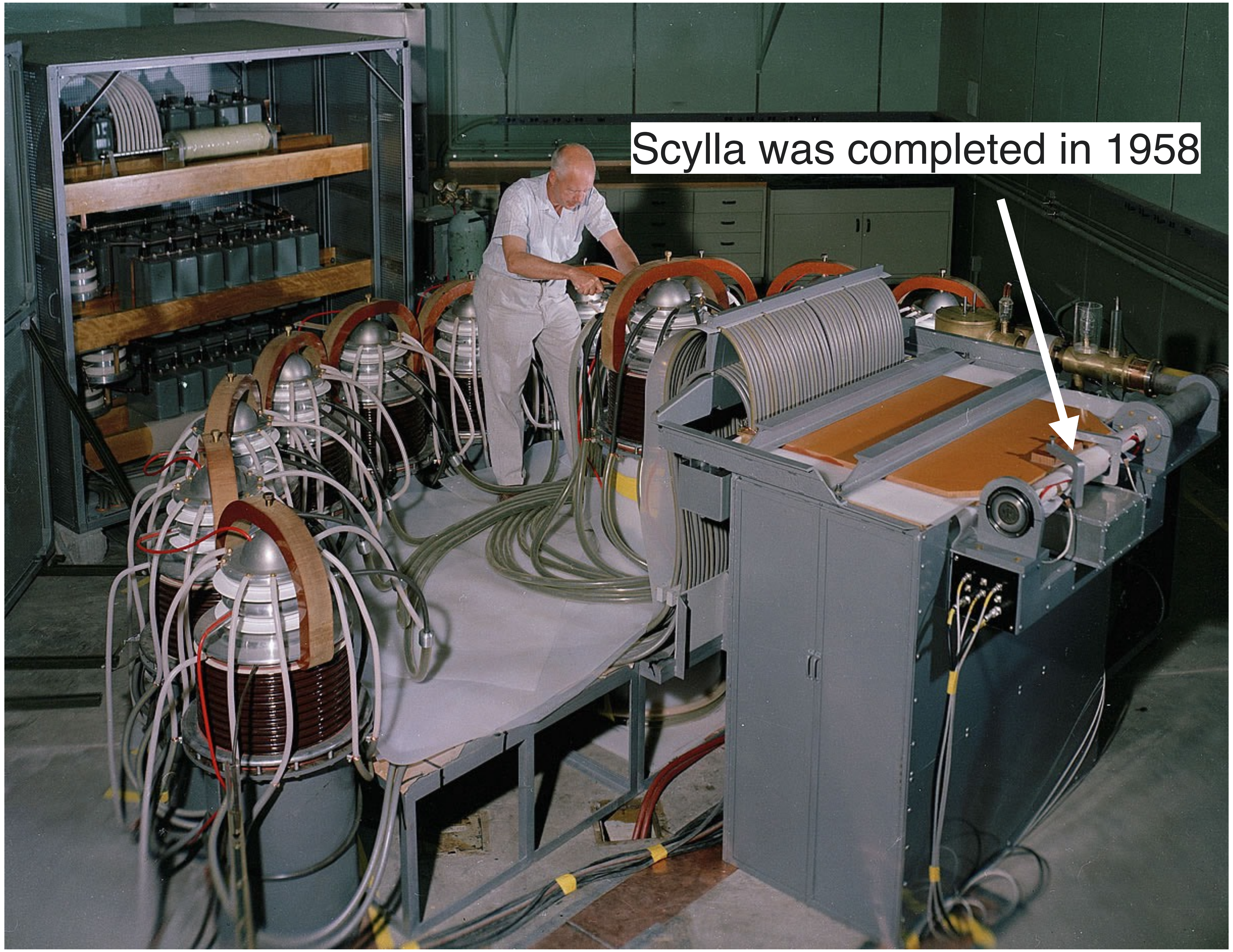
$$0 = \mathbf{j} \times \mathbf{B} - \nabla p$$



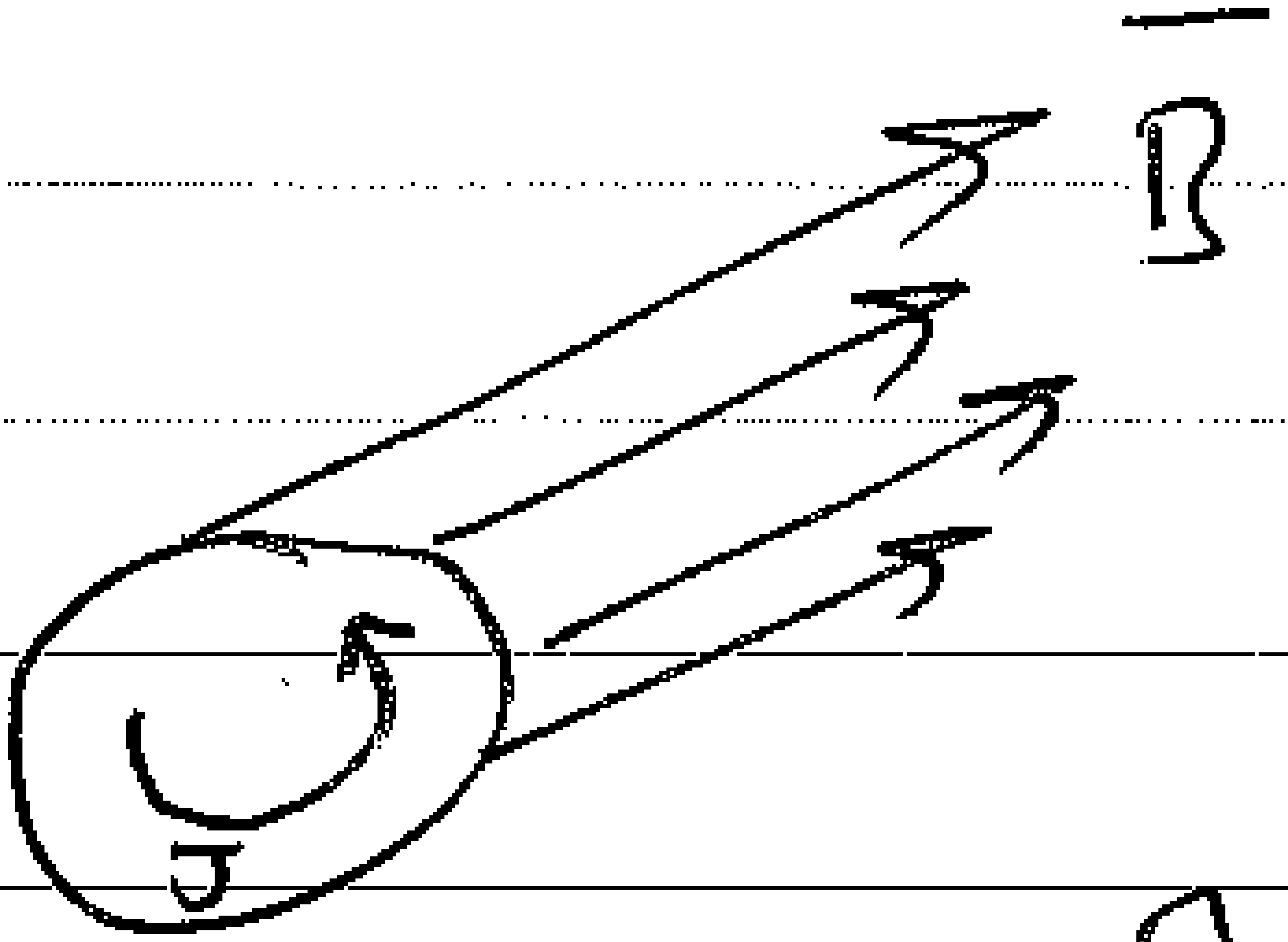
$$\nabla_\perp p = -\frac{1}{2\mu_0} \nabla_\perp B^2 + \frac{B^2}{\mu_0} \vec{\kappa}$$

$$\nabla_\perp \left(p + \frac{B^2}{2\mu_0} \right) = \frac{B^2}{\mu_0} \vec{\kappa}$$

(Theta) θ-Pinch



(Theta) θ-Pinch



$$\bar{B} = B(r)\,\hat{z}$$

$$\bar{J} = \hat{\theta}\,J(r)$$

$$\hat{b}\cdot\bar{\nabla}\,\hat{b} = \hat{z}\cdot\bar{\nabla}\,\hat{z} = 0$$

NO CURVATURE

STRAIGHT B-LINES

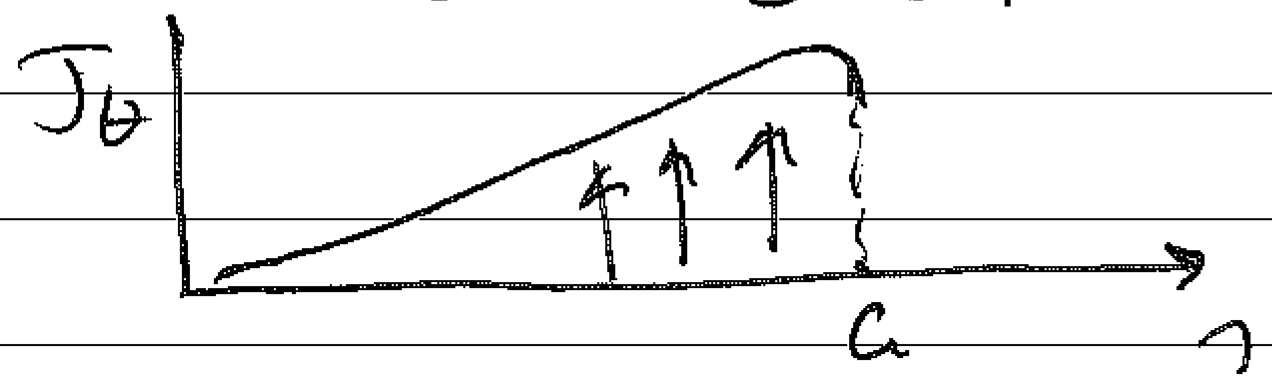
(Theta) θ-Pinch

$$\nabla\left(p + \frac{B^2}{2\mu_0}\right) \simeq 0$$

$$J \times B = \nabla p$$

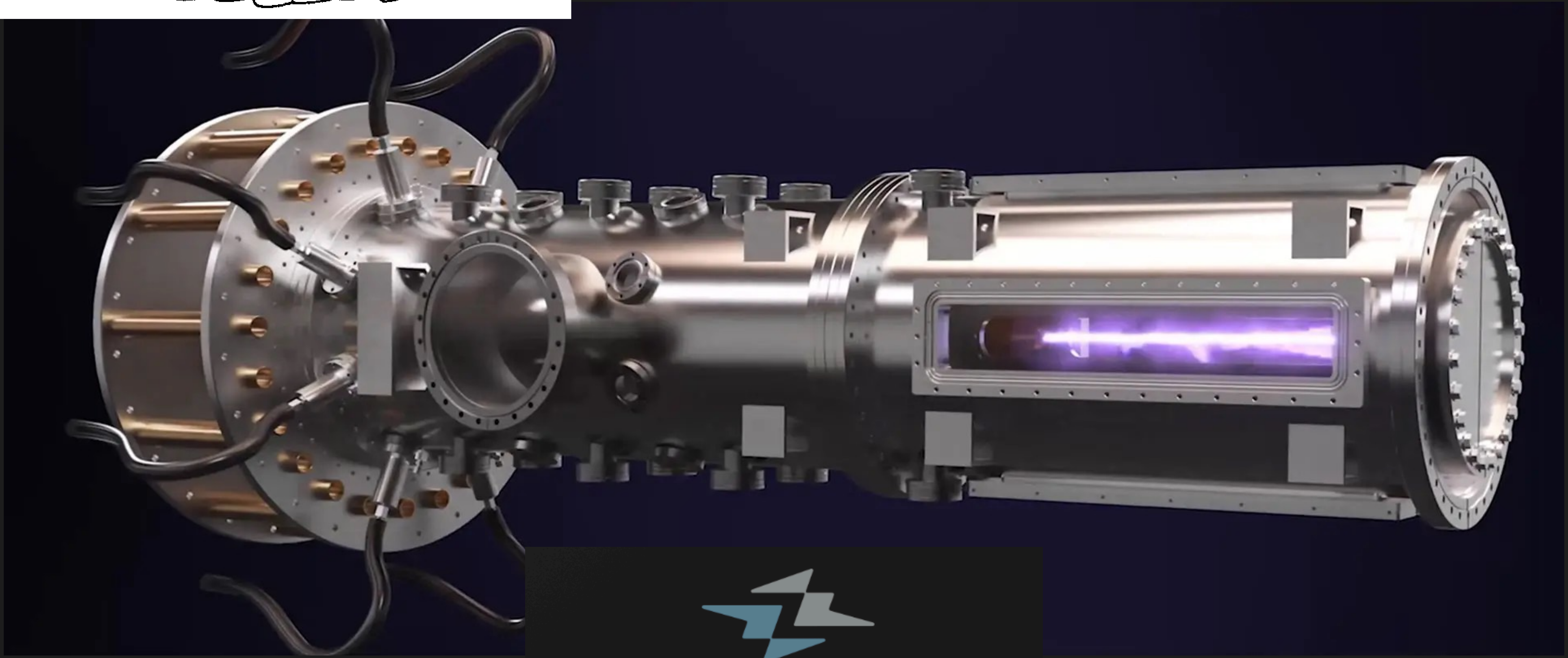
$$J_\perp = \frac{B \times \nabla p}{B^2} = \text{DIAMAGNETIC CURRENT}$$

$$J_\theta = \frac{1}{B}\frac{\partial p}{\partial r}$$



$$\nabla_\perp \left(p + \frac{B^2}{2\mu_0}\right) = \frac{B^2}{\mu_0}\bar{\kappa}$$

Z-Pinch

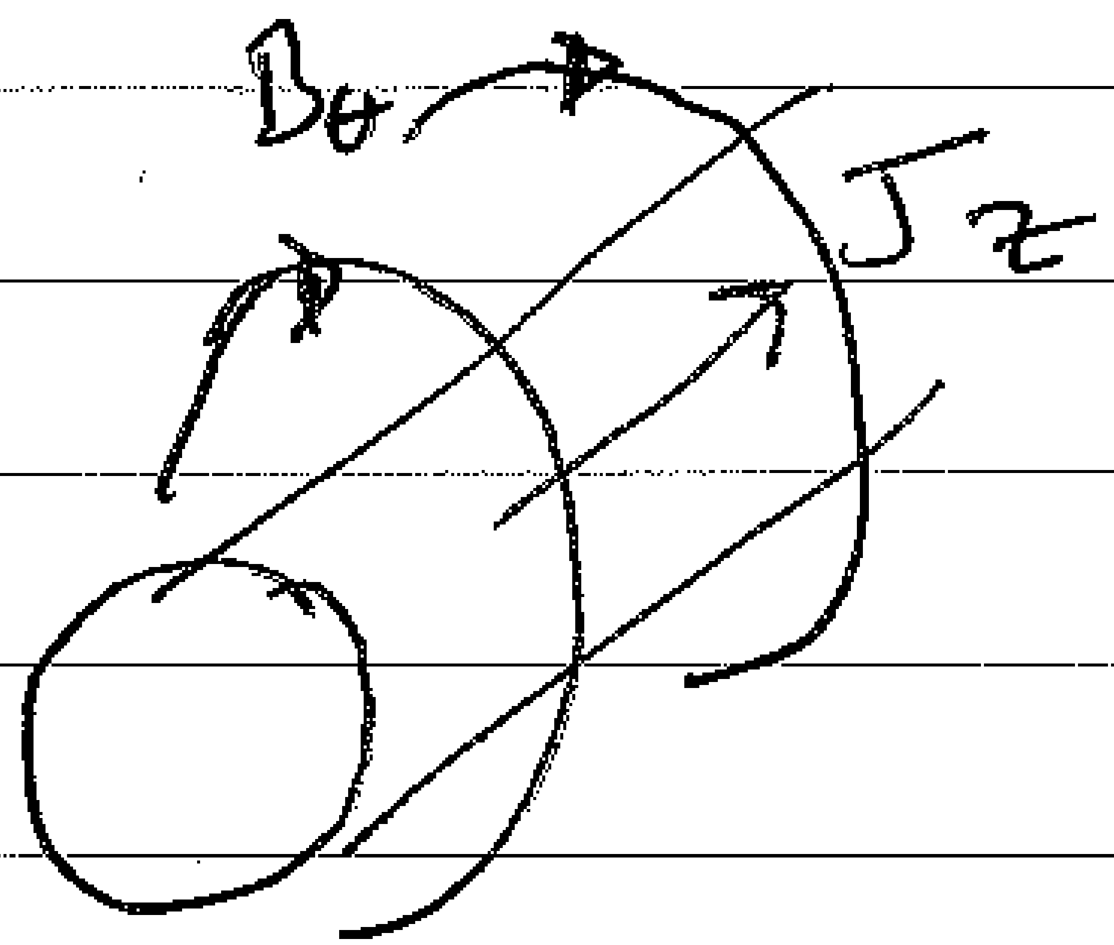


ZAP ENERGY

Fusion power. No magnets required.

$$\nabla_\perp \left(p + \frac{B^2}{2\mu_0} \right) = \frac{B^2}{\mu_0} \bar{\kappa}$$

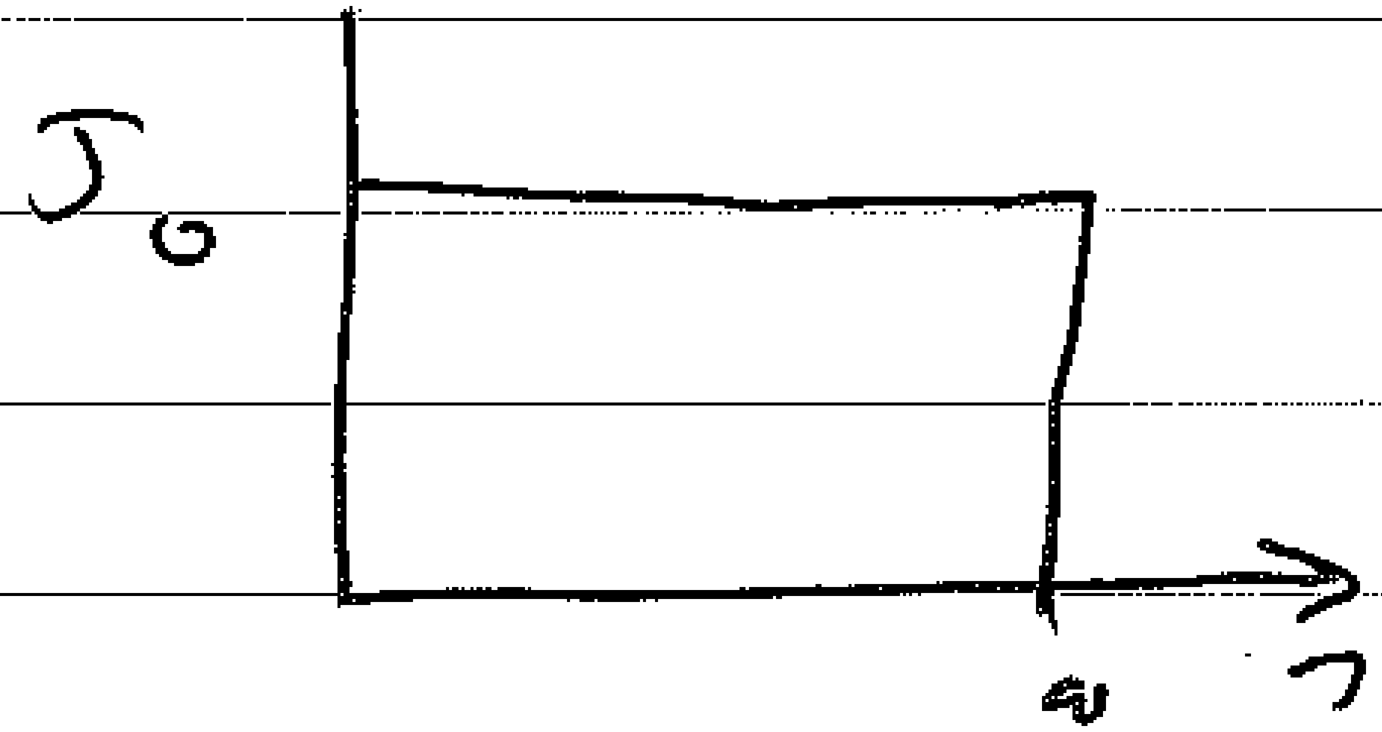
Z-Pinch



$$\bar{J} = \hat{z}\, J_z(r)$$

$$\bar{B} = \hat{\theta}\, B_\theta(r)$$

CONSTANT CURRENT DENSITY EXAMPLE



$$I_p = \pi a^2 J_0$$

$$B^* = \frac{\mu_0 I_p}{2\pi a} = \text{FIELD AT EDGE}$$

$$\nabla_\perp\left(p + \frac{B^2}{2\mu_0}\right) = \frac{B^2}{\mu_0}\bar{\kappa}$$

Z-Pinch

$$B_\theta(r) = \frac{1}{2\pi r}\int_0^r 2\pi r' \mu_0 J_0 \, dr' = \frac{r}{2}\mu_0 J_0 = B^*\left(\frac{r}{a}\right)$$

$$\bar{J}\times\bar{B} = \nabla p \Rightarrow -J_0 B^*\left(\frac{r}{a}\right) = \frac{\partial p}{\partial r}$$

$$\frac{\partial}{\partial r}p(r) = -\frac{2B^{*2}}{a\mu_0}\left(\frac{r}{a}\right)$$

$p(0) = 2\times$ EDGE PRESSURE

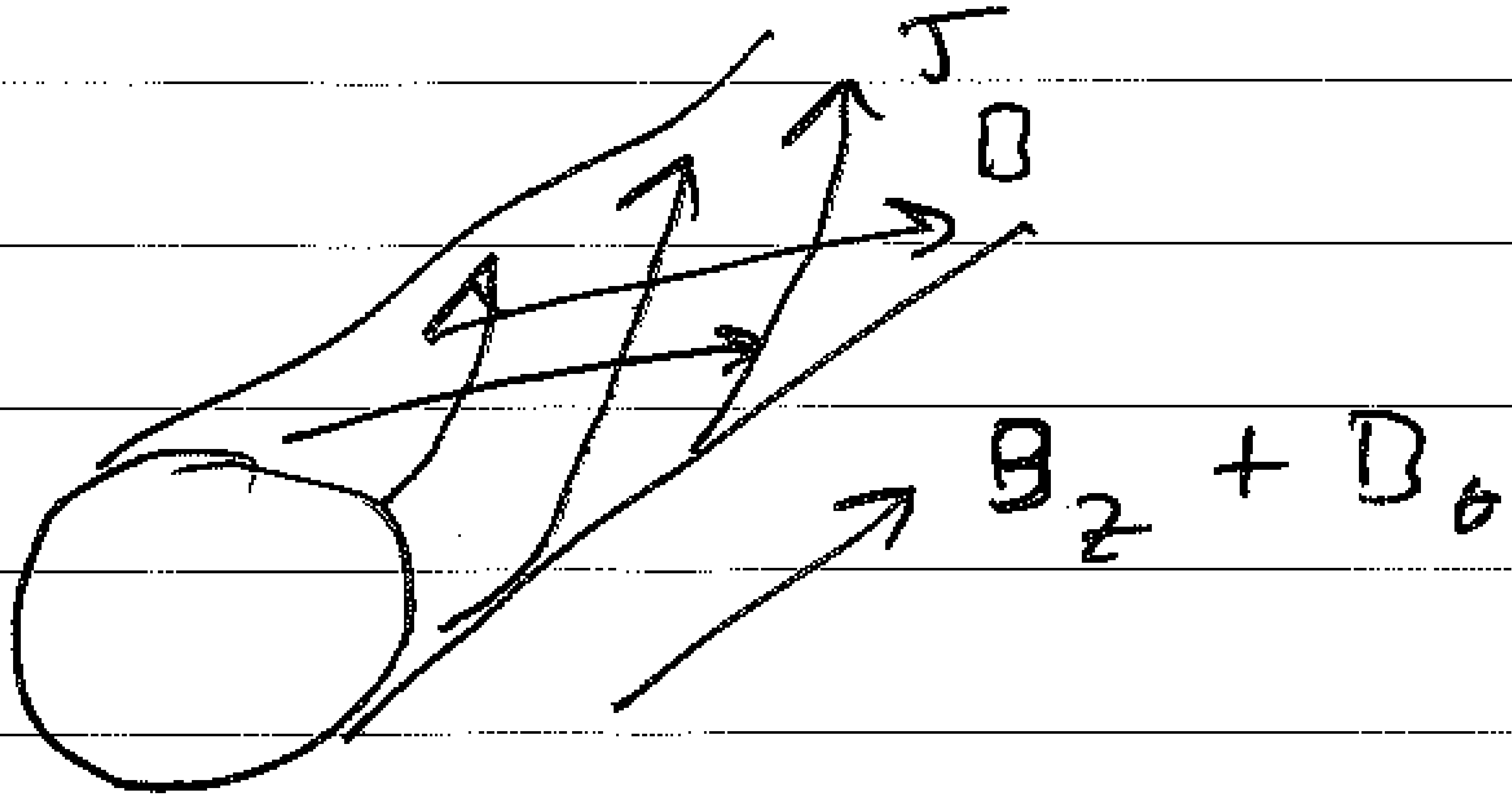
$$\boxed{p(r) = \frac{B^{*2}}{\mu_0}\left(1 - \frac{r^2}{a^2}\right)}$$

$$p(r) + \frac{B^2}{2\mu_0} = \frac{B^{*2}}{2\mu_0}\left(2 - \frac{r^2}{a^2}\right)$$

$$\frac{B^2}{\mu_0}\bar{\kappa} = -\frac{1}{r}\frac{B^{*2}}{\mu_0}\left(\frac{r^2}{a^2}\right)$$

$$\nabla_\perp\left(p + \frac{B^2}{2\mu_0}\right) = \frac{B^2}{\mu_0}\bar{\kappa}$$

"Screw-Pinch"
(a.k.a. "Straight Tokamak")

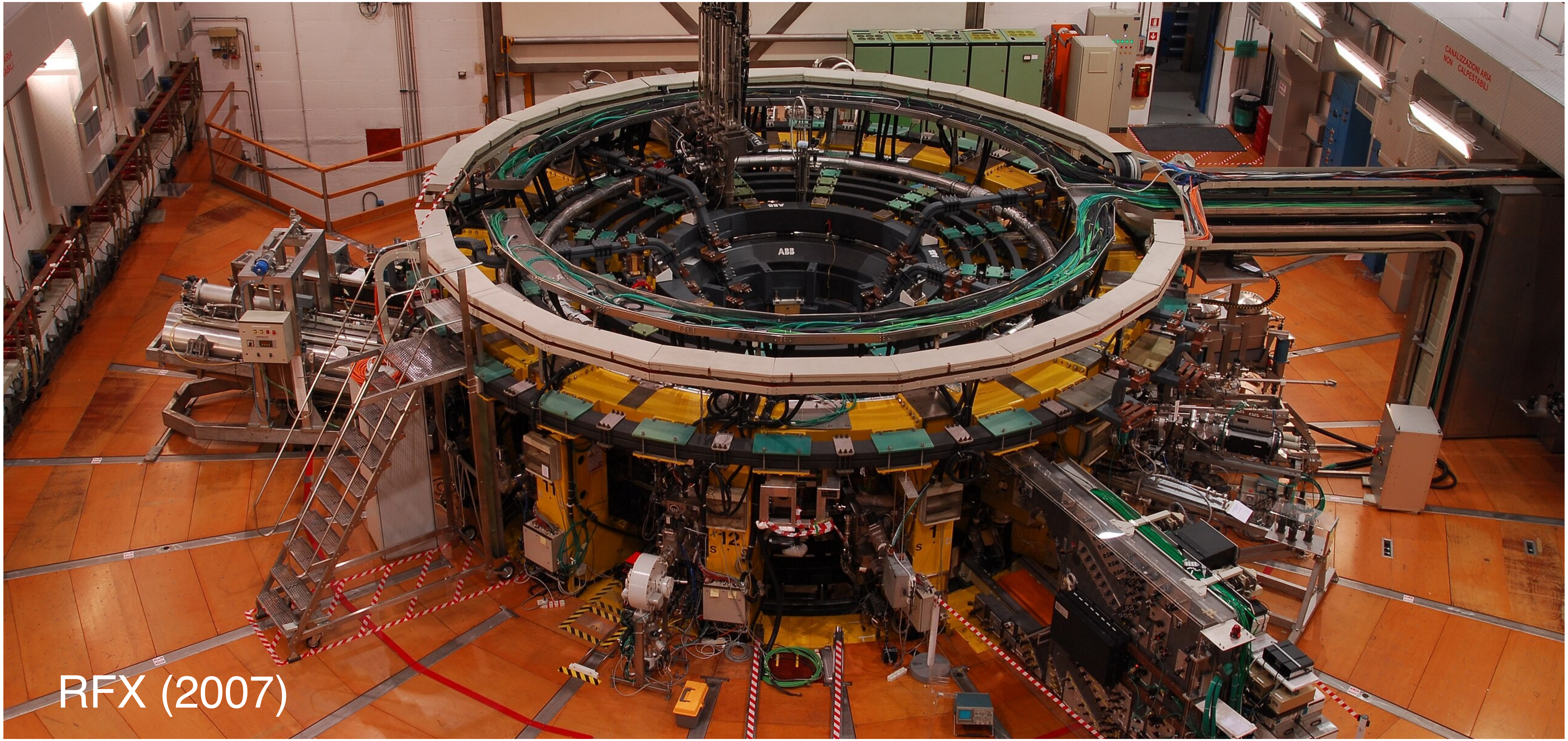


$$\bar{J} = \bar{J}_z(r) + \bar{J}_\theta(r)$$

$$\bar{B} = \bar{B}_z(r) + \bar{B}_\theta(r)$$

$$\nabla_\perp \left(p + \frac{B^2}{2\mu_0}\right) = \frac{B^2}{\mu_0}\overline{\kappa}$$

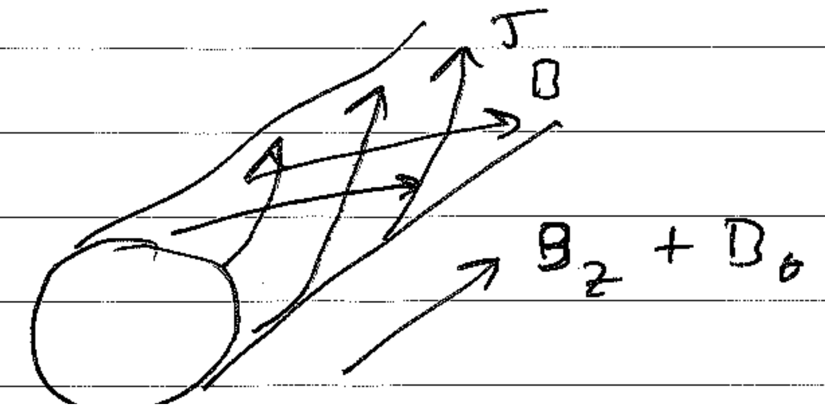
“Screw-Pinch”
(a.k.a. “Straight Tokamak” or “Straight RFP”)



RFX (2007)

$$\nabla_\perp\left(p+\frac{B^2}{2\mu_0}\right)=\frac{B^2}{\mu_0}\bar{\kappa}$$

“Screw-Pinch” (a.k.a. “Straight Tokamak”)



$$\bar{J} = \bar{J}_z(r) + \bar{J}_\theta(r)$$

$$\bar{B} = \bar{B}_z(r) + \bar{B}_\theta(r)$$

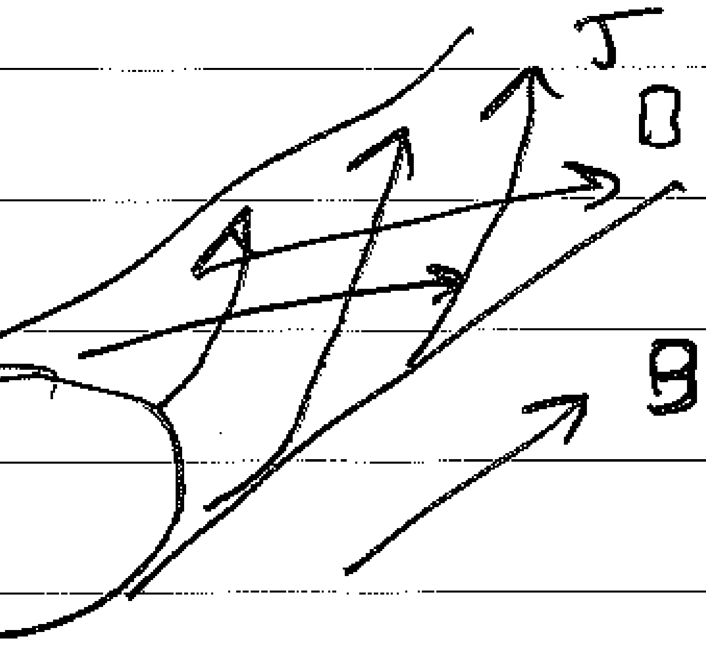
$$J_z = J_0 \qquad B_\theta = B_\theta^*\left(\frac{r}{a}\right)$$

$$\frac{B^2}{\mu_0}\bar{\kappa} = -\frac{\hat{r}}{r}\frac{B_\theta^2}{\mu_0} = -\frac{\hat{r}}{r}\frac{B_*^2}{\mu_0}\left(\frac{r}{a}\right)^2$$

$$\hat{r}\cdot\left(\bar{J}\times\bar{B}\right) = J_\theta B_z - J_z B_\theta = \frac{\partial p}{\partial r}$$

$$\nabla\left(p+\frac{B_z^2}{2\mu_0}+\frac{B_\theta^2}{2\mu_0}\right) = \frac{B_\theta^2}{\mu_0}\bar{\kappa} = -\frac{B^{*2}}{r\mu_0}\left(\frac{r}{a}\right)^2$$

"Screw-Pinch"
(a.k.a. "Straight Tokamak")



$B_z + B_\theta$

$$\bar{J} = \bar{J}_z(r) + \bar{J}_\theta(r)$$

$$\bar{B} = \bar{B}_z(r) + \bar{B}_\theta(r)$$

$$\nabla\left(p + \frac{B_z^2}{2\mu_0} + \frac{B_\theta^2}{2\mu_0}\right) = \frac{B_\theta^2}{\mu_0}\bar{K} = -\frac{B^{*2}}{r\mu_0}\left(\frac{r}{a}\right)^2$$

LET $p(r) = p_0\left(1 - \frac{r^2}{a^2}\right)$

$$\frac{B_z^2}{2\mu_0} = \frac{B_z^{*2}}{2\mu_0} - \frac{(\Delta B_z)^2}{2\mu_0}\left(\frac{r}{a}\right)^2$$

$$\frac{B_\theta^2}{2\mu_0} = \frac{B_\theta^{*2}}{2\mu_0}\left(\frac{r}{a}\right)^2$$

$$\frac{\partial}{\partial r}\left(\left(\frac{r^2}{a^2}\right)\left(-p_0 - \frac{(\Delta B_z)^2}{2\mu_0} + \frac{B_\theta^{*2}}{2\mu_0}\right)\right) = -\frac{B_\theta^{*2}}{r\mu_0}\left(\frac{r}{a}\right)^2$$

$$2\left(+p_0 + \frac{(\Delta B_z)^2}{2\mu_0} - \frac{B_\theta^{*2}}{2\mu_0}\right) = \frac{B_\theta^{*2}}{\mu_0}$$

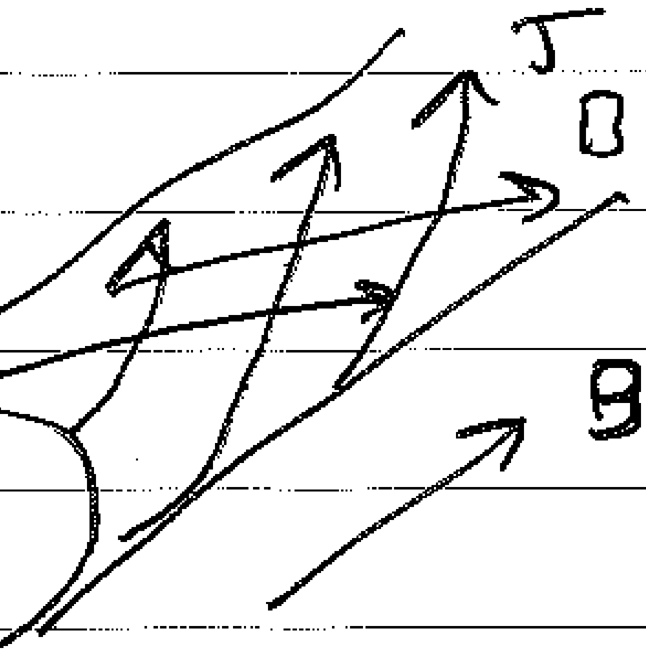
"Screw-Pinch" (a.k.a. "Straight Tokamak")

$$2\left(p_0 + \frac{(\Delta B_z)^2}{2\mu_0} - \frac{B_\theta^{*2}}{2\mu_0}\right) = \frac{B_\theta^{*2}}{\mu_0}$$

$$\boxed{\beta_p + \mu - 1 = 0}$$

$$\beta_p = \frac{p_0}{B_\theta^{*2}/2\mu_0}$$

$$\mu = \frac{(\Delta B_z)^2}{B_\theta^{*2}} = \begin{cases} >1 & \text{PARAMAGNETIC} \\ <1 & \text{DIAMAGNETIC} \end{cases}$$



$B_z + B_\theta$

$$\bar{J} = \bar{J}_z(r) + \bar{J}_\theta(r)$$

$$\bar{B} = \bar{B}_z(r) + \bar{B}_\theta(r)$$

$$\beta_p = 1 \implies \Delta B_z = 0 \qquad B_z = \text{CONSTANT}$$

Review paper

PLASMA EQUILIBRIUM IN A TOKAMAK

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ABSTRACT. The paper summarizes the basic information on the equilibrium of a toroidal plasma column in systems of the Tokamak type. It considers the methods of maintaining a plasma in equilibrium with the help of a conducting casing, an external maintaining field and the iron core of a transformer. Attention is paid to the role of the inhomogeneity of the maintaining field. It is shown in particular how the shape of the column cross-section depends on the curvature of the lines of force of the maintaining field. For the case (which has practical importance) weak asymmetry of the field distribution in the transverse cross-section, this paper describes a uniform method of consideration, which takes into account the influence of different factors on the equilibrium position of the column. This method is used for calculating plasma equilibrium in a Tokamak model with a conducting casing. Account is here taken of the effect of gaps in the casing and of finite electrical conductivity. Some cases of plasma equilibrium which are outside the standard Tokamak scheme are also considered, such as equilibrium in a conducting shell having the shape of a racetrack, equilibrium where the whole current is transferred by relativistic runaway electrons and equilibrium at high plasma pressure $\beta_I \sim R/a$.

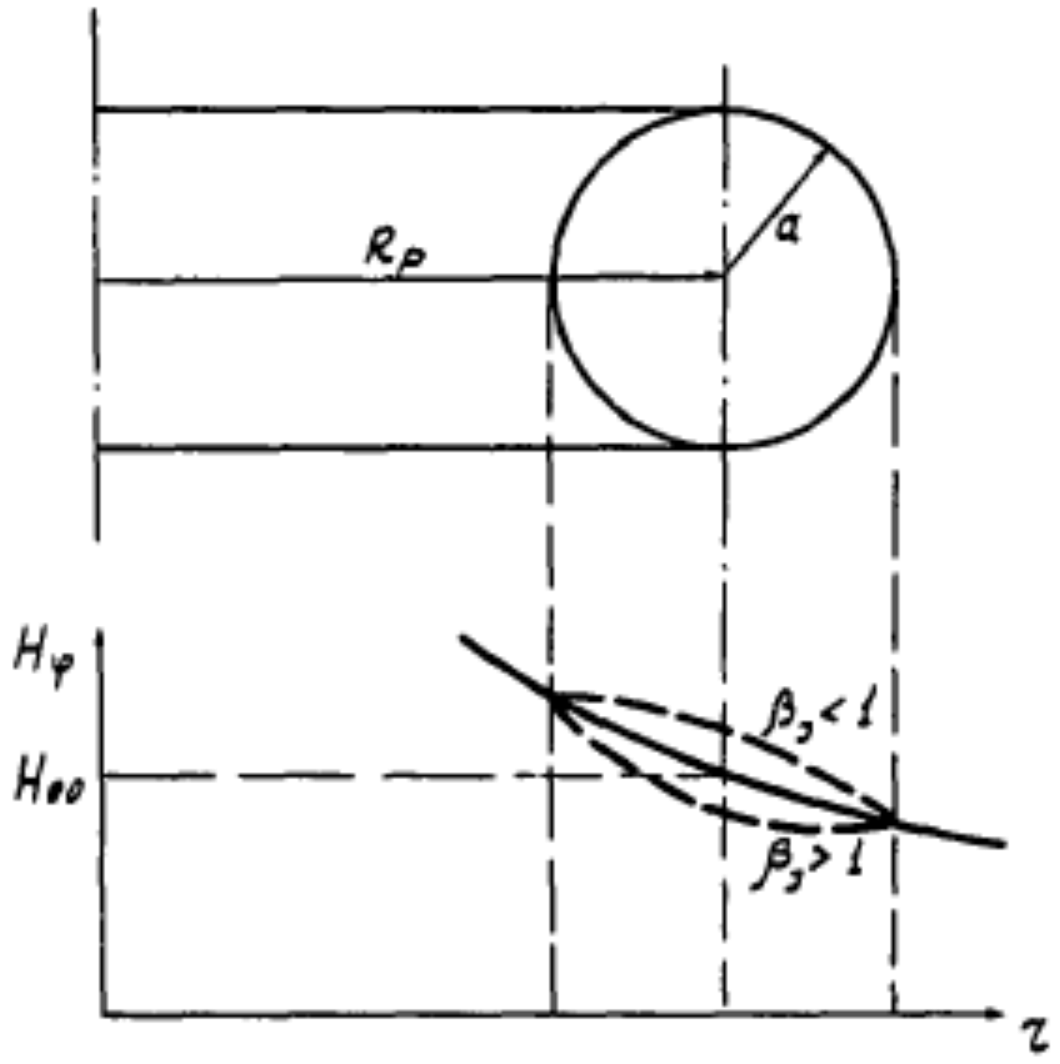

FIG.1. Distribution of a toroidal magnetic field.

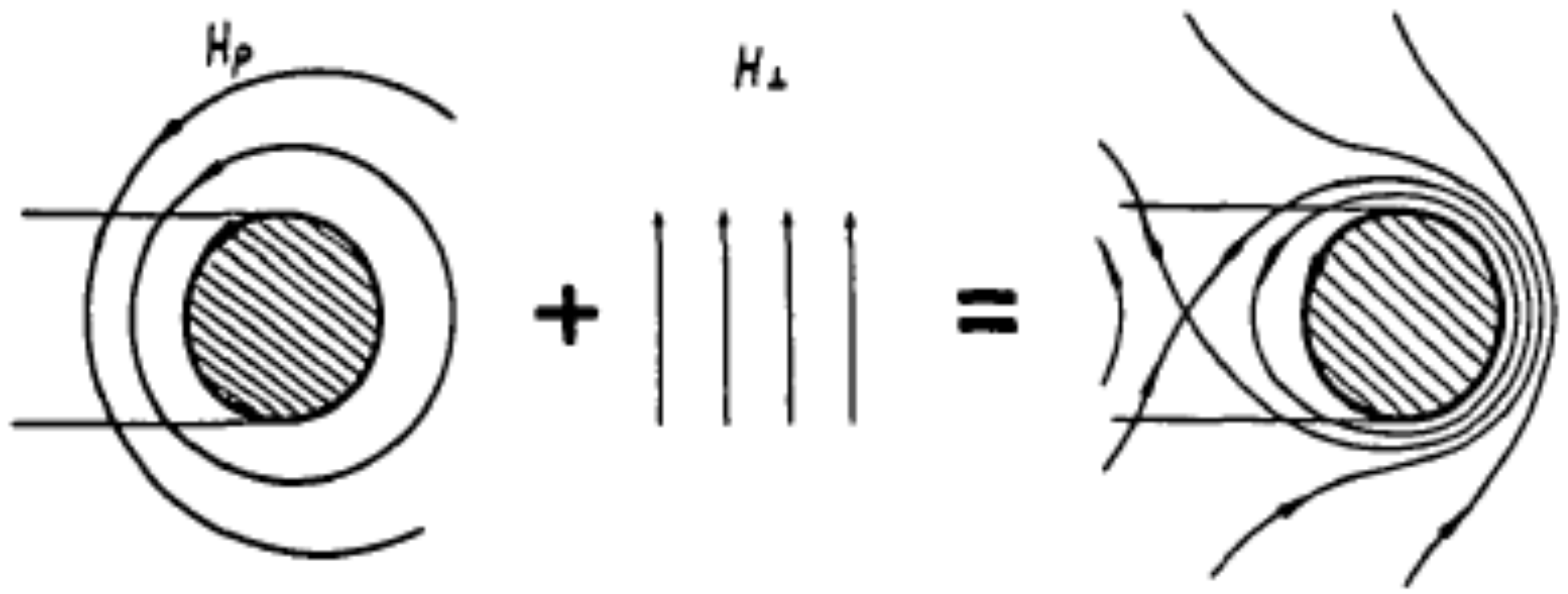

FIG.4. Diagram of the combination of the proper magnetic field of a ring current with transverse balancing magnetic field.

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Numerical Determination of Axisymmetric Toroidal Magnetohydrodynamic Equilibria

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Numerical schemes for the determination of stationary axisymmetric toroidal equilibria appropriate for modeling real experimental devices are given. Iterative schemes are used to solve the elliptic nonlinear partial differential equation for the poloidal flux function Ψ. The principal emphasis is on solving the free boundary (plasma-vacuum interface) equilibrium problem where external current-carrying toroidal coils support the plasma column, but fixed boundary (e.g., conducting shell) cases are also included. The toroidal current distribution is given by specifying the pressure and either the poloidal current or the safety factor profiles as functions of Ψ. Examples of the application of the codes to tokamak design at PPPL are given.

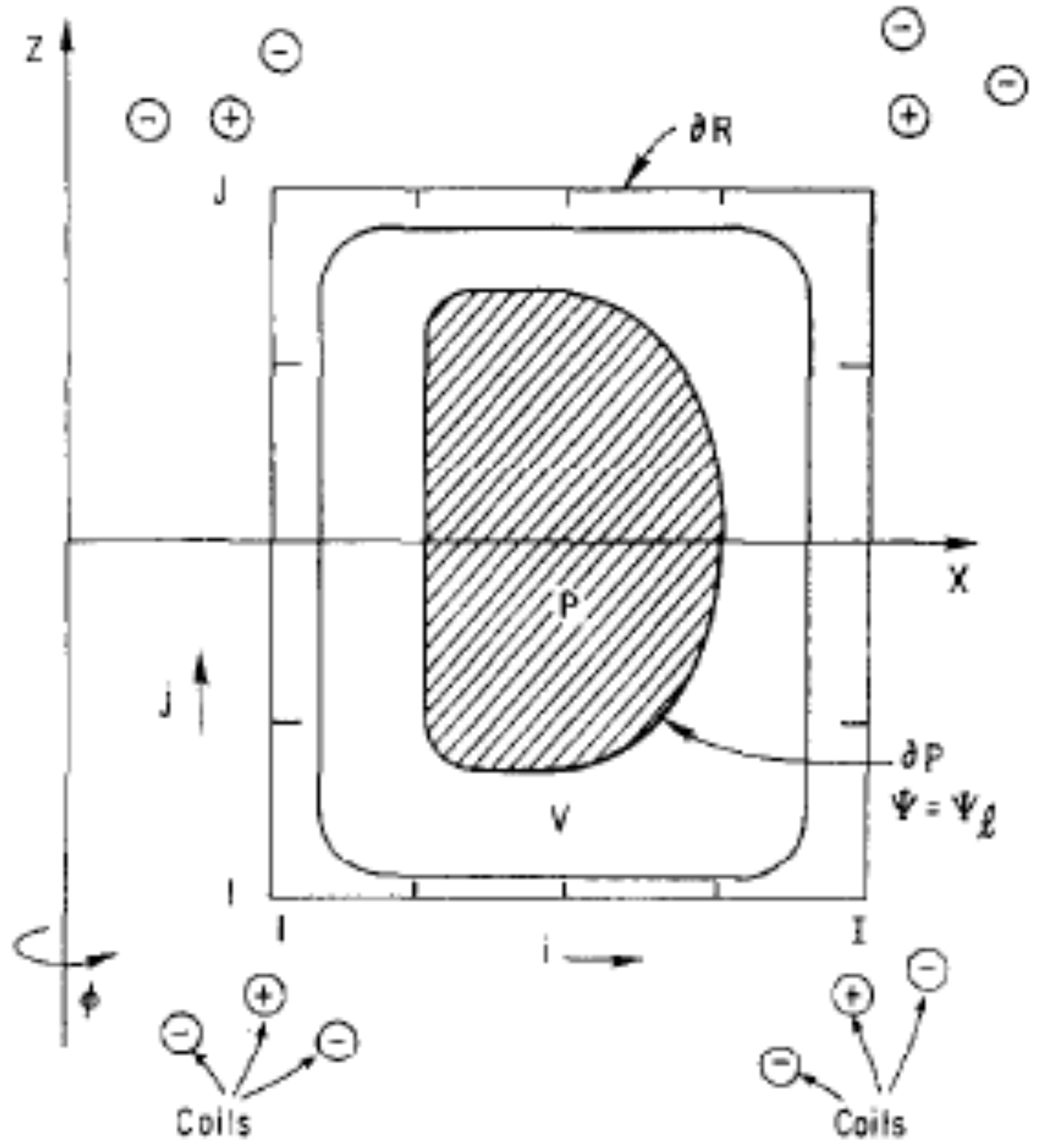


FIG. 1. Computational domain $\mathscr{R}$.

SOLUTION PROCEDURE WITH $p(\Psi)$ AND $g(\Psi)$ SPECIFIED

$$\nabla p = \mathbf{J} \times \mathbf{B},$$

$$\mathbf{J} = \nabla \times \mathbf{B},$$

$$\nabla \cdot \mathbf{B} = 0.$$

$$\mathbf{B} = \frac{1}{2\pi} \nabla\phi \times \nabla\Psi + RB_0 g \,\nabla\phi$$

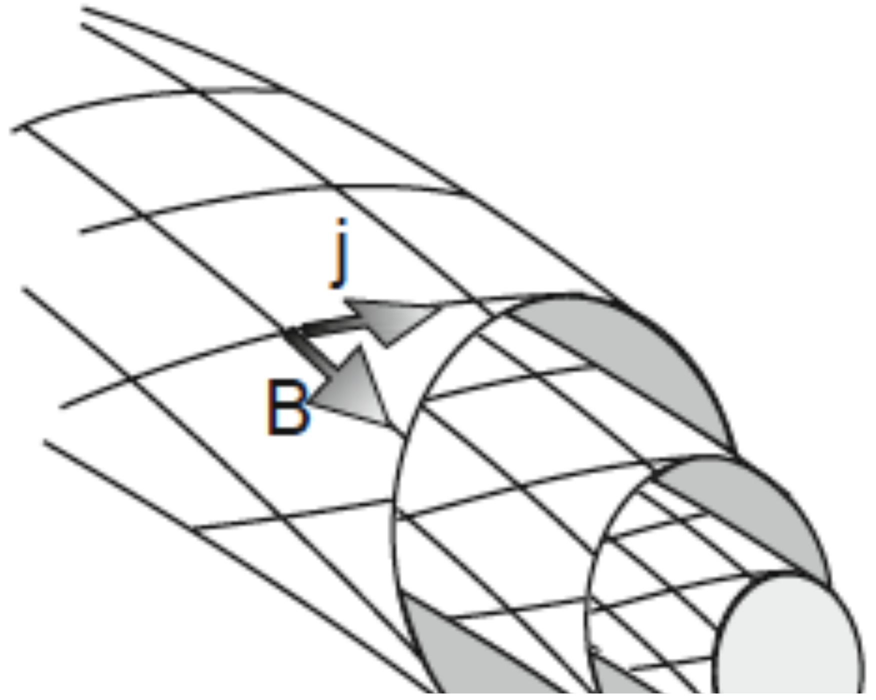
$$X \frac{\partial}{\partial X} \frac{1}{X} \frac{\partial \Psi}{\partial X} + \frac{\partial^2 \Psi}{\partial Z^2} = 2\pi X J_\phi ,$$

$$J_\phi = -2\pi \left(X \frac{dp}{d\Psi} + \frac{R^2 B_0{}^2}{2X} \frac{dg^2}{d\Psi} \right).$$

Grad-Shafranov Equation

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j}$$

$$0 = \mathbf{j} \times \mathbf{B} - \nabla p$$

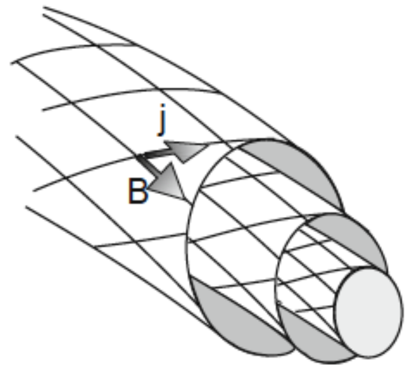


- $\bar{B} = \bar{\nabla}\varphi \times \bar{\nabla}\psi + G\bar{\nabla}\varphi$
- CYLINDRICAL SYMMETRY $\quad \bar{\nabla}\varphi = \dfrac{\hat{\varphi}}{R} \quad \bar{\nabla}\varphi \cdot \bar{\nabla}(\text{ANYTHING}) = 0$
- $P(\psi)$, $G(\psi)$ $\quad G =$ TOROIDAL FLUX $\quad \bar{B}_\varphi = \dfrac{G}{R}\hat{\varphi}$
- $\bar{B} \cdot \bar{B} = (\nabla\varphi \times \nabla\psi) \cdot (\nabla\varphi \times \nabla\psi) + G^2 |\nabla\varphi|^2$
  $= |\nabla\varphi|^2 \left(|\nabla\psi|^2 + G^2 \right)$

Grad-Shafranov Equation

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j}$$

$$0 = \mathbf{j} \times \mathbf{B} - \nabla p$$



- $$\mu_0 \bar{J} = \nabla \times \bar{B}$$

$$= \nabla \times (\nabla \varphi \times \nabla \psi) + \nabla \times (G \nabla \varphi)$$

$$= \bar{\nabla}\varphi\,(\nabla \cdot \nabla \psi) - \bar{\nabla}\psi\,(\nabla \cdot \bar{\nabla}\varphi) + (\nabla \psi \cdot \bar{\nabla})\bar{\nabla}\varphi$$

$$- (\nabla \varphi \cdot \bar{\nabla})\bar{\nabla}\psi + G\,\nabla \times \nabla \varphi + \bar{\nabla} G \times \bar{\nabla}\varphi$$

$$= \underbrace{\bar{\nabla}\varphi \left[\nabla^2 \psi - \frac{1}{\partial r}\frac{\partial \psi}{\partial r} \right]}_{J_\varphi} + \underbrace{\frac{\partial G}{\partial \psi}\,\bar{\nabla}\psi \times \bar{\nabla}\varphi}_{\text{poloidal}}$$

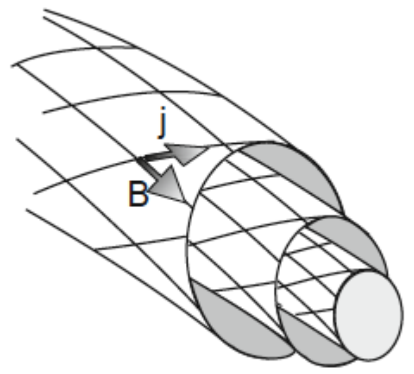
1 current within flux surface

$$(10)\quad \nabla \times (\mathbf{A} \times \mathbf{B}) = \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A}) + (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B}$$

Grad-Shafranov Equation

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j}$$

$$0 = \mathbf{j} \times \mathbf{B} - \nabla p$$



$$\bar\nabla p = \bar J \times \bar B$$

$$\mu_0 \bar\nabla\psi \cdot \bar\nabla p = \bar\nabla\psi \; \mu_0 \cdot (\bar J \times \bar B)$$

$$\mu_0 |\nabla\psi|^2 \frac{\partial p}{\partial\psi} = \mu_0 \bar J \cdot (\bar B \times \nabla\psi)$$

$$= \mu_0 \bar J \cdot \left[-\nabla\psi \times (\nabla\varphi \times \nabla\psi) + G \bar\nabla\varphi \times \nabla\psi \right]$$

$$= -\mu_0 J_\varphi |\nabla\varphi| |\nabla\psi|^2 + \mu_0 G \bar J \cdot \nabla\varphi \times \nabla\psi$$

$$= -\mu_0 J_\varphi |\nabla\varphi| |\nabla\psi|^2 + G \frac{\partial G}{\partial\psi} (\bar\nabla\psi \times \nabla\varphi) \cdot (\nabla\varphi \times \nabla\psi)$$

$$= -\mu_0 J_\varphi |\nabla\varphi| |\nabla\psi|^2 - G \frac{\partial G}{\partial\psi} |\nabla\varphi|^2 |\nabla\psi|^2$$

$$\boxed{\mu_0 J_\varphi |\nabla\varphi| = -\mu_0 \frac{\partial p}{\partial\psi} - |\nabla\varphi|^2 G \frac{\partial G}{\partial\psi}} \qquad \left(\nabla\varphi = \frac{1}{R}\right)$$

$$\Delta^* \psi = -\mu_0 R^2 \frac{\partial p}{\partial\psi} - G \frac{\partial G}{\partial\psi}$$

$$\Delta^* = R \frac{\partial}{\partial R}\left(\frac{1}{R} \frac{\partial\psi}{\partial R}\right) + \frac{\partial^2\psi}{\partial z^2}$$

(1) $\mathbf{A} \cdot \mathbf{B} \times \mathbf{C} = \mathbf{A} \times \mathbf{B} \cdot \mathbf{C} = \mathbf{B} \cdot \mathbf{C} \times \mathbf{A} = \mathbf{B} \times \mathbf{C} \cdot \mathbf{A} = \mathbf{C} \cdot \mathbf{A} \times \mathbf{B} = \mathbf{C} \times \mathbf{A} \cdot \mathbf{B}$

(2) $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{C} \times \mathbf{B}) \times \mathbf{A} = (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C}$

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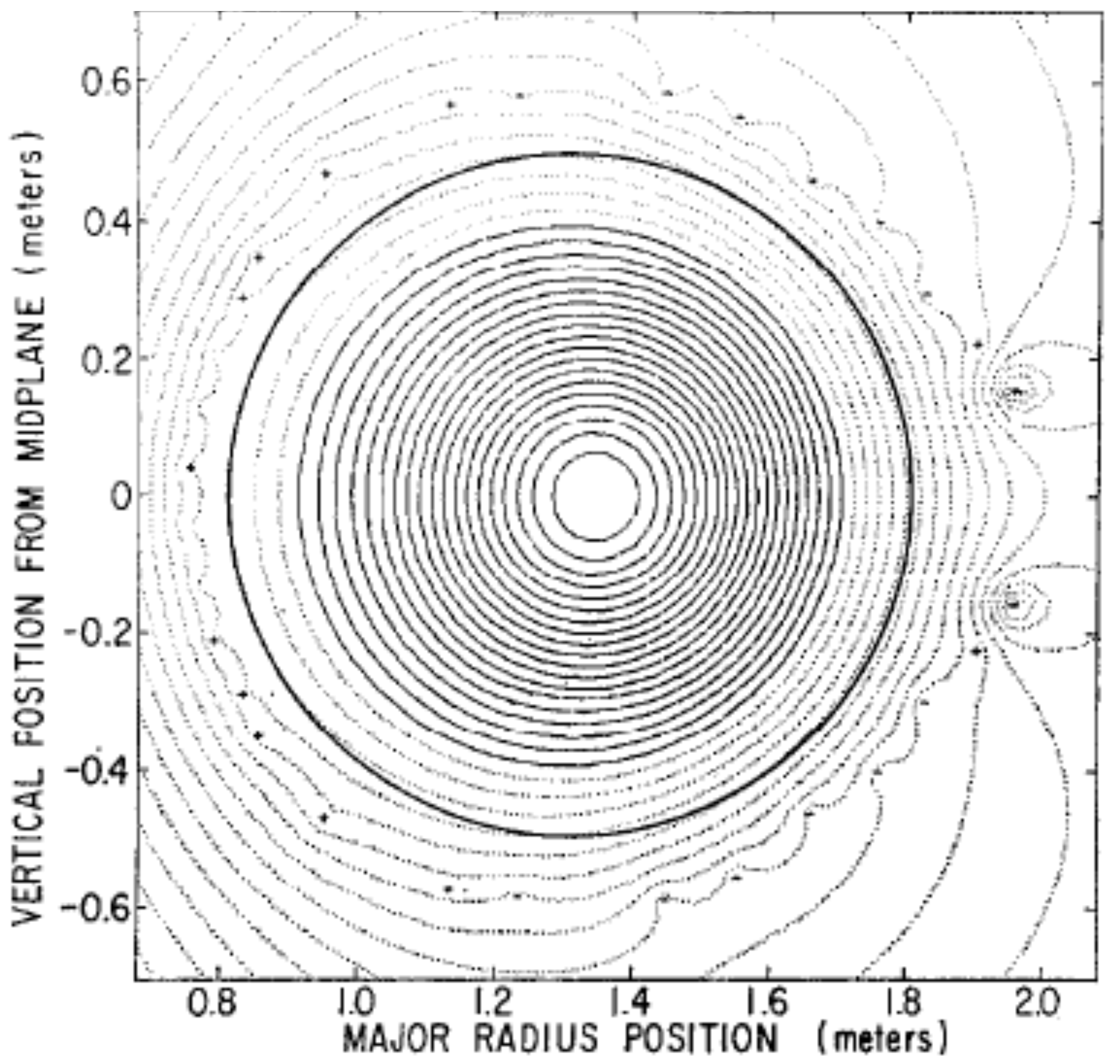
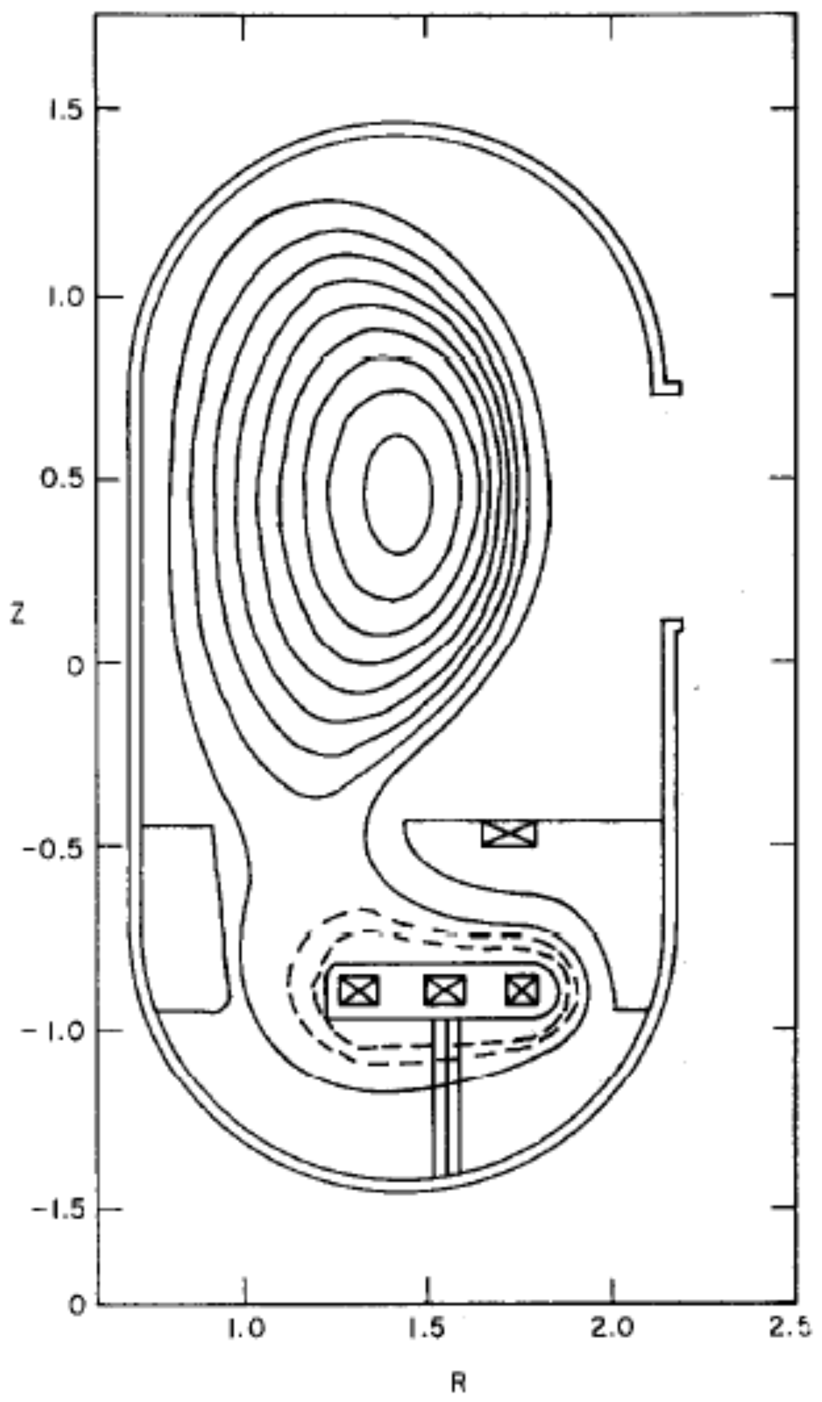
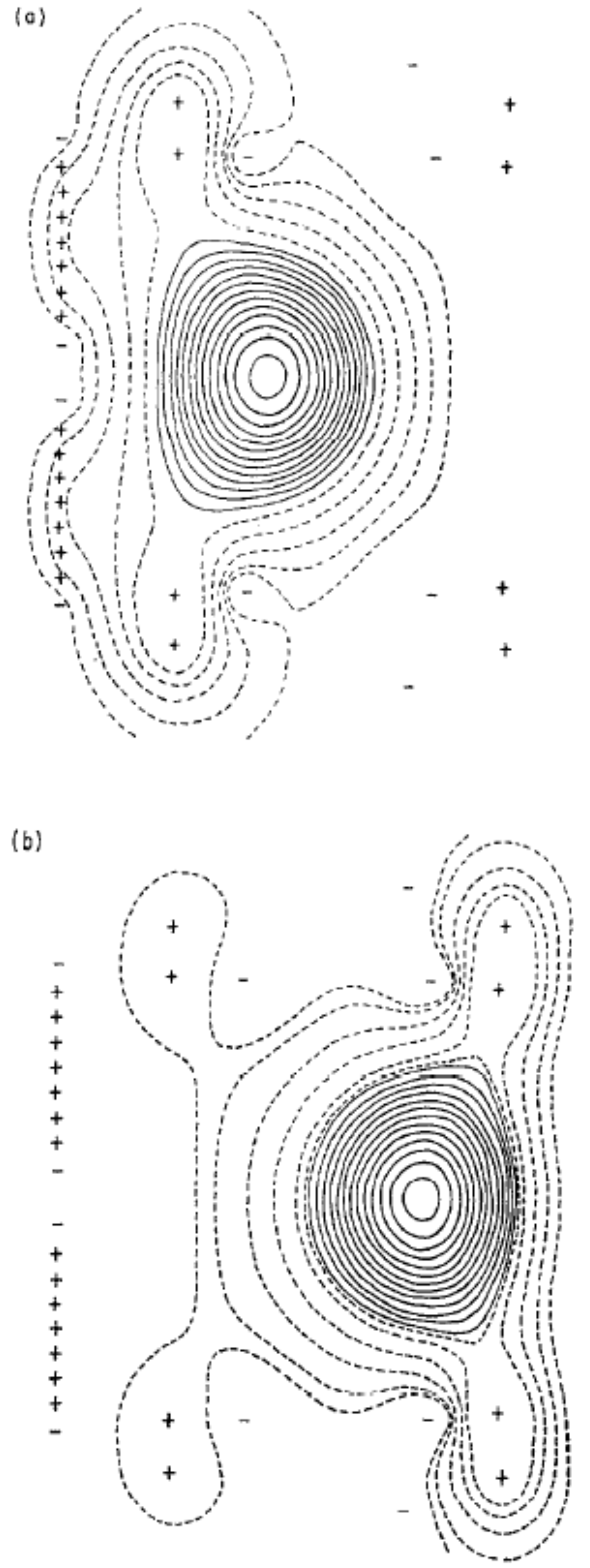


FIG. 4. A typical PLT equilibrium with $\beta_p = 0.23$ and $1.05 < q < 5.3$. The solid curve marks the position of the vacuum vessel. The pluses and minuses denote the poloidal field coils.





IG. 6. Typical PDX equilibria, showing that the plasma can be attac
s or (b) the outside divertor coils. Here $\alpha = \beta = 2$.


MACWORL

April 1989

The Macintosh® Ma

make it come true. The ny's HyperAnimator lets nt or scan in faces and pe whatever you want say; the text is automat- onverted to speech syn- zed with the lip move- Mom's voice won't like hers, however, un- u can get her to digitize give you the disk. If you type in her words, she'll distinctly computerish. If n't have time to draw or ces, you can use the sta- ine characters provided htStar.

erAnimator's best ad- man, however, is proba- ert, a talking head star- Disney's new version of bsent-Minded Professor," n Sunday nights as part World of Disney" TV se- oftware developers Jay n and Harry Anderson Albert to play the pro- electronic sidekick.

htStar has also added the Animator's audio feature -mail package called Mail, which lets your ani- coworkers deliver their es in persona.

erAnimator lists for . For further informa- ntact BrightStar Tech- in Bellevue, Washing- 206/885-5446.

Garrison

aMack

The U.S. Department

looks like rings of lightning and lasts for 200 millionths of a second each time it appears. During every flash of the light- ning ring, laser beams and magnetic sensors measure the dynamics of the ring at least once every millionth of a sec- ond, producing more than a megabyte of data.

The data is digitized, loaded into VAX computers, transferred to Macs, and then analyzed with TokaMack, an application Mauel wrote using Apple's Mac- intosh Programmer's Workshop (MPW) and MacApp. The analy- sis also involves four Cray com-

tohydrodynamic instabilities, called kinks, occurring in the plasma ring at very high pressure.

TokaMack is named after Tokamak, an earlier device de- veloped by the late Russian physicist L. A. Artsimovich for magnetically confining ring lightning. Mauel offers the soft- ware as freeware to scientists doing similar research through- out the world. TokaMack re- quires a Mac II.

For further information, con- tact Michael Mauel at Columbia University, at 212/854-4455.

–*Ann Garrison*

JONATHAN LEVINE

Columbia professor Michael Mauel leads a team using Macs to test the use of magnetic force fields to confine hydrogen plasma.

81 **Macworld News**

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- *And Now Presenting* . . . and More II connect to si services for quick turnaro
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Plus, Jasmine's new BackF dem support, SE/30 color cal imaging, C++ and a n and more.

Next Lecture

- More Piel / Chapter 5: “Fluid” Equations
- “Frozen-in” flux condition
- Alfvén wave
- Monday, October 9: Homework #5