Lecture7:

Plasma Physics 1

APPH E6101x
Columbia University

Last Lecture

- Coulomb collisions and the Coulomb logarithm
- Ambipolar diffusion
- (“Classical”) Transport in a magnetized plasma

Review: $e_z \times (A \times e_z) = A_\perp \sim \langle\rho\rho\rangle \cdot A$

FITZPATRICK #1

(A) $$\langle \rho\rho \rangle = \frac{\mu_\perp^2}{\Omega^2} \left\langle \begin{bmatrix} \cos^2\gamma & -\cos\gamma\sin\gamma \\ -\sin\gamma\cos\gamma & \sin^2\gamma \end{bmatrix} \right\rangle$$

$$= \frac{\mu_\perp^2}{2\Omega^2} \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{\overline{\overline{I}} - \hat{b}\hat{b}} \propto \frac{\mu}{e\Omega} \quad \left\{ \mu = \frac{e u_\perp^2}{2\Omega} \right.$$

Review: $e_z \times (A \times e_z) = A_\perp \sim \langle \rho\rho \rangle \cdot A$

(c) $e\langle \bar{u} \cdot (\bar{\rho}\cdot\bar{\nabla})\bar{E} \rangle = -e\Omega \langle \hat{b} \cdot \bar{\rho} \times (\bar{\rho}\cdot\bar{\nabla})\bar{E} \rangle$

BUT $\bar{\rho}\bar{\rho} = \frac{\mu}{e\Omega}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ SO

$$= -\mu \hat{b} \cdot (\nabla_\perp \times \bar{E})$$

$$= -\mu \frac{\partial |B|}{\partial t}$$

PERPENDICULAR IDENTITY MATRIX

Review: “Drift” with Collions

The elastic scattering of electrons on atoms is almost isotropic [68]. Therefore, on average, the electron loses its mean momentum $m_\mathrm{e}\bar{\mathbf{v}}_\mathrm{e}$ and we can write the equation of motion for an *average electron*

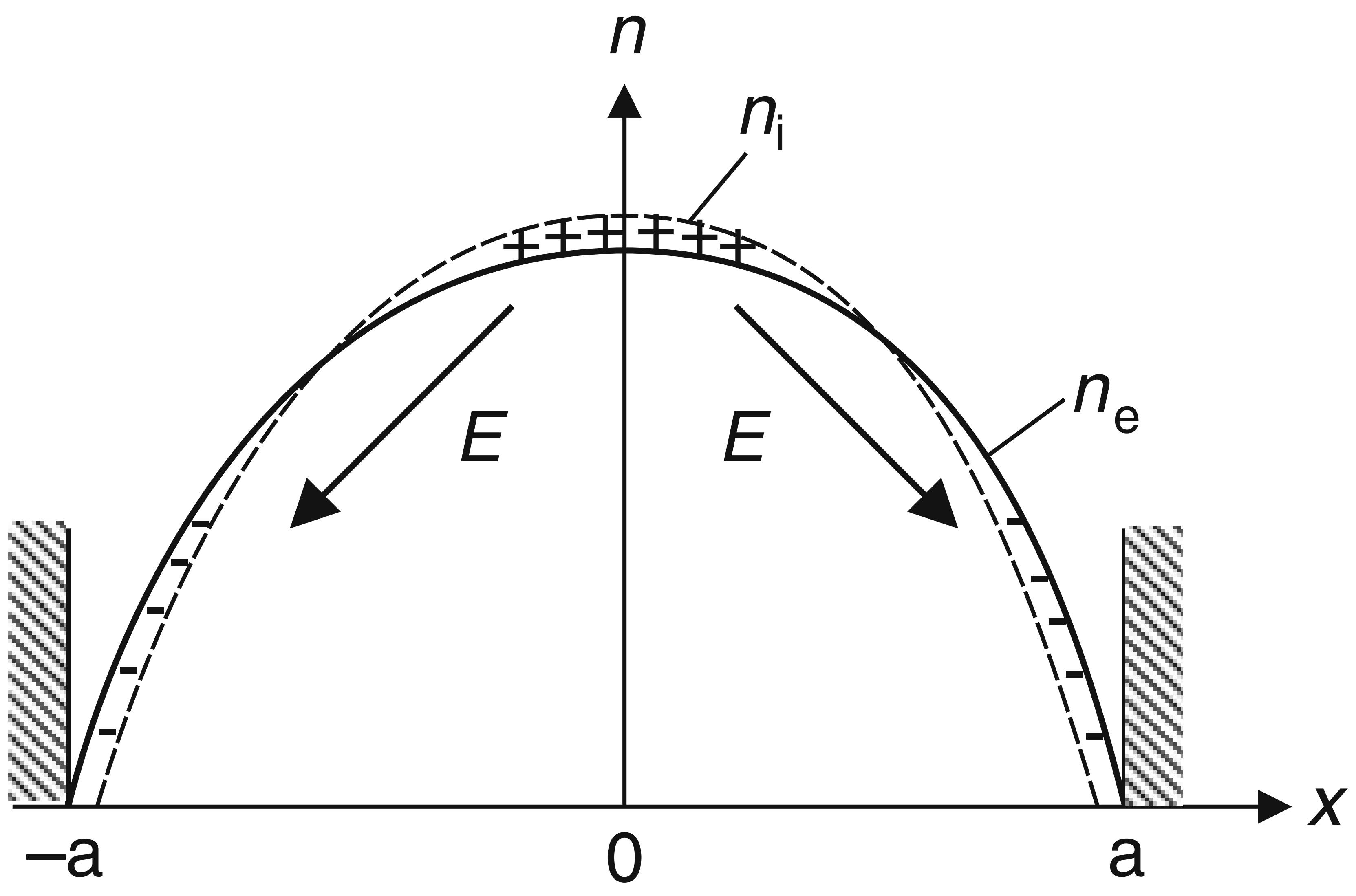
$$m\dot{\bar{v}} = -eE - m\bar{v}\nu_\mathrm{m}\,. \tag{4.25}$$

This average electron now moves in $-E$-direction. The quantity $\nu_\mathrm{m} = 1/\tau_\mathrm{coll}$ is the effective collision frequency for momentum transfer.

$$v_\mathrm{d} = -\frac{e}{m\nu_\mathrm{m}}E = -\mu_\mathrm{e}E$$

Review: Ambipolar Diffusion

$$\mu_{\mathrm{e}} = \frac{e}{m_{\mathrm{e}}\nu_{\mathrm{m,e}}} \quad ; \quad \mu_{\mathrm{i}} = \frac{e}{m_{\mathrm{i}}\nu_{\mathrm{m,i}}}$$



$$\boldsymbol{\Gamma}_{\mathrm{e,i}} = \pm n\mu_{\mathrm{e,i}}\mathbf{E} - D_{\mathrm{e,i}}\nabla n$$

$$\frac{D}{\mu} = \frac{k_{\mathrm{B}}T}{e}$$

$$\Gamma_e = \Gamma_i$$

$$-n\mu_e E - D_e\nabla n = n\mu_i E - D_i\nabla n$$

OR

$$E = -\frac{1}{n}\nabla n\left(\frac{D_e - D_i}{\mu_e + \mu_i}\right)$$

$$\approx -\frac{kT}{e}\frac{\nabla n}{n}$$

$$= +\mu_e\left(\frac{D_e - D_i}{\mu_e + \mu_i}\right)\nabla n - D_e\nabla n$$

$$= -\left[\frac{D_e\mu_i}{\mu_e + \mu_i} + \frac{\mu_e D_i}{\mu_e + \mu_i}\right]\nabla n$$

$$\approx -\left[2D_i\right]\nabla n$$

Review: Collisional "Drift" with a Magnetic Field

$$\bar{J} = q n \bar{v} = \bar{\bar{\sigma}} \cdot \bar{E}$$

FIND AVERAGE DRIFT WITH STATIC E, B AND COLLISIONS

$$\left.\begin{aligned} \frac{d\bar{v}}{dt} \approx 0 &\approx \frac{q}{m}\bar{E} + \omega_c \bar{V}_\perp \times \hat{b} - \nu_m \bar{V}_\perp \\ 0 &\approx \frac{q}{m}\bar{E} \times \bar{B} - \omega_c \bar{V}_\perp B - \nu_m \bar{V}_\perp \times \bar{B} \end{aligned}\right\} \perp B$$

$$\left. 0 \approx \frac{q}{m} E_\parallel - \nu_m V_\parallel \right\} \parallel B$$

Review: Collisional "Drift" with a Magnetic Field

$$\bar{V}_\perp - \left(\frac{\omega_c}{\nu_m}\right) \bar{V}_\perp \times \vec{b} = \frac{q}{m\nu_m} \bar{E}_\perp$$

$e_z \times (A \times e_z) = A_\perp$

$$\left(\frac{\omega_c}{\nu_m}\right)^2 \left[\bar{V}_\perp + \left(\frac{\nu_m}{\omega_c}\right) \bar{V}_\perp \times \vec{b} = \frac{q}{m \cancel{\nu_m}} \frac{\bar{E} \times \vec{b}}{\omega_c} \right]$$

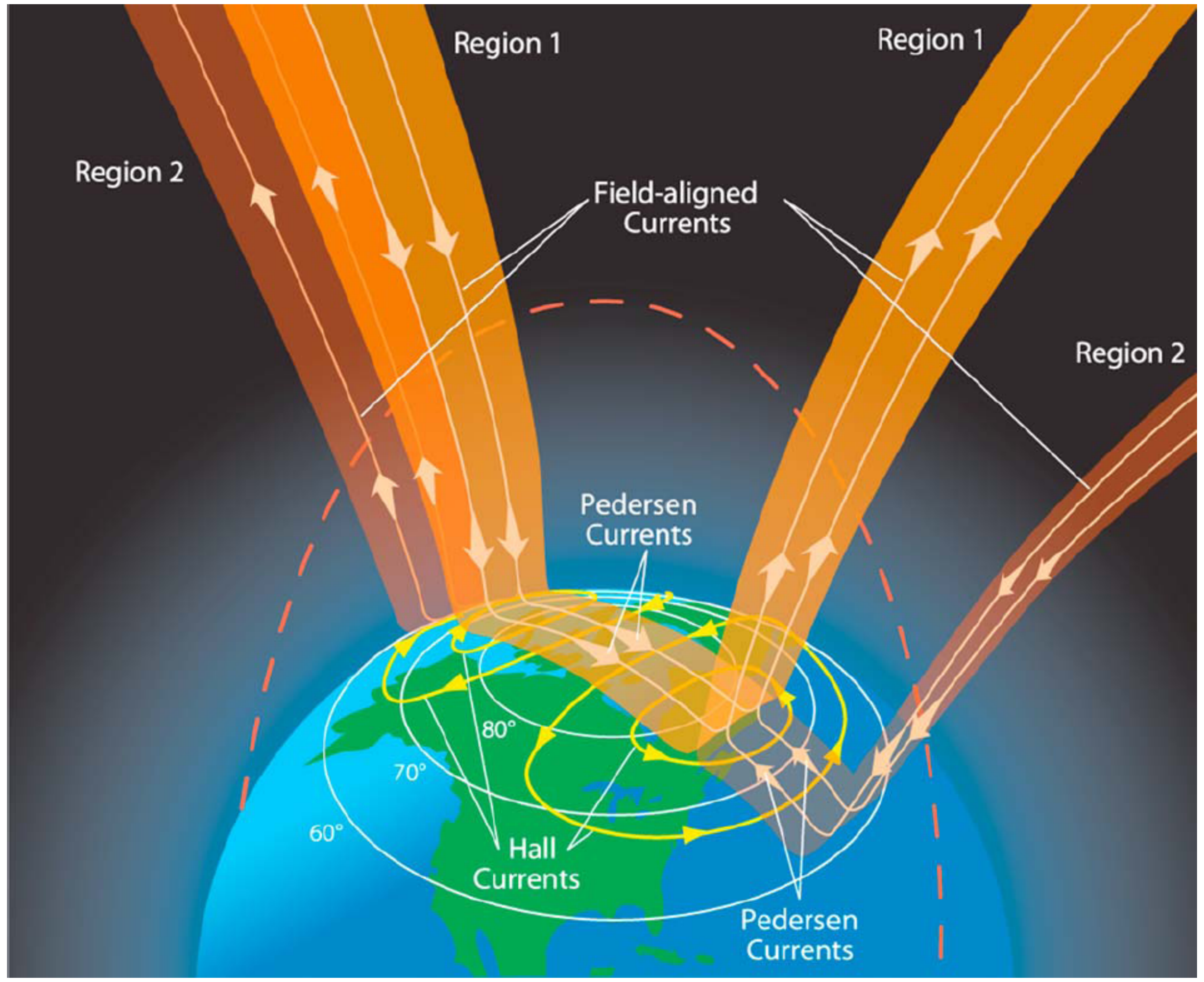
$$\left(1 + \left(\frac{\omega_c}{\nu_m}\right)^2\right) \bar{V}_\perp = \frac{q}{m\nu_m} \bar{E}_\perp + \frac{q}{m\nu_m} \left(\frac{\omega_c}{\nu_m}\right) \bar{E}_\perp \times \vec{b}$$

$$\left\{ V_\perp = \frac{q}{m\nu_m} \begin{pmatrix} \frac{1}{1 + \left(\frac{\omega_c}{\nu_m}\right)^2} & \frac{\omega_c / \nu_m}{1 + \left(\frac{\omega_c}{\nu_m}\right)} \\ -\frac{\omega_c/\nu_m}{(\ )} & \frac{1}{1 + (\ )} \end{pmatrix} \cdot \bar{E}_\perp \right.$$

QED

Conductivity in a Magnetized Plasma

1879 EDWIN HALL
(PHD DISSERTATION)



$$
\begin{array}{cc} & \begin{array}{cc} \text{Pedersen} & \text{Hall} \end{array} \\
\begin{pmatrix} j_x \\ j_y \\ j_z \end{pmatrix} = \sigma_{\mathrm{i}} &
\begin{pmatrix} \dfrac{1}{1+(\omega_{\mathrm{ci}}/\nu_{\mathrm{m,i}})^2} & \dfrac{\omega_{\mathrm{ci}}/\nu_{\mathrm{m,i}}}{1+(\omega_{\mathrm{ci}}/\nu_{\mathrm{m,i}})^2} & 0 \\ \dfrac{-\omega_{\mathrm{ci}}/\nu_{\mathrm{m,i}}}{1+(\omega_{\mathrm{ci}}/\nu_{\mathrm{m,i}})^2} & \dfrac{1}{1+(\omega_{\mathrm{ci}}/\nu_{\mathrm{m,i}})^2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}
\end{array}
$$

$\dfrac{q^2 n}{m \nu_m}$

$\omega_c = \dfrac{qB}{m}$

$\nu_m = v_{TH}/\lambda_{mfp}$

UNMAGNETIZED $\omega_{ci} \ll \nu_m$

$\lambda_{mfp} \ll \rho_c$

$$
\bar{J} = \sigma \begin{pmatrix} 1 & \dfrac{\omega_{ci}}{\nu_m} & 0 \\ -\dfrac{\omega_c}{\nu_m} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}
$$

← SMALL

MAGNETIZED $\omega_{ci} \gg \nu_m$

$\lambda_{mfp} \gg \rho_c$

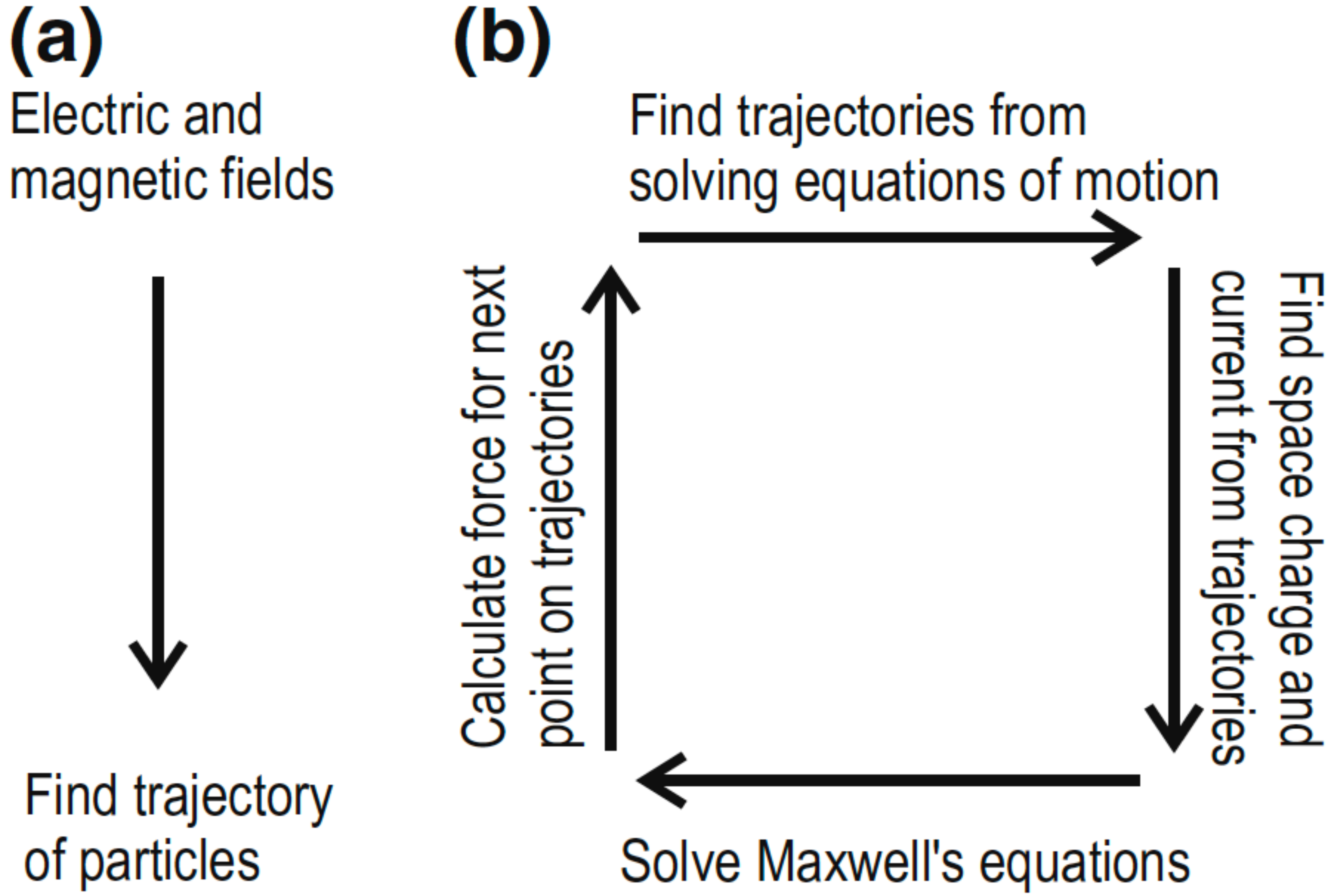
$$
\bar{J} = \sigma \begin{pmatrix} 0 & \dfrac{\nu_m}{\omega_{ci}} & 0 \\ -\dfrac{\nu_m}{\omega_{ci}} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}
$$

$$
\frac{\nu_m \sigma}{\omega_c} = qn\left(\frac{1}{B}\right) \quad (E \times B)
$$

Outline Today

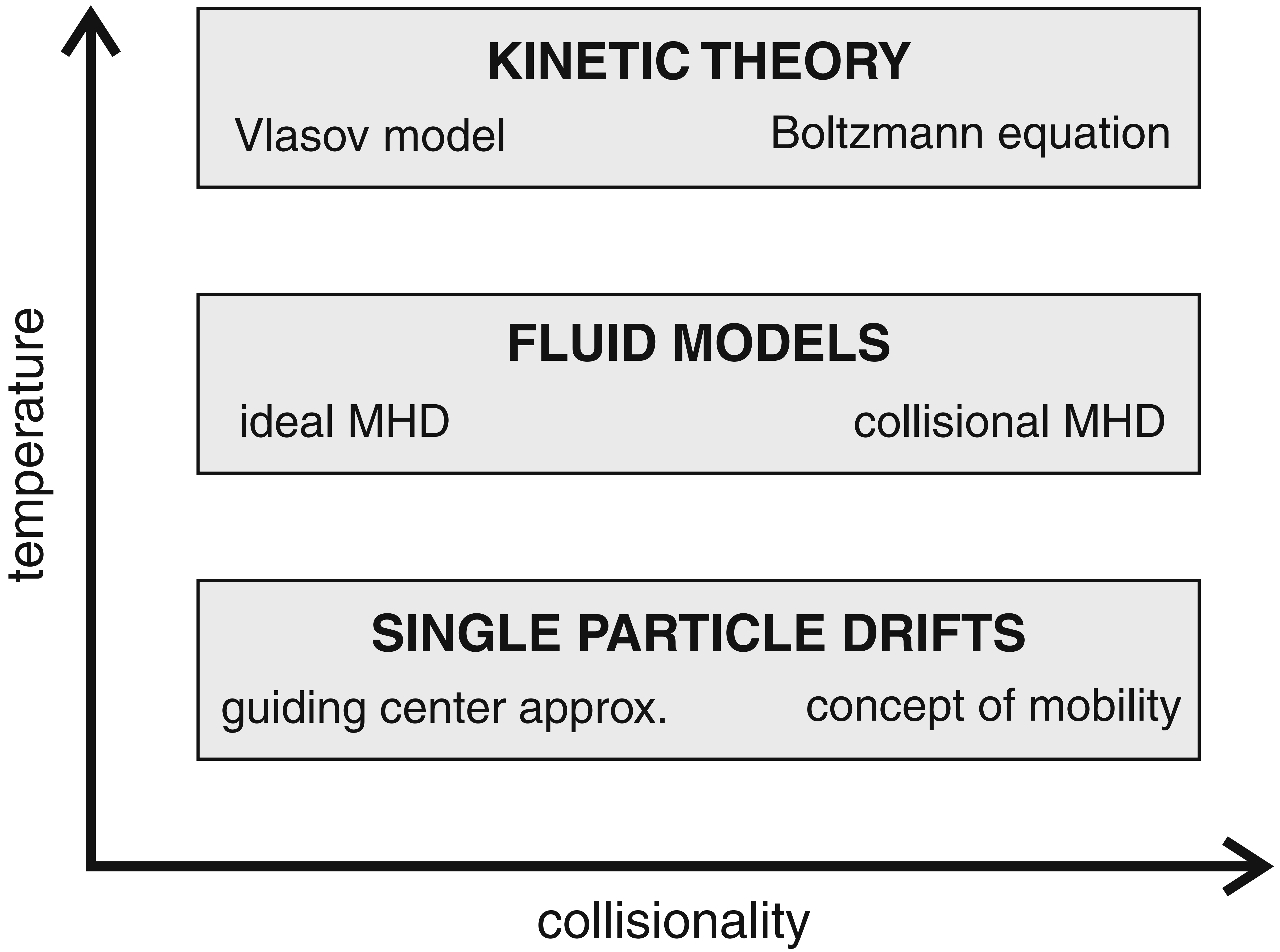
- Moments of the distribution function
- Fluid equations (“two fluid”)
- The “closure problem”
- MHD equations (“single fluid”)

Piel, Ch. 5: Fluid Theory



(over simplified)

Piel, Ch. 9: Kinetic Theory



(over simplified)

Particle Phase Space

$$\Delta N^{(\alpha)} = f^{(\alpha)}(\mathbf{r}, \mathbf{v}, t)\Delta x \Delta y \Delta z \, \Delta v_x \Delta v_y \Delta v_z$$

$$f^{(\alpha)}(\mathbf{r}.\mathbf{v}, t) = \sum_k \delta(\mathbf{r} - \mathbf{r}_k(t))\delta(\mathbf{v} - \mathbf{v}_k(t))$$

but single particle effects are relatively small
when the plasma parameter is large

So f(x,v,t) becomes "*nearly smooth*"

Particle Phase Space

$$N^{(\alpha)} = \iint f^{(\alpha)}(\mathbf{r}, \mathbf{v}, t)\, \mathrm{d}^3 r\, \mathrm{d}^3 v$$

$$n^{(\alpha)}(\mathbf{r}, t) = \int f^{(\alpha)}(\mathbf{r}, \mathbf{v}, t) \mathrm{d}^3 v$$

$$u = (1/n) \int v f\, \mathrm{d}v$$

$$p = \int m (v - u)^2 f\, \mathrm{d}v$$

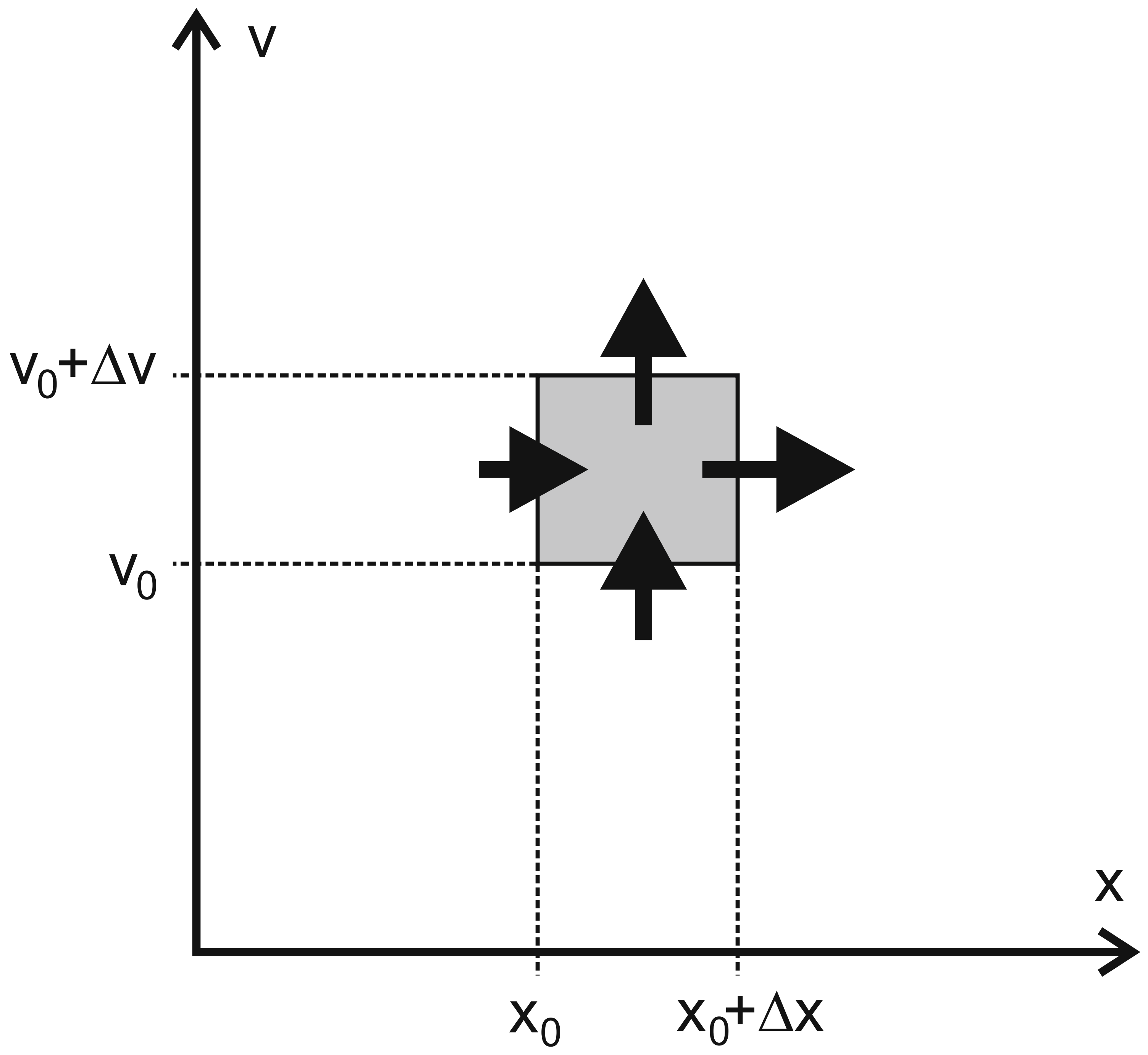
plus higher-order
velocity-space
moments

Vlasov Equation (6 dimensions)

$$\frac{\partial f}{\partial t} = -\frac{\partial}{\partial x}(f v_x) - \frac{\partial}{\partial v_x}(f a)$$

$$\frac{\partial f}{\partial t} + v_x \frac{\partial f}{\partial x} + a \frac{\partial f}{\partial v_x} = 0$$

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla_r f + \mathbf{a} \cdot \nabla_v f = 0$$



What is the "Boltzman Eq"?
How does it differ from the Vlasov Eq?

Conservation Property of Vlasov Equation

$$\frac{\partial N}{\partial t} = \frac{\partial}{\partial t}\iint f \,\mathrm{d}x\mathrm{d}v = -\iint v\frac{\partial f}{\partial x}\,\mathrm{d}x\mathrm{d}v - \iint a\frac{\partial f}{\partial v}\,\mathrm{d}x\mathrm{d}v$$

$$= -\int_{-\infty}^{\infty}\mathrm{d}v\left\{\left[vf\right]_{x=-\infty}^{x=\infty} - \int_{-\infty}^{\infty} f\,\frac{\mathrm{d}v}{\mathrm{d}x}\,\mathrm{d}x\right\}$$

$$-\int_{-\infty}^{\infty}\mathrm{d}x\left\{\left[af\right]_{v=-\infty}^{v=\infty} - \int_{-\infty}^{\infty} f\,\frac{\mathrm{d}a}{\mathrm{d}v}\,\mathrm{d}v\right\} = 0$$

by definition…

$$\frac{\partial f}{\partial t} = -\frac{\partial}{\partial x}(f v_x) - \frac{\partial}{\partial v_x}(f a)$$

Particle Trajectories and the Vlasov Equation

$$\frac{\mathrm{d}f(\mathbf{x}(t), \mathbf{v}(t), t)}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial \mathbf{x}} \cdot \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} + \frac{\partial f}{\partial \mathbf{v}} \cdot \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}$$
$$= \frac{\partial f}{\partial t} + \frac{\partial f}{\partial \mathbf{x}} \cdot \mathbf{v} + \frac{\partial f}{\partial \mathbf{v}} \cdot \frac{q}{m}(\mathbf{E} + \mathbf{v} \times \mathbf{B}) = 0$$

Also…

The Vlasov equation is invariant under time reversal, $(t \to -t)$, $(\mathbf{v} \to -\mathbf{v})$.

Velocity-Space Moments of the Vlasov Equation

$$\frac{\partial f}{\partial t} = -\frac{\partial}{\partial x}(f v_x) - \frac{\partial}{\partial v_x}(f a)$$

$$\rho = \sum_\alpha q_\alpha \int f_\alpha \mathrm{d}^3 v$$

$$0 = \frac{\partial}{\partial t} \int f \,\mathrm{d}v + \frac{\partial}{\partial x} \int v\, f \,\mathrm{d}v + a[f]_{-\infty}^{\infty} = \frac{\partial n}{\partial t} + \frac{\partial}{\partial x}(nu)$$

$$\mathbf{j} = \sum_\alpha q_\alpha \int \mathbf{v}_\alpha f_\alpha \mathrm{d}^3 v$$

$$0 = \frac{\partial}{\partial t} \int m v f \,\mathrm{d}v + \frac{\partial}{\partial x} \int v^2 f \,\mathrm{d}v + a \int v \frac{\partial f}{\partial v} \,\mathrm{d}v$$

(Moment Hierarchy!)

The Fluid Closure Problem…

$$
\begin{aligned}
0 &= \frac{\partial}{\partial t}\int m v f \, \mathrm{d}v + \frac{\partial}{\partial x}\int v^2 f \, \mathrm{d}v + a \int v \frac{\partial f}{\partial v} \, \mathrm{d}v \\
&= \frac{\partial}{\partial t}\int m v f \, \mathrm{d}v + \frac{\partial}{\partial x}\left[\int m(v-u)^2 f \, \mathrm{d}v + nmu^2\right] \\
&\quad + a \left([vf]_{-\infty}^{\infty} - \int f \frac{\mathrm{d}v}{\mathrm{d}v} \, \mathrm{d}v \right) \\
&= \frac{\partial}{\partial t}(nmu) + \frac{\partial p}{\partial x} + u \frac{\partial}{\partial t}(nmu) + (nmu)\frac{\partial u}{\partial x} - nma \\
&= nm\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right) + \frac{\partial p}{\partial x} - nma \, ,
\end{aligned}
$$

(See Chapter 5 Piel)

$$
p = \int m(v-u)^2 f \, \mathrm{d}v
$$

What is the dynamics of p?

Chapter 5
Fluid Models

Chapter 5
Fluid Models

$$-\frac{\partial N}{\partial t} = I_N(x+\Delta x) - I_N(x) \approx \frac{\partial I_N}{\partial x}\Delta x \,. \tag{5.6}$$

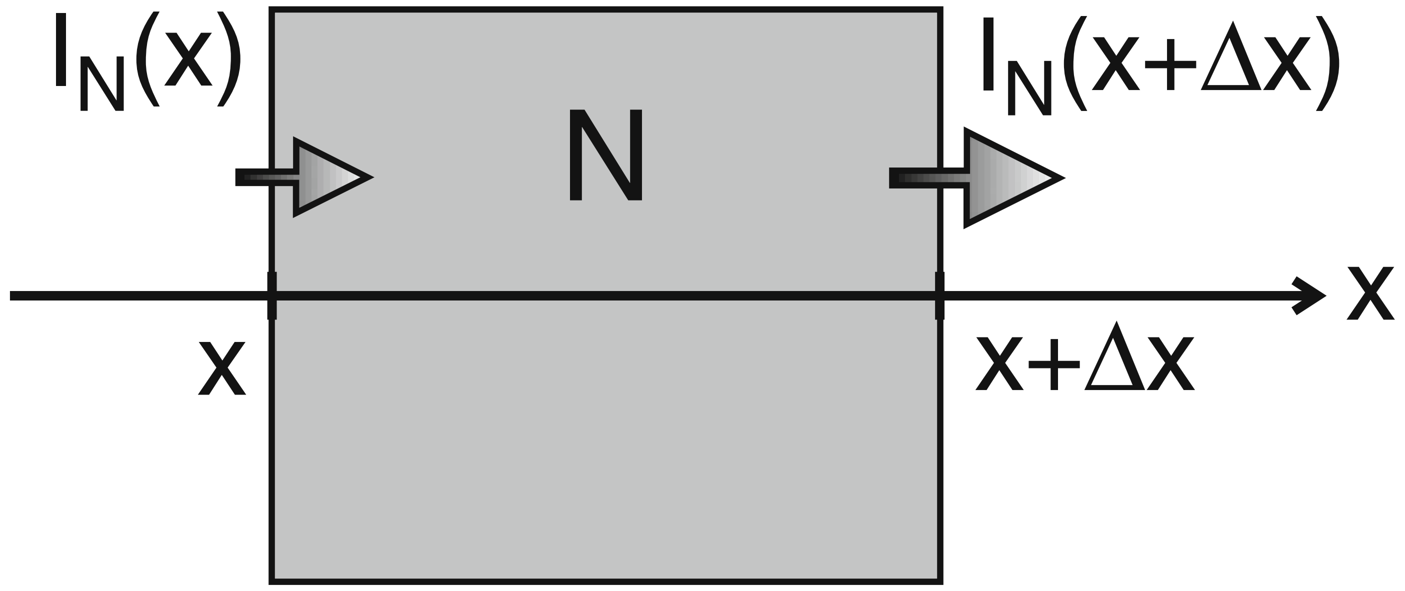


Fig. 5.3 Definitions used to derive the continuity equation

$$\frac{\partial n}{\partial t} + \frac{\partial (n\, u_x)}{\partial x} = 0$$

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\mathbf{u}) = 0$$

Convective Derivative

$$\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} = \frac{\partial \mathbf{u}}{\partial t} + \frac{\partial \mathbf{u}}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial \mathbf{u}}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\partial \mathbf{u}}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}t}$$

$$\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} = \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u}$$

Eulerian Description

(Field Variables)

FLUID DYNAMICS IS MOST COMMONLY DESCRIBED USING FIELD QUANTITIES

$$\rho(\bar{x}, t) \quad \bar{u}(\bar{x}, t) \quad T(\bar{x}, t)$$

A PROPERTY AT $\bar{x}$ EVOLVES IN TIME.

THIS IS THE EULERIAN DESCRIPTION.

LAGRANGIAN $\Longleftrightarrow$ EULERIAN

ARE DIFFERENT WAYS TO DESCRIBE THE SAME THING. THEY ARE EQUIVALENT!

EULERIAN	LAGRANGIAN
$\rho(\bar{x}, t)$	$\rho(\bar{x}_0, t) = \rho[x_0(\bar{x}, t), t]$
$\rho(\bar{x}, t) = \rho[\bar{x}(\bar{x}_0, t), t]$	$\rho(\bar{x}_0, t)$

Integral Relations

$$\frac{d}{dt}\int_{V(t)} f\,dV = \begin{cases} \int_V \frac{\partial f}{\partial t} dV \text{ (fixed V)} \\ \int_V \frac{\partial f}{\partial t} dV + \int_A d\mathbf{A}\cdot\mathbf{U}\,f \end{cases}$$

Osborn Reynolds

(Reynolds Transport Theory)

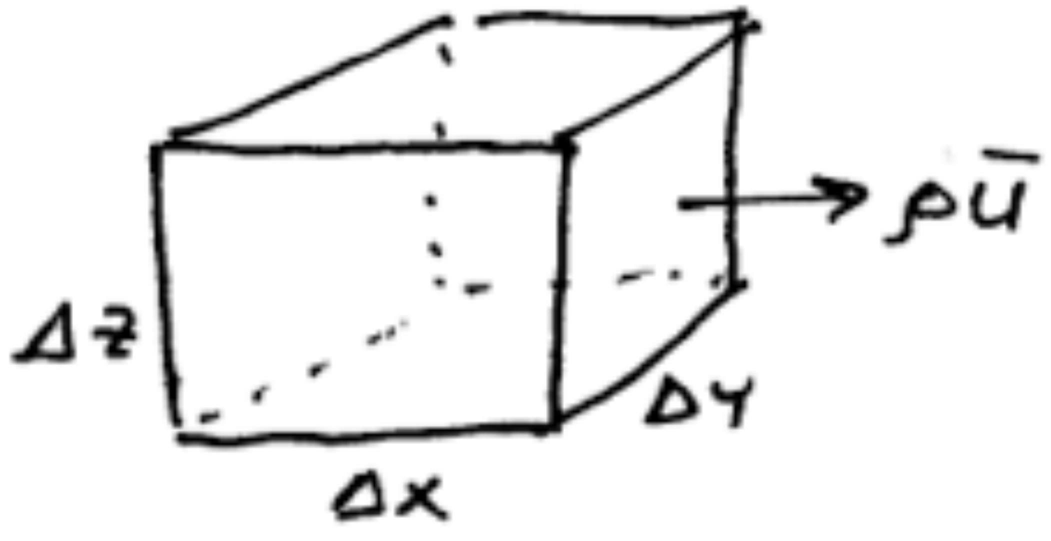
1842-1912

Integral Relations

(Section 4.2)

$$\begin{aligned}
\frac{d}{dt}\int_{V(t)} (\rho f) dV &= \int_V \frac{\partial(\rho f)}{\partial t} dV + \int_A d\mathbf{A}\cdot\mathbf{U}(\rho f) \\
&= \int_V \left(\rho\frac{\partial f}{\partial t} + f\frac{\partial \rho}{\partial t} + \nabla\cdot\rho f\mathbf{U}\right) dV \\
&= \int_V \left(\rho\frac{\partial f}{\partial t} + \boxed{f\frac{\partial \rho}{\partial t} + f\nabla\cdot\rho\mathbf{U}} + \rho\mathbf{U}\cdot\nabla f\right) dV \\
&= \int_V \rho\left(\frac{\partial f}{\partial t} + \mathbf{U}\cdot\nabla f\right) dV \\
&= \int_V \rho\frac{df}{dt} dV
\end{aligned}$$

CONSERVATION OF Mass



$$\frac{\partial \rho}{\partial t} + \bar{\nabla}\cdot(\rho\bar{u}) = 0$$

$$\frac{\partial}{\partial t}(\Delta V \rho) = -\sum_{\substack{\text{ALL} \\ \text{SURFACES}}} \rho\bar{u}\cdot\Delta\bar{A}$$

$$= -\frac{\partial}{\partial x_i}(\rho u_i)\Delta x_j \Delta x_k$$

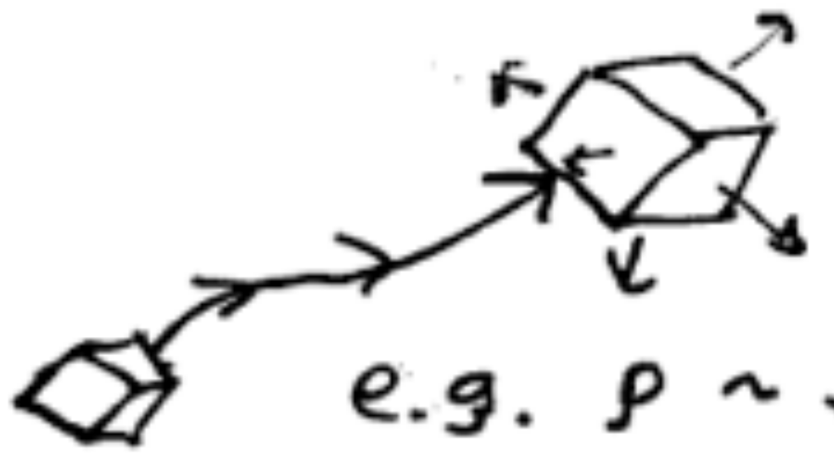
$$= -\Delta V\, \nabla\cdot(\rho\bar{u})$$

$$\underbrace{\frac{\partial \rho}{\partial t} + (\bar{u}\cdot\bar{\nabla})\rho}_{\displaystyle\frac{D\rho}{Dt}} = -\rho\nabla\cdot\bar{u}$$

$$\frac{D\rho}{Dt} = -\rho\nabla\cdot\bar{u}$$

$$\frac{D\rho}{\rho} = -Dt\,(\nabla\cdot\bar{u})$$

!!



e.g. $\rho \sim e^{-\int dt' \nabla\cdot\bar{u}}$

Continuity

Newton's Law

NEWTON'S LAW FOR A PARTICLE

$$\bar{F} = m\bar{a}$$

$$\bar{F} = \frac{d}{dt}(m\bar{v})$$

NEWTON'S LAW FOR A FLUID:

$$\bar{F} = \frac{D}{Dt}(\rho\bar{u})$$

$(\rho\bar{u})u_j$

FLUX OF MOMENTUM FROM FLUID ELEMENT

$$= \frac{\partial}{\partial t}(\rho\bar{u}) + \bar{\nabla}\cdot(\rho\bar{u}\bar{u})$$

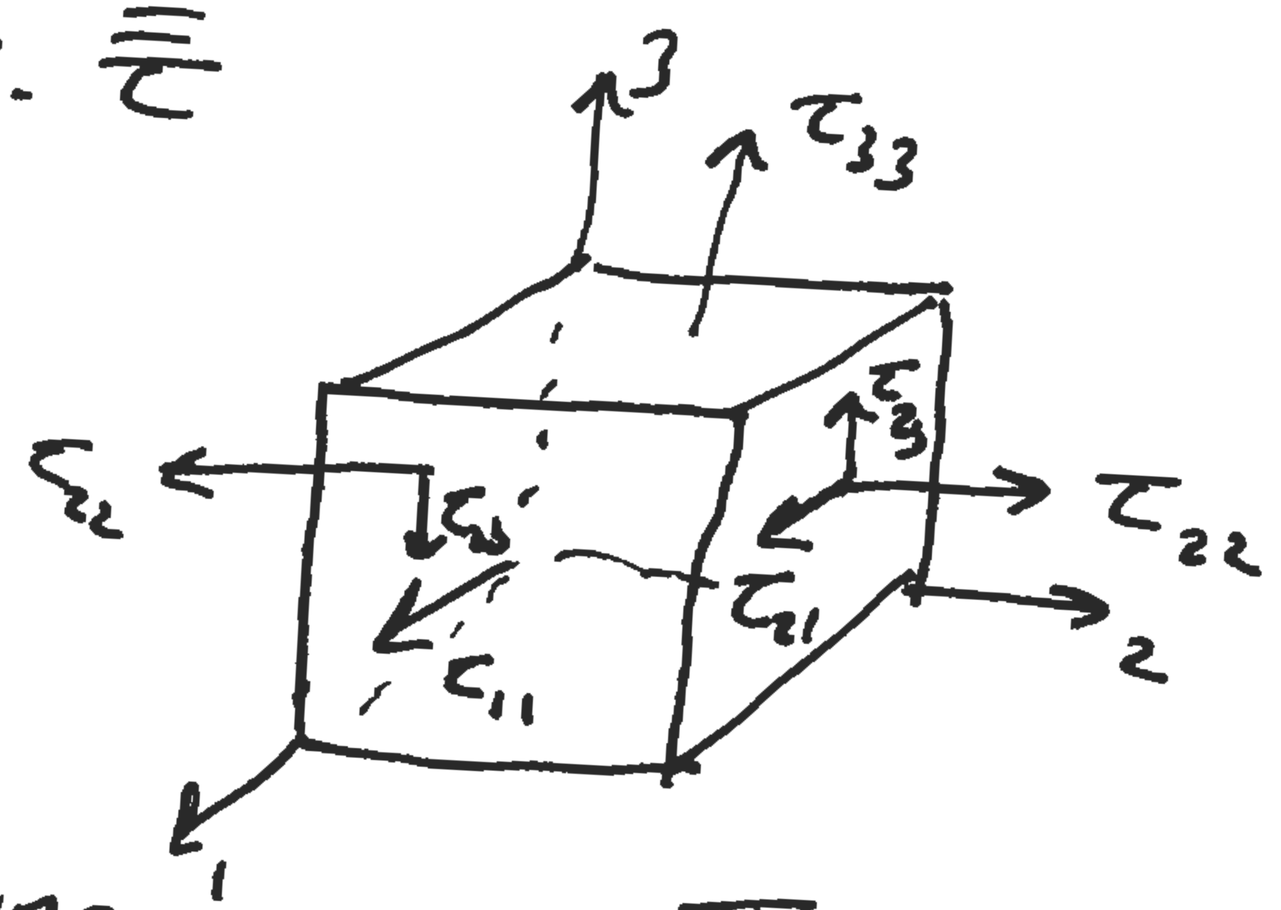
$$= \frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_j}(\rho u_i u_j)$$

$$= \rho\left[\frac{\partial\bar{u}}{\partial t} + (\bar{u}\cdot\bar{\nabla})\bar{u}\right] + \bar{u}\underbrace{\left[\frac{\partial\rho}{\partial t} + \bar{\nabla}\cdot(\rho\bar{u})\right]}_{=0 \text{ CONSERVATION OF MASS}}$$

$$= \rho\frac{D\bar{u}}{Dt}$$

Momentum

$$\rho\left(\frac{\partial \bar{u}}{\partial t} + (\bar{u}\cdot\nabla)\bar{u}\right) = \rho\bar{g} + \nabla\cdot\bar{\bar{\tau}}$$



$\bar{\bar{\tau}}$ = STRESS TENSOR

= USUALLY SYMMETRIC

• = HAS NORMAL STRESS ~ PRESSURE

= HAS SHEAR STRESS ~ (OFF DIAGONAL)

<u>GRADIENTS</u> OF STRESS PRODUCE <u>FORCE</u>

τ_{ij} ← DIRECTION
↑
WHICH FACE

τ_{ij} IS POSITIVE WHEN DIRECTED IN DIRECTION OF AXIS

$\tau_{ii} > 0$ IMPLIES TENSILE STRESS

$\tau_{ii} < 0$ IMPLIES COMPRESSIVE STRESS

τ_{ij} $(i \neq j)$ ARE SHEAR STRESSES

Fluid Mechanics
(Eulerian and Lagrangian Description part 1)

Watch me!

https://youtu.be/zyyzEiKftIM?si=dBo1EKIqRpKLTRAx

Momentum/Force Equation

$$\Delta I_{\mathrm{P}} = (m v_x) \Delta n(v_x) |v_x| \Delta y \Delta z$$

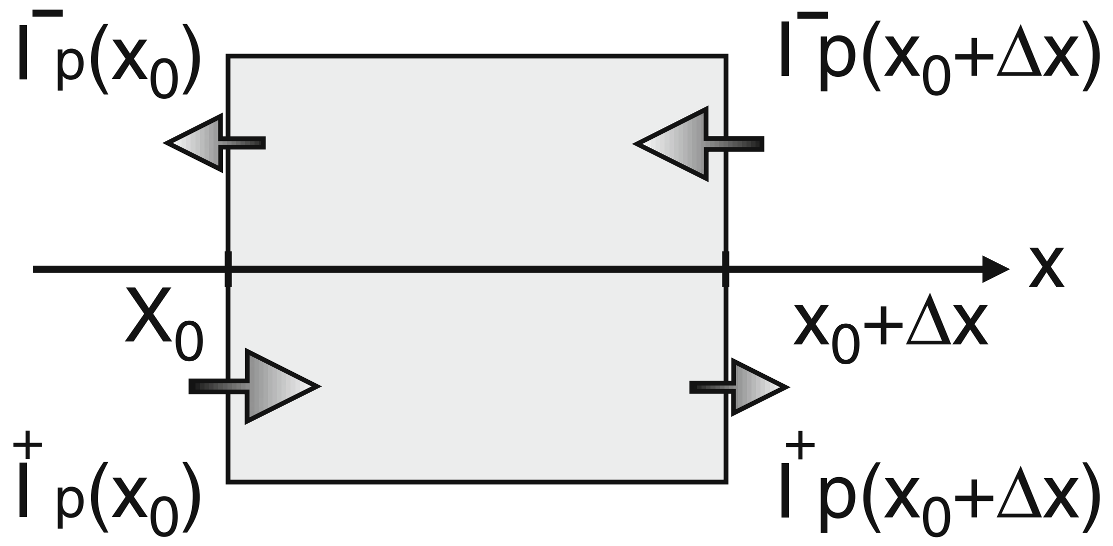


Fig. 5.4 Calculation of pressure forces

$$\text{Gain at } x_0 : \quad I_{\mathrm{P}}^{+}(x_0) = \sum_{v_x > 0} \left[\Delta n(v_x)(m v_x)|v_x| \right]_{x_0} \Delta y \Delta z$$

$$\text{Loss at } x_0 : \quad I_{\mathrm{P}}^{-}(x_0) = \sum_{v_x < 0} \left[\Delta n(v_x)(m v_x)|v_x| \right]_{x_0} \Delta y \Delta z$$

$$\text{Gain at } x_0 + \Delta x_0 : \; I_{\mathrm{P}}^{-}(x_0 + \Delta x) = \sum_{v_x < 0} \left[\Delta n(v_x)(m v_x)|v_x| \right]_{x_0 + \Delta x} \Delta y \Delta z$$

$$\text{Loss at } x_0 + \Delta x_0 : \; I_{\mathrm{P}}^{+}(x_0 + \Delta x) = \sum_{v_x > 0} \left[\Delta n(v_x)(m v_x)|v_x| \right]_{x_0 + \Delta x} \Delta y \Delta z$$

$$\frac{\partial P_x}{\partial t} = I_{\mathrm{P}}^{+}(x_0) - I_{\mathrm{P}}^{+}(x_0 + \Delta x) + I_{\mathrm{P}}^{-}(x_0 + \Delta x) - I_{\mathrm{P}}^{-}(x_0)$$

Pressure Gradient Force

$$\frac{\partial P_x}{\partial t} = -m \sum_{v_x=-\infty}^{\infty} \left([\Delta n(v_x) v_x^2]_{x_0+\Delta x} - [\Delta n(v_x) v_x^2]_{x_0} \right)$$

$$= -m \frac{\partial}{\partial x} \left(n \langle v_x^2 \rangle \right) \Delta x \Delta y \Delta z$$

$$\frac{\partial}{\partial t}(nmu_x) = -m \frac{\partial}{\partial x} \left[n \left(\langle u_x^2 \rangle + 2u_x \langle \tilde{v}_x \rangle + \langle \tilde{v}_x^2 \rangle \right) \right]$$

$$n \langle v_x^2 \rangle = \int f(v_x) v_x^2 \mathrm{d}v_x$$

~Pressure

$$(1/2)\, m \langle \tilde{v}_x^2 \rangle = (1/2)\, k_\mathrm{B} T$$

$$v_x = u_x + \tilde{v}_x$$

$$\frac{\partial}{\partial t}(nmu_x) = -\frac{\partial}{\partial x} \left[nmu_x^2 + nk_\mathrm{B} T \right]$$

Momentum/Force Equation

Memorize me!

Isotropic P

$$nm\left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}\right) = nq(\mathbf{E} + \mathbf{u}\times\mathbf{B}) - \nabla p$$

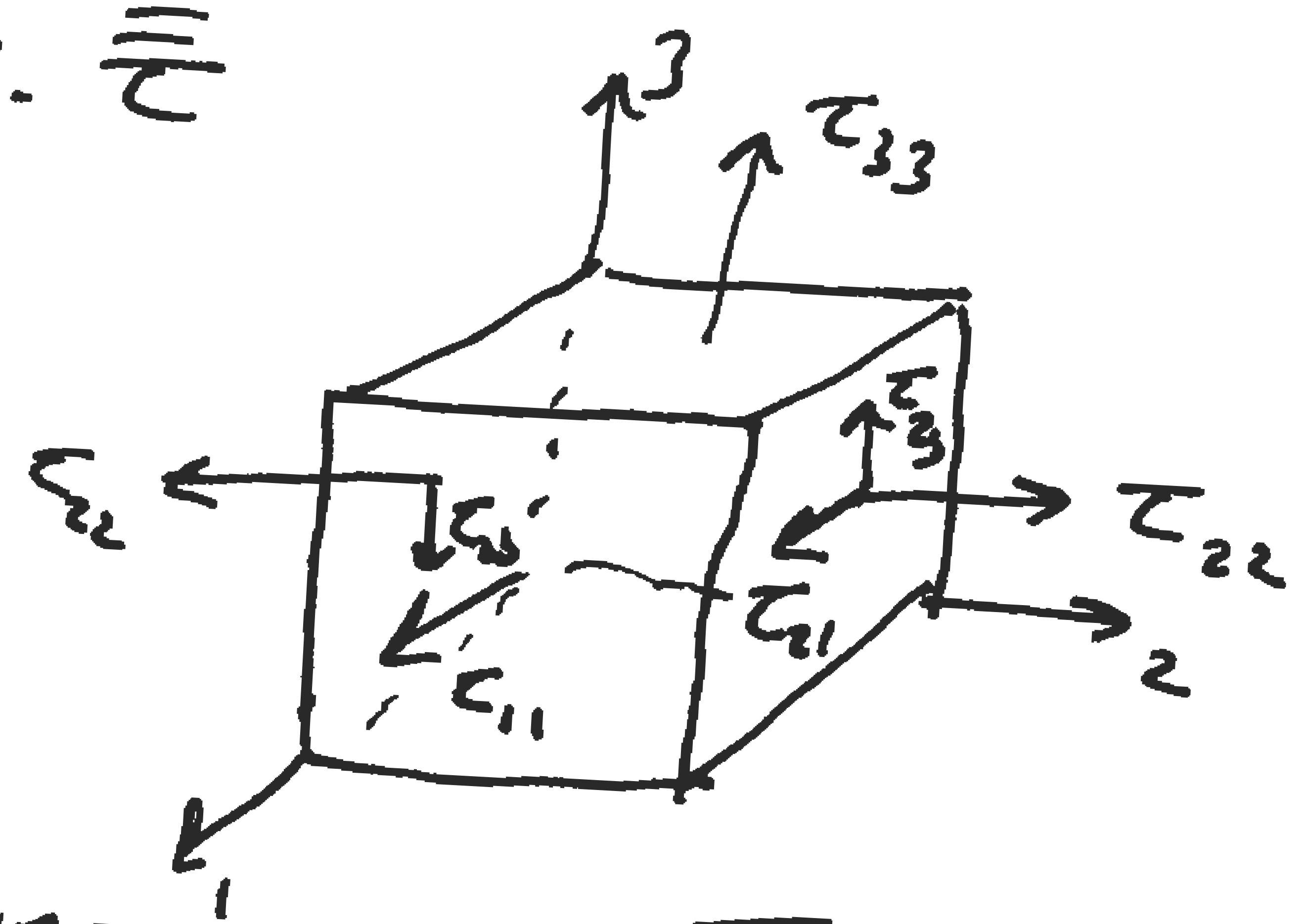
Stress Tensor

$$P_{ij} = nm\langle \tilde{v}_i \tilde{v}_j \rangle$$

Momentum

Stress Tensor

$$\rho\left(\frac{\partial \bar{u}}{\partial t} + (\bar{u}\cdot\nabla)\bar{u}\right) = \rho\bar{g} + \nabla\cdot\bar{\bar{\tau}}$$



$\bar{\bar{\tau}}$ = STRESS TENSOR

= USUALLY SYMMETRIC

= HAS NORMAL STRESS ~ PRESSURE

= HAS SHEAR STRESS ~ (OFF DIAGONAL)

GRADIENTS OF STRESS PRODUCE FORCE

τ_{ij}

i ← WHICH FACE

j ← DIRECTION

τ_{ij} IS POSITIVE WHEN DIRECTED IN DIRECTION OF AXIS

$\tau_{ii} > 0$ IMPLIES TENSILE STRESS

$\tau_{ii} < 0$ IMPLIES COMPRESSIVE STRESS

τ_{ij} $(i \neq j)$ ARE SHEAR STRESSES

"Two Fluid"

$$nm_{\mathrm{i}}\frac{\partial \mathbf{u}_{\mathrm{i}}}{\partial t} = ne(\mathbf{E} + \mathbf{u}_{\mathrm{i}} \times \mathbf{B}) - \nabla p_{\mathrm{i}} + nm_{\mathrm{i}}\mathbf{g} + n\nu_{\mathrm{ei}}m_{\mathrm{e}}(\mathbf{u}_{\mathrm{e}} - \mathbf{u}_{\mathrm{i}})$$

$$nm_{\mathrm{e}}\frac{\partial \mathbf{u}_{\mathrm{e}}}{\partial t} = -ne(\mathbf{E} + \mathbf{u}_{\mathrm{e}} \times \mathbf{B}) - \nabla p_{\mathrm{e}} + nm_{\mathrm{e}}\mathbf{g} + n\nu_{\mathrm{ei}}m_{\mathrm{e}}(\mathbf{u}_{\mathrm{i}} - \mathbf{u}_{\mathrm{e}})$$

isotropic

MHD: "Single Fluid" from "Two Fluid"

$$nm_{\mathrm{i}}\frac{\partial \mathbf{u}_{\mathrm{i}}}{\partial t} = ne(\mathbf{E}+\mathbf{u}_{\mathrm{i}}\times\mathbf{B}) - \nabla p_{\mathrm{i}} + nm_{\mathrm{i}}\mathbf{g} + n\nu_{\mathrm{ei}}m_{\mathrm{e}}(\mathbf{u}_{\mathrm{e}}-\mathbf{u}_{\mathrm{i}})$$

$$nm_{\mathrm{e}}\frac{\partial \mathbf{u}_{\mathrm{e}}}{\partial t} = -ne(\mathbf{E}+\mathbf{u}_{\mathrm{e}}\times\mathbf{B}) - \nabla p_{\mathrm{e}} + nm_{\mathrm{e}}\mathbf{g} + n\nu_{\mathrm{ei}}m_{\mathrm{e}}(\mathbf{u}_{\mathrm{i}}-\mathbf{u}_{\mathrm{e}})$$

Sum:

$$\rho_{\mathrm{m}}\frac{\partial \mathbf{v}_{\mathrm{m}}}{\partial t} = \mathbf{j}\times\mathbf{B} - \nabla p + \rho_{\mathrm{m}}\mathbf{g}$$

$$\rho_{\mathrm{m}} = n(m_{\mathrm{i}}+m_{\mathrm{e}}) \qquad p = p_{\mathrm{e}}+p_{\mathrm{i}} \qquad \mathbf{v}_{\mathrm{m}} = \frac{(m_{\mathrm{i}}\mathbf{u}_{\mathrm{i}}+m_{\mathrm{e}}\mathbf{u}_{\mathrm{e}})}{m_{\mathrm{e}}+m_{\mathrm{i}}}$$

$$\mathbf{j} = ne(\mathbf{u}_{\mathrm{i}}-\mathbf{u}_{\mathrm{e}})$$

Difference:

$$\frac{m_{\mathrm{i}}m_{\mathrm{e}}}{e}\frac{\partial \mathbf{j}}{\partial t} = e\rho_{\mathrm{m}}\left(\mathbf{E}+\mathbf{v}_{\mathrm{m}}\times\mathbf{B} - \frac{\nu_{\mathrm{ei}}m_{\mathrm{e}}}{ne^2}\mathbf{j}\right) - m_{\mathrm{i}}\mathbf{j}\times\mathbf{B} - m_{\mathrm{e}}\nabla p_{\mathrm{i}} + m_{\mathrm{i}}\nabla p_{\mathrm{e}}$$

$$\mathbf{E}+\mathbf{v}_{\mathrm{m}}\times\mathbf{B} = \eta\mathbf{j} + \frac{1}{ne}(\mathbf{j}\times\mathbf{B} - \nabla p_{\mathrm{e}})$$

Next Lecture

- More: Piel / Chapter 5: “Fluid” Equations
- Equilibrium and force balance

➡Alfvén wave