Lecture 5:

Plasma Physics 1

APPH E6101x
Columbia University

Outline

✓ Homework #2

✓ Charged Particle Drifts (Summary)

✓ Gyro-averaging

✓ More details from last week

➡ Adiabatic Invariants (Part 1)

➡ Examples (Part 2):

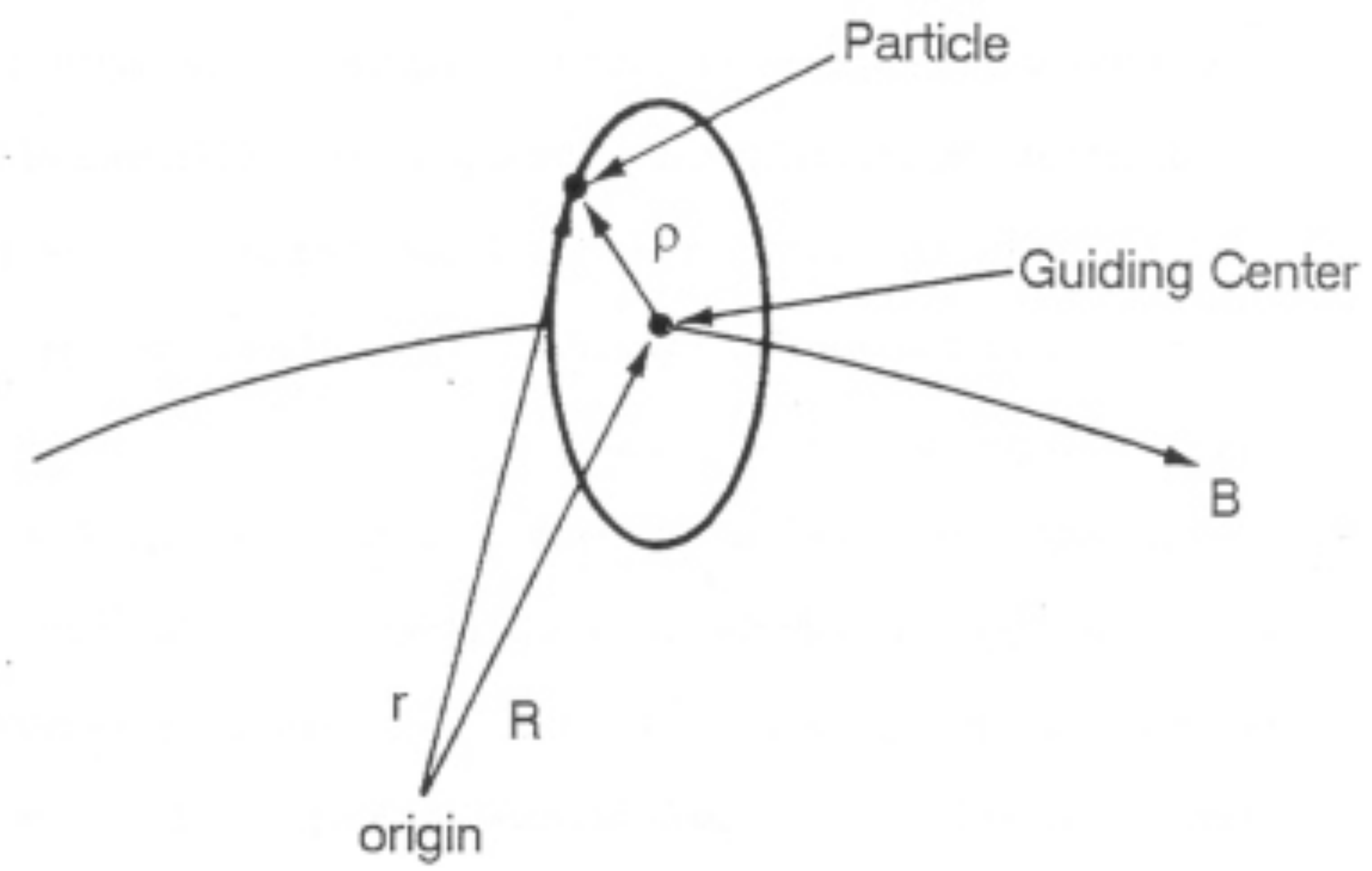
- Mirror
- Magnetosphere (dipole)
- Tokamak

Adiabatic Invariants (Part 1)

$$(\mu,\ J,\ \psi)$$

See Fitzpatrick Sec. 2.8

Motion with weakly inhomogeneous B…



Use Bernstein's notation…

(See "handout": http://sites.apam.columbia.edu/courses/apph6101x/Plasma1-Adiabatic-Handout.pdf)

Magnetic Force does not Change Kinetic Energy

$$\bar{v} \cdot \left[\frac{d\bar{v}}{dt} = \frac{q}{m} \bar{v} \times \bar{B} \right]$$

$$\frac{d}{dt}\left(\frac{1}{2} v^2\right) = \frac{q}{m} \bar{v} \cdot (\bar{v} \times \bar{B}) = 0$$

$$v^2 = \bar{v} \cdot \bar{v} = v_x^2 + v_y^2 + v_z^2$$

$B(x)$ changing slowly…

HERE, WE START WITH . . .

$$\frac{d\dot{\bar{r}}}{dt} = \dot{\bar{r}} \times \Big(\underbrace{\bar{\Omega} + (\bar{\rho}\cdot\bar{\nabla})\bar{\Omega} + \cdots}_{\text{TAYLOR EXPAND ABOUT GYRO CENTER}} \Big)$$

$$\bar{\Omega}(\bar{r}) = \bar{\Omega}(\bar{R} + \bar{\rho})$$

$$\approx \bar{\Omega}(\bar{R}) + (\bar{\rho}\cdot\bar{\nabla})\bar{\Omega} + \cdots$$

$$\underset{(1)}{\frac{d\dot{\bar{r}}}{dt}} \approx \underset{(2)}{\dot{\bar{r}} \times \bar{\Omega}} + \underset{(3)}{\dot{\bar{r}} \times (\bar{\rho}\cdot\bar{\nabla})\bar{\Omega}} + \cdots$$

Drifts (separately)

NOTE THAT THERE ARE OSCILLATING (GYRATING) AND NON-OSCILLATING TERMS.

SO LET'S LOOK ONLY AT THE <u>DRIFTS</u>. SO WE LOOK AT THE <u>GYRO-AVERAGED</u> DRIFT MOTION

1ST TWO ARE EASY..

$$\left\langle \frac{d\dot{\vec{r}}}{dt} \right\rangle_\theta = \frac{d\bar{V}}{dt} \quad (1) \qquad ; \qquad \left\langle \dot{\vec{r}} \times \vec{\Omega} \right\rangle_\theta = \bar{V} \times \vec{\Omega} \quad (2)$$

Drifts (separately)

(3) LAST IS THE <u>PRODUCT</u> OF TWO OSCILLATING TERMS...

$$\frac{1}{2\pi}\int_0^{2\pi} d\theta \; \dot{\bar{r}} \times (\bar{\rho}\cdot\bar{\nabla})\bar{r} = \frac{1}{2\pi}\int_0^{2\pi} d\theta \; \bar{w} \times (\bar{\rho}\cdot\bar{\nabla})\bar{r}$$

WHICH GIVES SOMETHING LIKE A <u>CONSTANT</u> PLUS SOMETHING THAT GOES LIKE $2\omega_c$! WE WANT THE CONSTANT PART...

NOW, I WANT TO AVERAGE THE

$\bar{w}\,\bar{\rho}$ TERMS

SO I GOING TO DO THE FOLLOWING

$$\bar{w} \times (\bar{\rho}\cdot\bar{\nabla})\bar{r} = \bar{w}\,\underbrace{(\bar{\rho}\cdot\bar{\nabla})} \times \bar{r}$$

THIS IS JUST AN OPERATOR WHICH ACTS ON $\bar{r}$ (IT'S THE SAME THING)

Drifts (separately)

THEN USING

$$\bar{\omega} = \omega\left[\hat{e}_1 \cos\theta - \hat{e}_2 \sin\theta\right]$$

$$\bar{\rho} = \frac{\omega}{\Omega}\left[\hat{e}_1 \sin\theta + \hat{e}_2 \cos\theta\right]$$

$$\langle \bar{\omega}\bar{\rho} \rangle_\theta = \frac{\omega^2}{\Omega} \frac{1}{2\pi}\int_0^{2\pi} d\theta \left[\hat{e}_1 \cos\theta - \hat{e}_2 \sin\theta\right]\left[\hat{e}_1 \sin\theta + \hat{e}_2 \cos\theta\right]$$

$$\to \frac{\omega^2}{\Omega} \frac{1}{2\pi}\int_0^{2\pi} d\theta \left[\hat{e}_1\hat{e}_2 \cos^2\theta - \hat{e}_2\hat{e}_1 \sin^2\theta\right]$$

OR

$$\langle \bar{\omega}\bar{\rho} \rangle_\theta = \frac{\omega^2}{2\Omega}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\langle \dot{\hat{1}} \times (\vec{\rho}\cdot\vec{\nabla})\vec{\Omega} \rangle_\theta = \frac{\omega^2}{2\Omega}\left[\hat{e}_1\hat{e}_2 - \hat{e}_2\hat{e}_1\right]\cdot\vec{\nabla}\times\vec{\Omega}$$

$$= -\frac{\omega^2}{2\Omega}\left(\hat{b}\times\vec{\nabla}\right)\times\vec{\Omega}$$

since $\quad \hat{b}\times\vec{\nabla} = (\hat{e}_1\times\hat{e}_2)\times\vec{\nabla}$ OK

$$= \hat{e}_2\,\hat{e}_1\cdot\vec{\nabla} - \hat{e}_1\,\hat{e}_2\cdot\vec{\nabla}$$

$$\frac{d\vec{V}}{dt} = \vec{V}\times\vec{\Omega} - \frac{\omega^2}{2\Omega}\left(\hat{b}\times\vec{\nabla}\right)\times\vec{\Omega}$$

$$= \vec{V}\times\vec{\Omega} - \frac{\omega^2}{2\Omega}\left[\vec{\nabla}(\vec{\Omega}\cdot\hat{b}) - \hat{b}\,\underbrace{\vec{\nabla}\cdot\vec{\Omega}}_{\to\, 0}\right]$$

$\nabla\cdot\vec{B} = 0$

NOW DEFINE $\boxed{\mu = \dfrac{\frac{1}{2}m\omega^2}{B}}$

$$\boxed{\frac{d\vec{V}}{dt} = \vec{V}\times\vec{\Omega} - \frac{\mu}{m}\vec{\nabla}|B|}$$

!!

Magnetic Force does not Change Kinetic Energy

$$\hat{b} \cdot \left[\frac{d\bar{v}}{dt} = \bar{v} \times \bar{\Omega} - \frac{\mu}{m} \bar{\nabla} |B| \right]$$

$$\frac{dv_\parallel}{dt} = -\frac{\mu}{m} \hat{b} \cdot \nabla |B| = -\frac{\mu}{m} \frac{\partial |B|}{\partial s}$$

WHERE $\hat{b} \cdot \bar{\nabla} = \frac{\partial}{\partial s}$

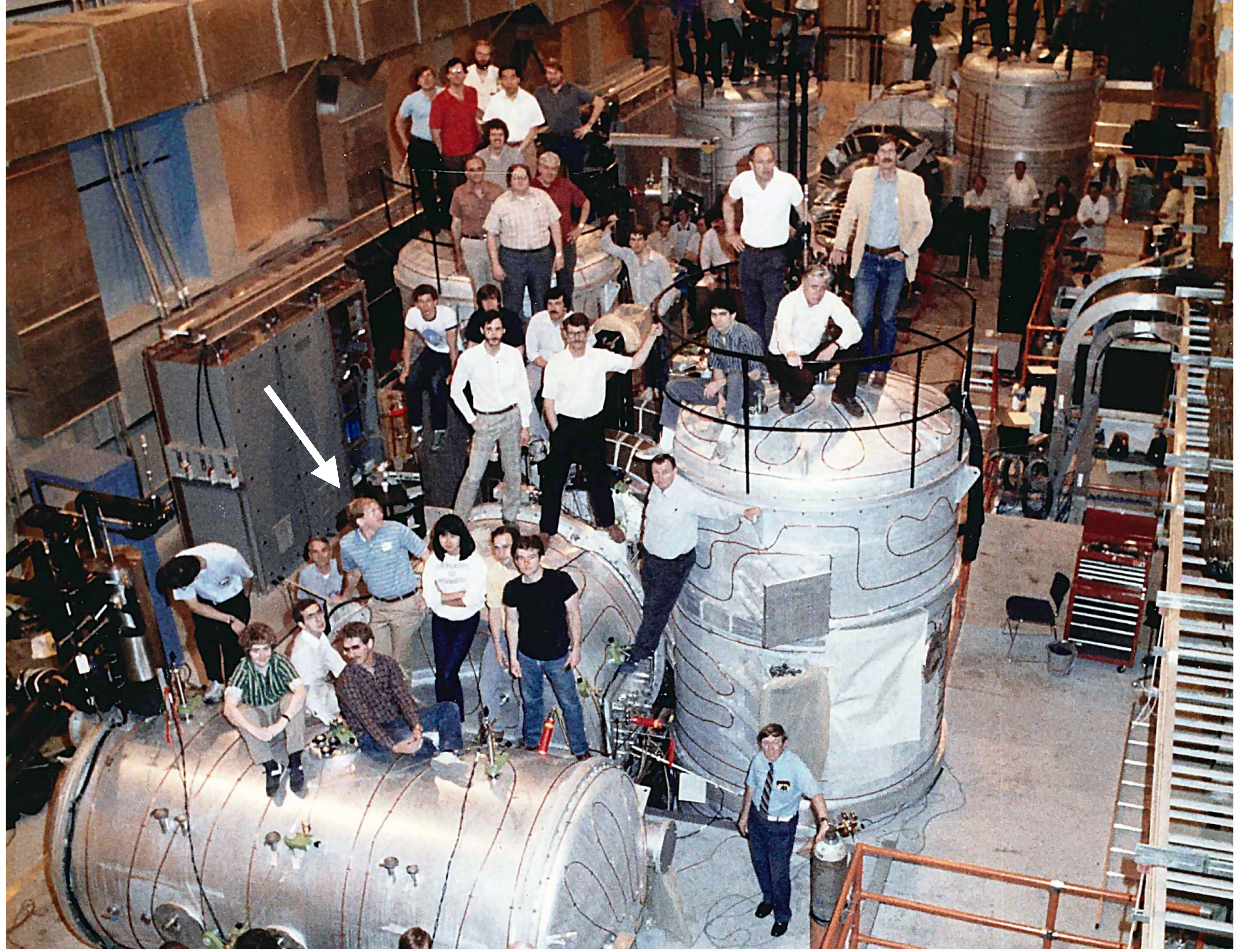
MULTIPLY BY $v_\parallel$ ...

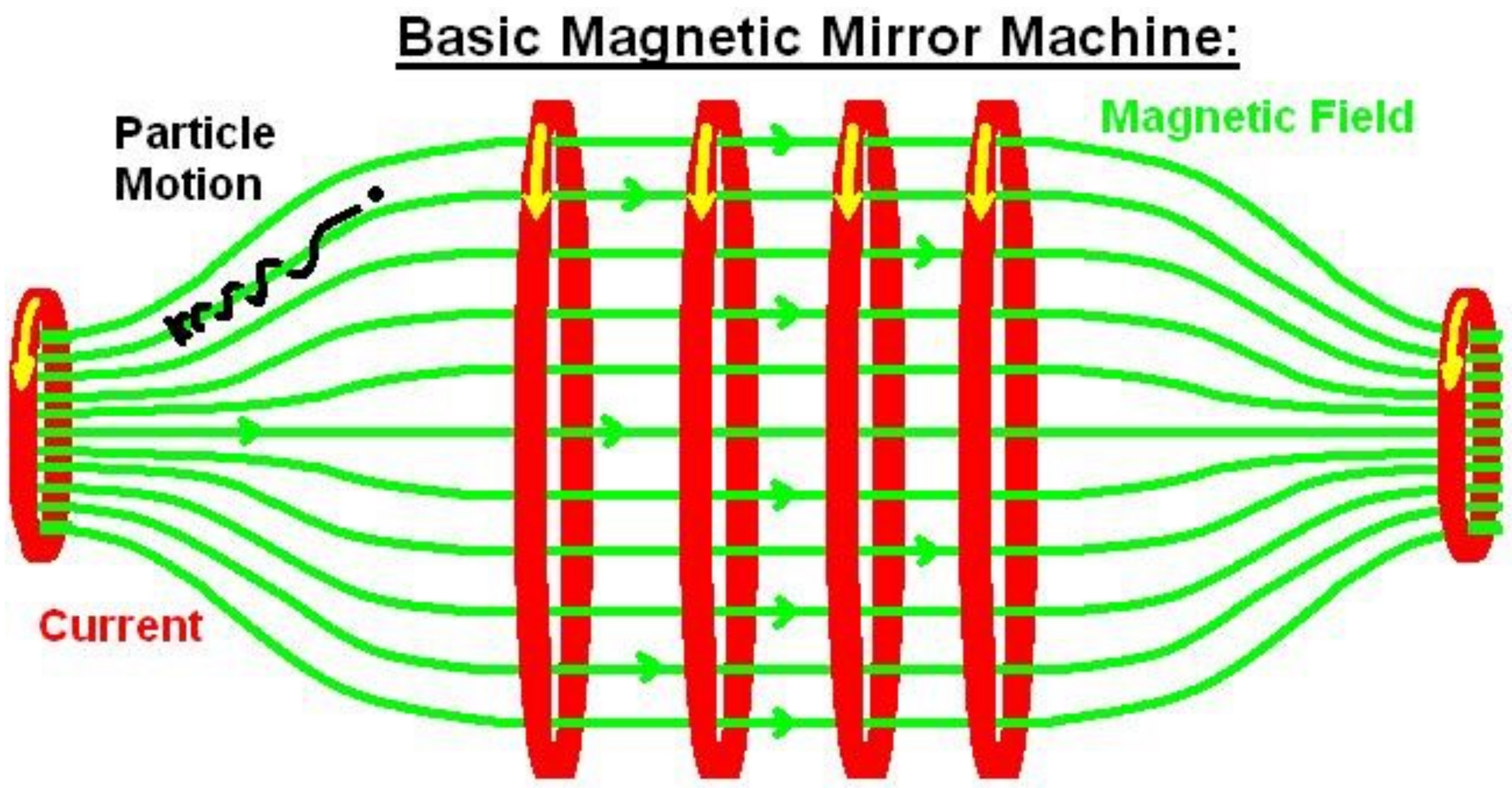
$$v_\parallel \frac{dv_\parallel}{dt} = -\frac{\mu}{m} v_\parallel \frac{\partial |B|}{\partial s} = -\frac{\mu}{m} \frac{\partial |B|}{\partial t}$$

THEN, IF μ = CONSTANT

$$\frac{d}{dt}\left(\frac{1}{2} v_\parallel^2 + \frac{\mu}{m} B\right) = \frac{d}{dt}\left(\frac{1}{2} v_\parallel^2 + \frac{1}{2} v_x^2 + \frac{1}{2} v_y^2\right) = 0$$

Magnetic Mirror(s)





$$\frac{d}{dt}\left(\frac{1}{2}V_{||}^{2}(s)+\frac{\mu}{m}B(s)\right)=0$$

Adiabatic Invariants (Part 2)

$$(\mu,\ J,\ \psi)$$

Harmonic Oscillator $\frac{d^2x}{dt^2} + \omega^2 x = 0$

- Separate descriptions of perpendicular and parallel motion, IF ω = CONSTANT

$$x(t) = x_0 \sin(\omega t + \phi)$$
$$\frac{dx}{dt} = \omega x_0 \cos(\omega t + \phi)$$

- Fast gyration around B

- Slow perpendicular drift of gyro center

KINETIC ENERGY PER UNIT MASS $= \frac{1}{2}\left(\frac{dx}{dt}\right)^2$

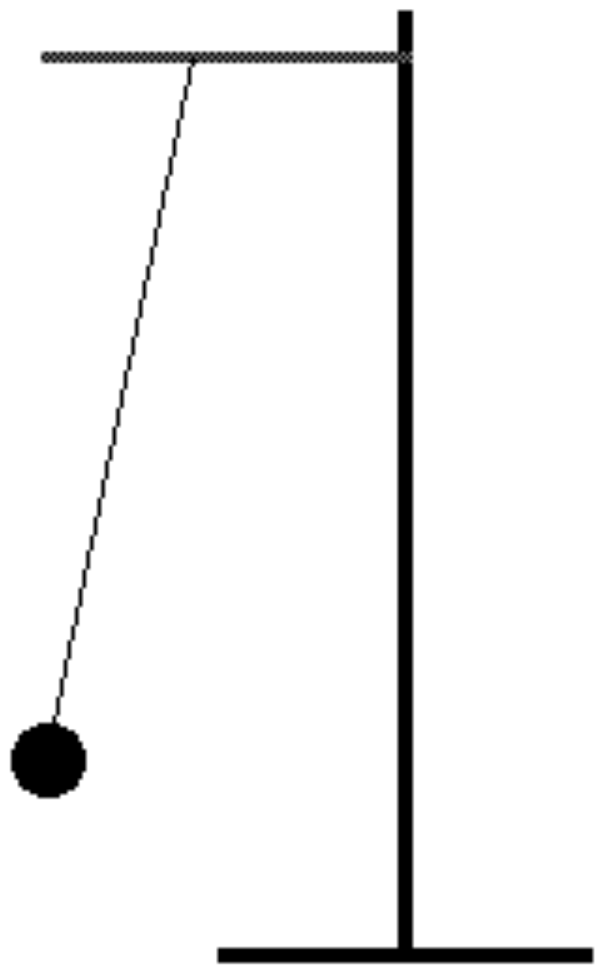
FORCE PER UNIT MASS $= -\nabla V = -\frac{\partial V}{\partial x}$

V = POTENTIAL PER UNIT MASS

SO $\quad V(x) = \frac{1}{2}\omega^2 x^2$

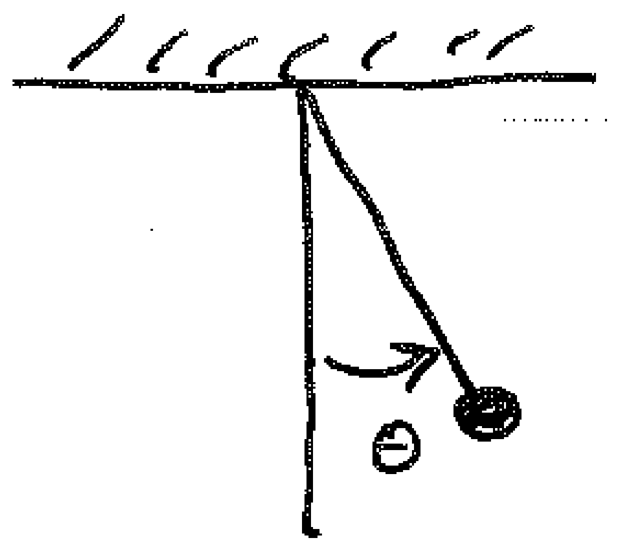
TOTAL ENERGY PER UNIT MASS

$$W = \frac{1}{2}\left(\frac{dx}{dt}\right)^2 + V(x)$$
$$= \frac{1}{2}\omega^2 x_0^2$$



Hamiltonian for an Oscillator

CANNONICAL TRANSFORMATION



PENDULUM

$$(P_\theta, \theta) \Longrightarrow (J, \varphi)$$

ACTION-ANGLE COORDINATES

REAL (PHYSICAL) COORDINATES

WE USE A "GENERATING FUNCTION" WELL-KNOWN IN CLASSICAL, HAMILTONIAN MECHANICS

$$H_{OLD} = P_\theta \dot{\theta} - L = \frac{P_\theta^2}{2m} + V(\theta)$$

$$P_\theta = m\dot{\theta}$$

$$V(\theta) = -\frac{mg}{\ell}\cos\theta$$

Hamiltonian for an Oscillator

DURING THE CANNONICAL TRANSFORMATION, WE REQUIRE

$$\delta \oint L\,dt = \delta \oint (P_\theta \dot{\theta} - H_{OLD})\,dt = \delta \oint (P_J \dot{\varphi} - H_{NEW})\,dt$$

AND THESE TWO TERMS MUST DIFFER ONLY IN A TOTAL DERIVATIVE OF THE GENERATING FUNCTION, F

SINCE $\frac{dF}{dt} = \dot{\theta}\frac{\partial F}{\partial \theta} + \dot{\varphi}\frac{\partial F}{\partial \varphi} + \frac{\partial F}{\partial t}$, WE MUST HAVE

$$H_{NEW} = H_{OLD} + \dot{\varphi}\left(P_J + \frac{\partial F}{\partial \varphi}\right) - \dot{\theta}\left(P_\theta - \frac{\partial F}{\partial \theta}\right) + \frac{\partial F}{\partial t}$$

WE ARE FREE TO CHOOSE $F(\theta, \varphi, t)$ AS WE WISH PROVIDED THAT IT SATISFIES OUR REQUIREMENTS.

Hamiltonian for an Oscillator

$$\frac{dq}{dt} = \frac{\partial H}{\partial p}$$
$$\frac{dp}{dt} = -\frac{\partial H}{\partial q}$$

$$\boxed{\frac{d\phi}{dt} = \frac{\partial H_{new}}{\partial P_J} \qquad \frac{dP_J}{dt} = -\frac{\partial H_{new}}{\partial \phi}}$$

WE CHOOSE $P_\theta = \frac{\partial F}{\partial \theta}$

$P_J = -\frac{\partial F}{\partial \varphi}$

THEN, $H_{NEW} = H_{OLD} + \frac{\partial F}{\partial t}$ (VERY NICE)

FOR A PENDULUM, IF $F(\theta, \varphi, t) = \frac{1}{2} m \omega_B(t) \theta^2 \frac{\cos\varphi}{\sin\varphi}$

THEN

$P_\theta = m\dot{\theta} = \frac{\partial F}{\partial \theta} = m \omega_B \theta \frac{\cos\varphi}{\sin\varphi}$ BUT $\theta = \theta_m \sin\varphi$

$= m \omega_B \theta_m \cos\varphi$ ← SO

$P_J = -\frac{\partial F}{\partial \varphi} = \frac{1}{2} m \omega_B \theta_m^2 = J$

THUS $\boxed{H_{NEW} = \omega_B J + \underbrace{\tfrac{1}{2} J \sin 2\varphi}_{\text{HARMONIC OF OSCILLATOR FREQUENCY}}\ \underbrace{\frac{1}{\omega_B} \frac{\partial \omega_B}{\partial t}}_{\text{SMALL TERM}}}$

IF $\dot{\omega}_B = 0$, THEN

$\dot{J} = 0$

$\dot{\varphi} = \omega_B$

Very nice!

Adiabatic Invariants (Part 3)

$$(\mu,\ J,\ \psi)$$

For an oscillator x(t),
What happens when ω(t) changes slowly with time?

- Separate descriptions of perpendicular and parallel motion
- Fast gyration around B
- Slow perpendicular drift of gyro center

LET'S SAY

AND WE DEMAND

$x(\theta)$ TO BE PERIODIC

$$x(t) = x(t, \theta) \qquad \frac{d\theta}{dt} = \nu(t)$$

$$\frac{dx}{dt} = \frac{\partial x}{\partial t} + \frac{d\theta}{dt}\frac{\partial x}{\partial \theta} = \frac{\partial x}{\partial t} + \nu \frac{\partial x}{\partial \theta}$$

$$\frac{d^2x}{dt^2} = \frac{\partial^2 x}{\partial t^2} + \nu \frac{\partial^2 x}{\partial \theta \partial t} + \frac{\partial}{\partial t}\left(\nu \frac{\partial x}{\partial \theta}\right) + \nu \frac{\partial}{\partial \theta}\left(\nu \frac{\partial x}{\partial \theta}\right)$$

$$= \frac{\partial^2 x}{\partial t^2} + 2\nu \frac{\partial^2 x}{\partial \theta \partial t} + \frac{d\nu}{dt}\frac{\partial x}{\partial \theta} + \nu^2 \frac{\partial^2 x}{\partial \theta^2}$$

Order by Order with $\dot{\nu}/\nu \ll 1$

$$\frac{d^2x}{dt^2} = \omega^2 x$$

PUTTING THIS INTO OUR EQUATION

$$\underbrace{\nu^2 \frac{\partial^2 x}{\partial \theta^2} + \omega^2 x}_{0^{TH}} = \underbrace{-\frac{\partial^2 x}{\partial t^2}}_{2^{ND}} \underbrace{- 2\nu \frac{\partial^2 x}{\partial \theta \partial t} - \frac{d\nu}{dt}\frac{\partial x}{\partial \theta}}_{1^{ST}}$$

$\omega(t)$!

NOW, WE'RE GOING TO SOLVE THIS AS IF

$\frac{d\nu}{dt}$ IS VERY SLOW $\quad \left|\frac{\dot{\nu}}{\nu}\right| \ll \nu$

$\frac{\partial x}{\partial t}$ IS VERY SLOW $\quad \frac{|\dot{x}|}{|x|} \ll \nu$

MOST RAPID MOTION IS IN THE OSCILLATION.

Order by Order with $\dot{\nu}/\nu \ll 1$

THEN, TO LEADING ORDER ...

WKB ✓

$$\nu^2 \frac{\partial^2 x}{\partial \theta^2} + \omega^2 x = 0 \quad \text{or} \quad \frac{d\theta}{dt} = \nu = \omega$$

$$x = x_0(t)\sin(\theta(t)) \; ; \quad \theta(t) = \int_0^t dt' \omega(t') + \phi$$

but, how does energy and amplitude change with frequency?

Understanding Energy/Amplitude for a Slowly Changing Oscillator

LET'S MULTIPLY BY $\frac{\partial x}{\partial \theta}$

$$\frac{\partial}{\partial \theta}\left(\frac{1}{2} \gamma^2 \left(\frac{\partial x}{\partial \theta}\right)^2 \right) + \frac{\partial}{\partial \theta}\left(\frac{1}{2} \omega^2 x^2 \right)$$

$$= - \frac{\partial x}{\partial \theta} \frac{\partial^2 x}{\partial t^2} - \gamma \frac{\partial}{\partial t}\left(\left(\frac{\partial x}{\partial \theta}\right)^2 \right) - \frac{d\gamma}{dt}\left(\frac{\partial x}{\partial \theta}\right)^2$$

$$\frac{\partial}{\partial \theta}\left(\frac{1}{2} \gamma^2 \left(\frac{\partial x}{\partial \theta}\right)^2 + \frac{1}{2} \omega^2 x^2 \right) = - \frac{\partial x}{\partial \theta} \frac{\partial^2 x}{\partial t^2} - \frac{\partial}{\partial t}\left(\gamma \left(\frac{\partial x}{\partial \theta}\right)^2 \right)$$

NOW, THE TERM $\frac{\partial x}{\partial \theta} \frac{\partial^2 x}{\partial t^2}$ IS <u>VERY</u> <u>VERY</u> <u>SMALL</u>.

Understanding Energy/Amplitude for a Slowly Changing Oscillator

NOW, THE TERM $\frac{\partial x}{\partial \theta} \frac{\partial^2 x}{\partial t^2}$ is <u>VERY</u> <u>VERY</u> <u>SMALL</u>.

SO IN THE SPIRIT OF OUR PERTURBATION ANALYSIS, WE'RE going TO IGNORE THIS TERM.

THEN

$$\frac{\partial}{\partial t}\left(\gamma \left(\frac{\partial x}{\partial \theta}\right)^2\right) = -\frac{\partial}{\partial \theta}\left(\frac{1}{2}\gamma^2\left(\frac{\partial x}{\partial \theta}\right)^2 + \frac{1}{2}\omega^2 x^2\right)$$

BUT $x(\theta)$ MUST BE <u>PERIODIC</u>.

SO we can integrate over θ $\qquad \frac{1}{2}\gamma^2\left(\frac{\partial x}{\partial \theta}\right)^2 + \frac{1}{2}\omega^2 x^2 = \text{PERIODIC}$

THUS! $\qquad \boxed{\left\langle \gamma \left(\frac{\partial x}{\partial \theta}\right)^2 \right\rangle_\theta \cong \text{CONSTANT}}$ $\qquad$ d/dt (…) = 0

Understanding Energy/Amplitude for a Slowly Changing Oscillator

$$\left\langle \nu \left(\frac{\partial x}{\partial \theta}\right)^2 \right\rangle_\theta = \frac{\nu}{2\pi}\int_0^{2\pi} d\theta \; x_0^2 \cos^2\theta$$

$$= \frac{1}{2}\,\nu\, x_0^2 = \underline{\underline{\frac{1}{2}\,\omega x_0^2}} = \frac{(1/2)\omega^2 x_0^2}{\omega} = \frac{\text{Energy}}{\text{frequency}}$$

$$\oint \nu \frac{\partial x}{\partial \theta}\frac{\partial x}{\partial \theta}\,\frac{d\theta}{2\pi}$$

$$p = \frac{dx}{dt} \sim \nu \frac{\partial x}{\partial \theta}$$

$$dq = \frac{\partial x}{\partial \theta}\, d\theta$$

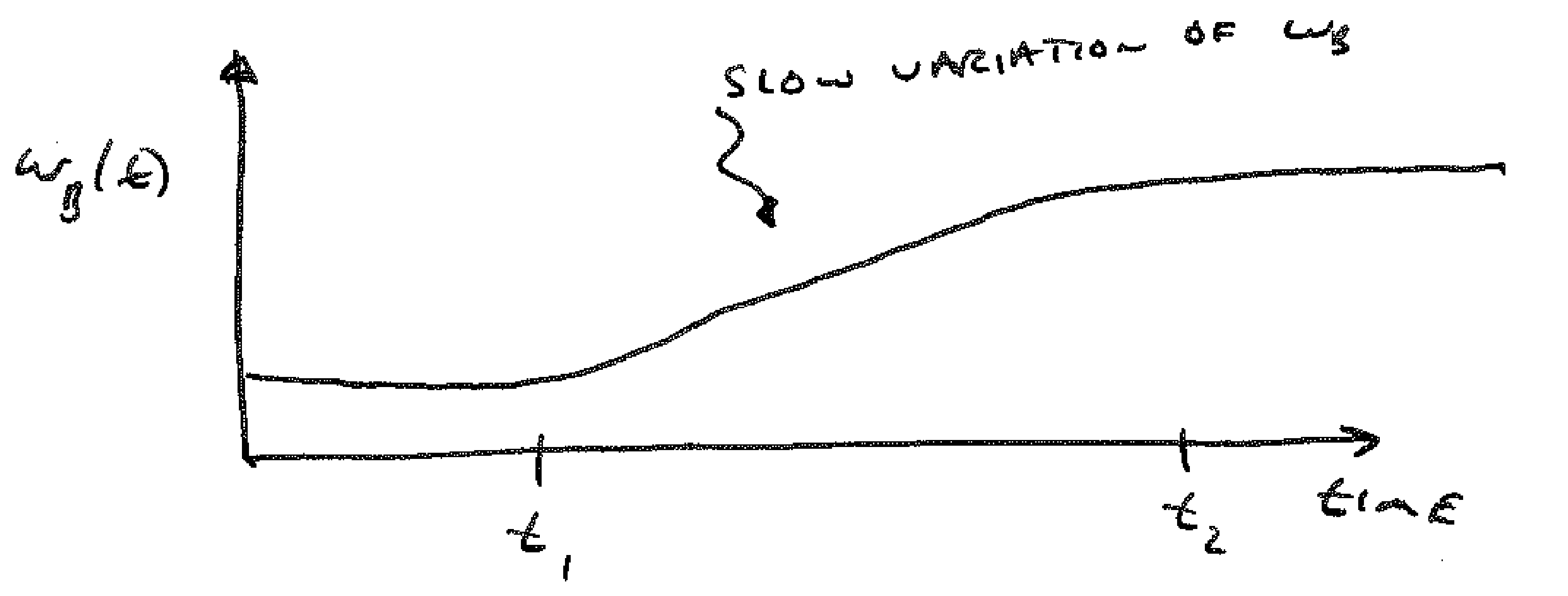
$$\text{ACTION} = \oint_{0\le\theta\le 2\pi} P\,dq = \boxed{\frac{\text{ENERGY}}{\text{Frequency}}}\,.$$

How Good are Adiabatic Invariants?

EXACT EQUATIONS OF MOTION

$$\begin{cases} \dot{J} = -\dfrac{\partial H_{NEW}}{\partial \varphi} = -J\cos 2\varphi \, \dfrac{1}{\omega_B}\dfrac{\partial \omega_B}{\partial t} \\ \dot{\varphi} = \omega_B + \dfrac{1}{2}\sin 2\varphi \, \dfrac{1}{\omega_B}\dfrac{\partial \omega_B}{\partial t} \end{cases}$$

FROM MECHANICS, LANDAU + LIFSHITZ . . .



$$\Delta J = J(t_2) - J(t_1) = + \int_{t_1}^{t_2} J\cos 2\varphi \, \frac{1}{\omega_B}\frac{\partial \omega_B}{\partial t}\, dt$$

BUT $\dfrac{J\cos 2\varphi}{\omega_B} = \sum_{p=-\infty}^{\infty} \Lambda_p e^{ip\varphi}$!

MUST BE A PERIODIC FUNCTION OF φ

$$= 2\,\mathrm{Re}\left\{ \sum_{p=1}^{\infty} e^{ip\varphi} \Lambda_p \right\}$$

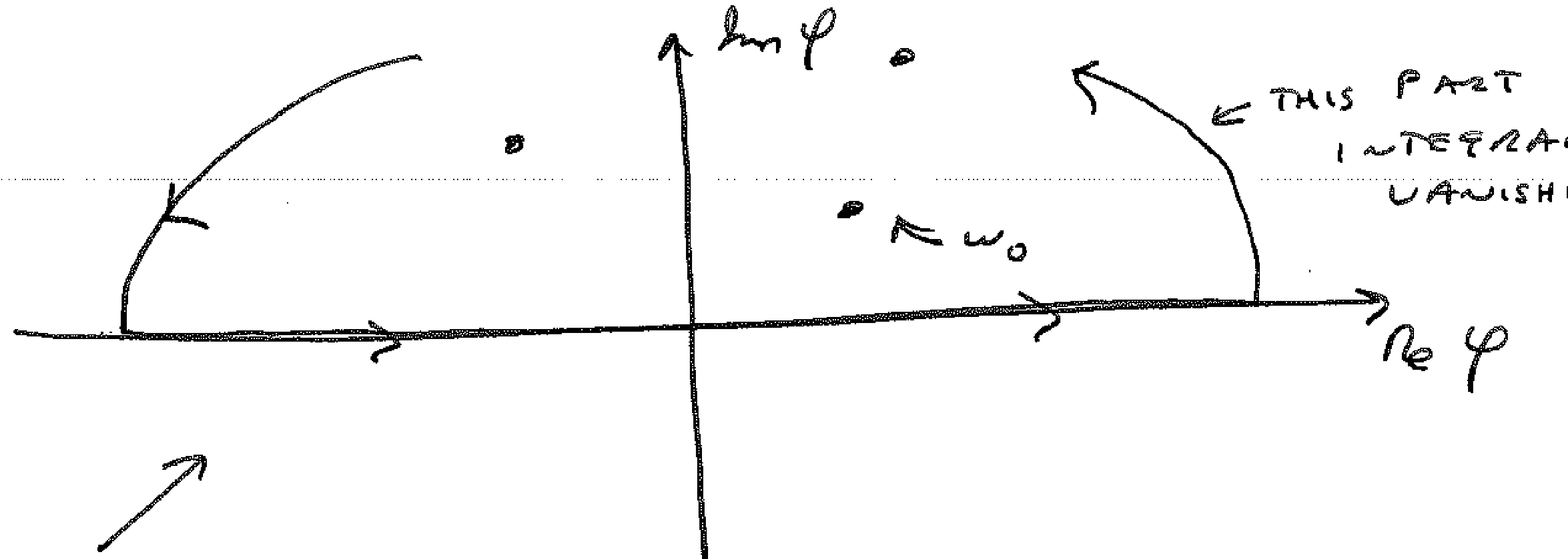
How Good are Adiabatic Invariants?

Answer:

Exponentially good

$$\Delta J = 2\,\mathrm{Re}\left\{ \int_{-\infty}^{\infty} dt\, \Lambda_p\, e^{ip\varphi}\, \frac{\partial \omega_B}{\partial t} \right\}$$

$$= 2\,\mathrm{Re}\left\{ \int_{-\infty}^{\infty} d\varphi\, \Lambda_p\, e^{ip\varphi}\, \frac{\partial \omega_B}{\partial \varphi} \right\} \qquad \left(dt = \frac{d\varphi}{\omega_B}\right)$$



THIS PART OF INTEGRAL VANISHES

WE CAN DO THE INTEGRAL IN THE COMPLEX φ-PLANE

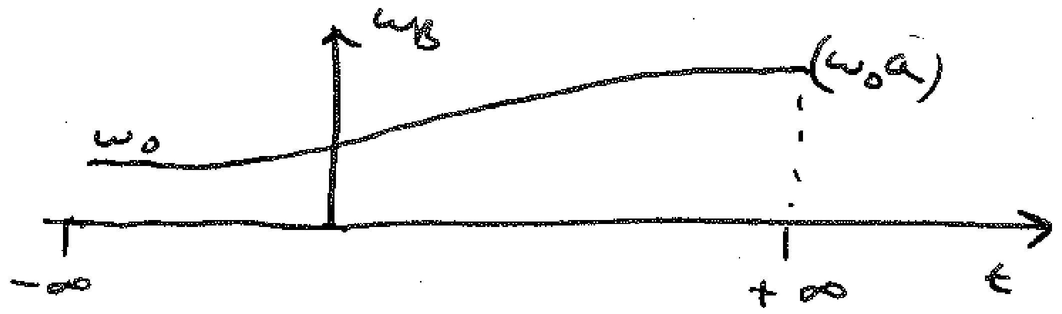
$$\Delta J \sim e^{ip\omega_0}$$

AN ADIABATIC INVARIANT IS EXP. GOOD

WHERE ω_0 = LOWEST SINGULARITY IN UPPER HALF PLANE

ω_0 IS SINGULARITY OF $\omega_B(t)$

EXAMPLE: $\omega_B(t) = \omega_0^2 \dfrac{1 + a^2 e^{\alpha t}}{1 + e^{\alpha t}}$ $\qquad \alpha \ll \omega_0$, $a > 0$



THEN $\dfrac{d\omega_B}{dt} = \frac{1}{2}\alpha\,\omega_B \left(\dfrac{a^2}{e^{-\alpha t} + a^2} - \dfrac{1}{e^{-\alpha t} + 1} \right)$ $\qquad \mathrm{Im}\,\omega_0 \approx \dfrac{\pi \omega_0 a}{\alpha}$

Nonadiabaticity in mirror machines

Ronald H. Cohen, George Rowlands,[a] and James H. Foote

Lawrence Livermore Laboratory, University of California, Livermore, California 94550

(Received 14 February 1977; final manuscript received 25 October 1977)

Phys. Fluids 21(4), April 1978

See:

Hamiltonian_Method_I.nb
(Mathematica Notebook with charged-particle orbits in a magnetic mirror.)

Hamiltonian_Method_II.nb
(Mathematica Notebook with charged-particle orbits in a point dipole.)

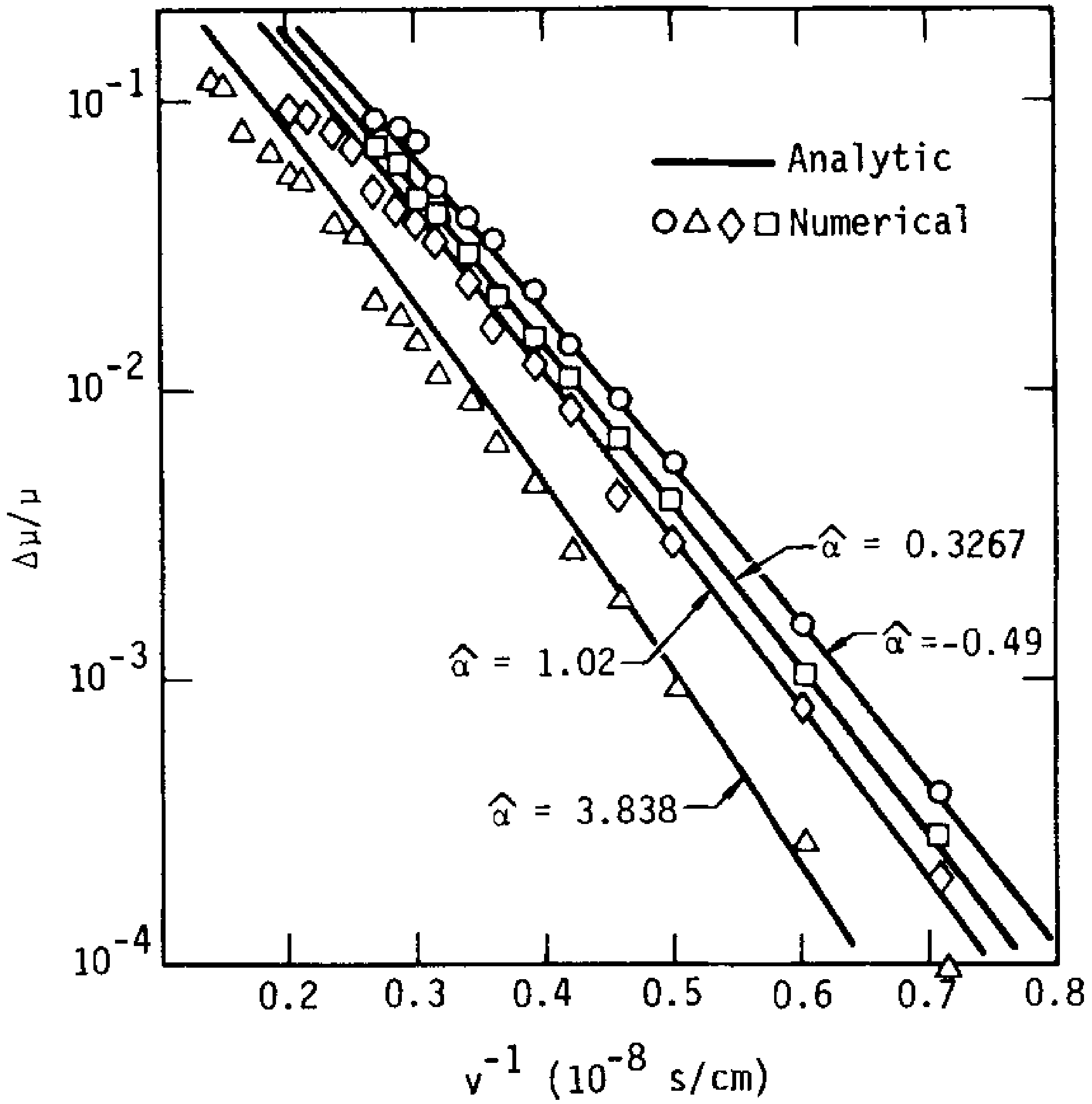
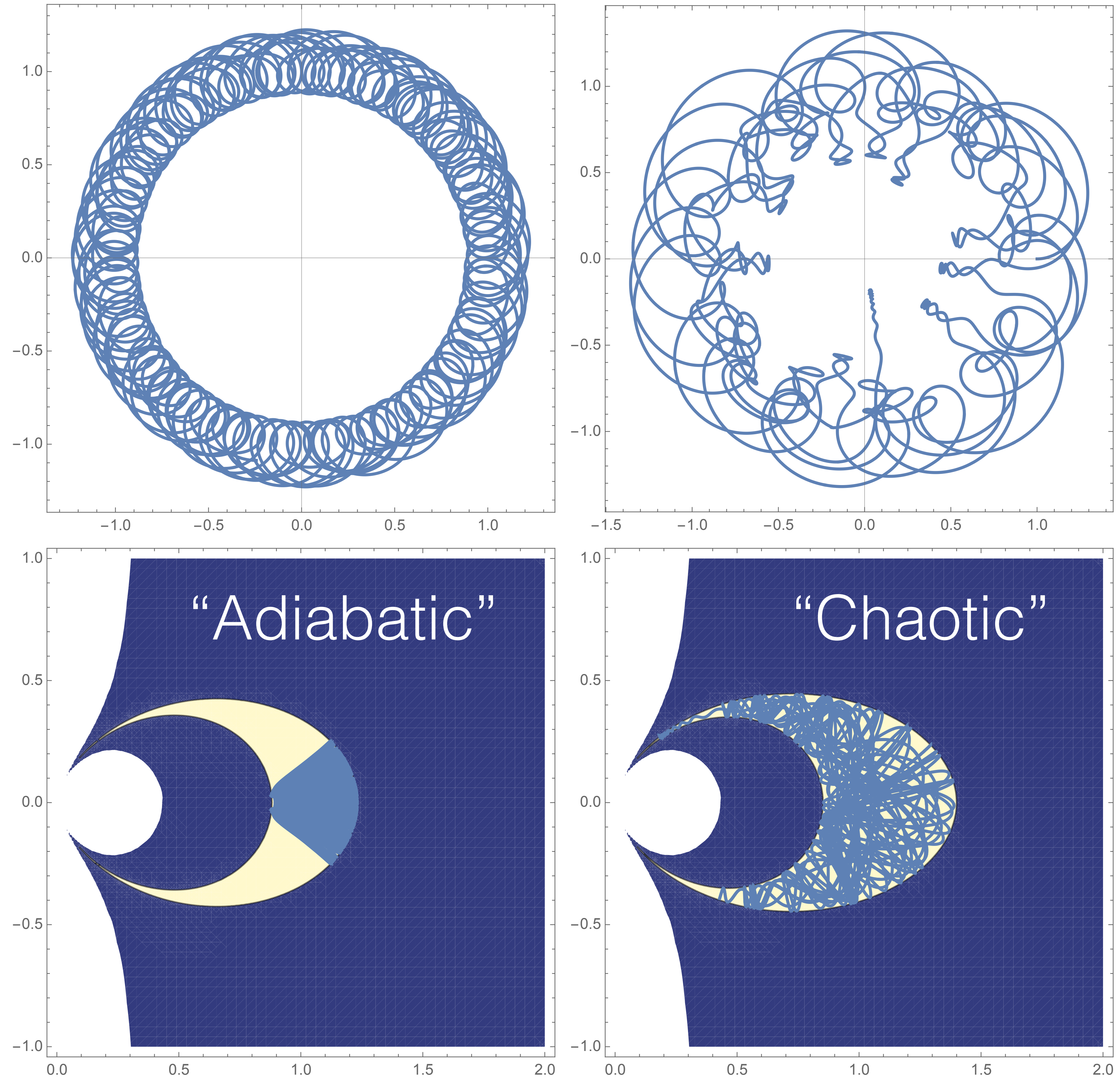


FIG. 1. $\Delta\mu/\mu$ versus v^{-1} for the model field defined by Eqs. (21) and (29) with $L = 20.2$ cm, $B_{00} = 10$ kG. Protons are started at $z = 0$, $r_0 = 3$ cm with $\mathbf{v}_\perp$ radial, $v_\perp/v = 0.544$.



Drift Hamiltonian (Famous!)

$$H(\rho_\parallel, \alpha, \psi, \chi) = \frac{1}{2}\rho_\parallel^2 \frac{eB^2}{mc} + \frac{\mu c}{e} B + c\Phi;$$

$$\mathbf{B} = \nabla\alpha \times \nabla\psi ,$$

$$\mathbf{B} = \nabla\chi + \beta\nabla\psi + \gamma\nabla\alpha .$$

$$\nabla\chi \cdot (\nabla\alpha \times \nabla\psi) = B^2$$

then

$$\frac{d\chi}{dt} = \frac{\partial H}{\partial \rho_\parallel}, \quad \frac{d\rho_\parallel}{dt} = -\frac{\partial H}{\partial \chi}, \quad \frac{d\psi}{dt} = \frac{\partial H}{\partial \alpha}, \quad \frac{d\alpha}{dt} = -\frac{\partial H}{\partial \psi}.$$

The adiabatic invariance of J,

$$J = \frac{e}{c}\int \rho_\parallel \, d\chi = \int m v_\parallel \, dl ,$$

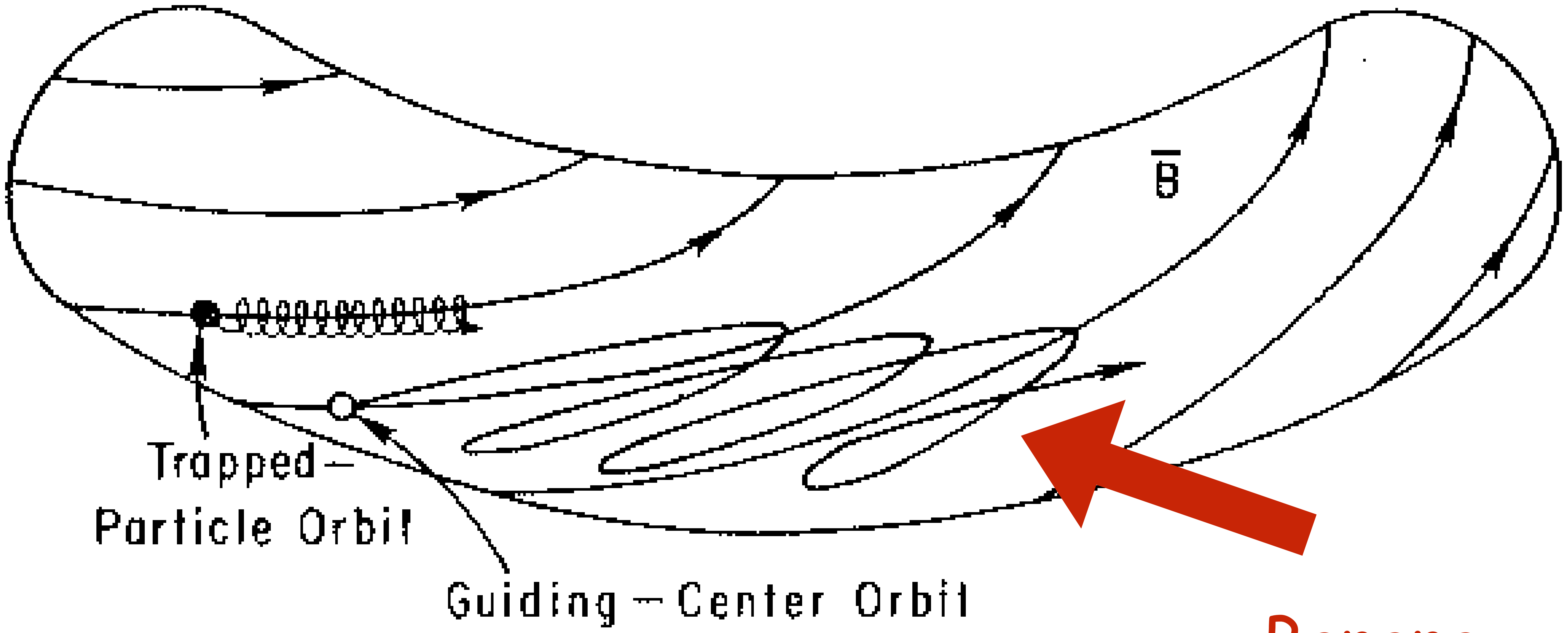
then follows from the standard classical mechanics[4] treatment. The Hamiltonian is just the energy E times c/e and it is conserved.

Guiding center drift equations

Allen H. Boozer

Phys. Fluids 23(5), May 1980

Examples of Confined Orbits (Part 2)



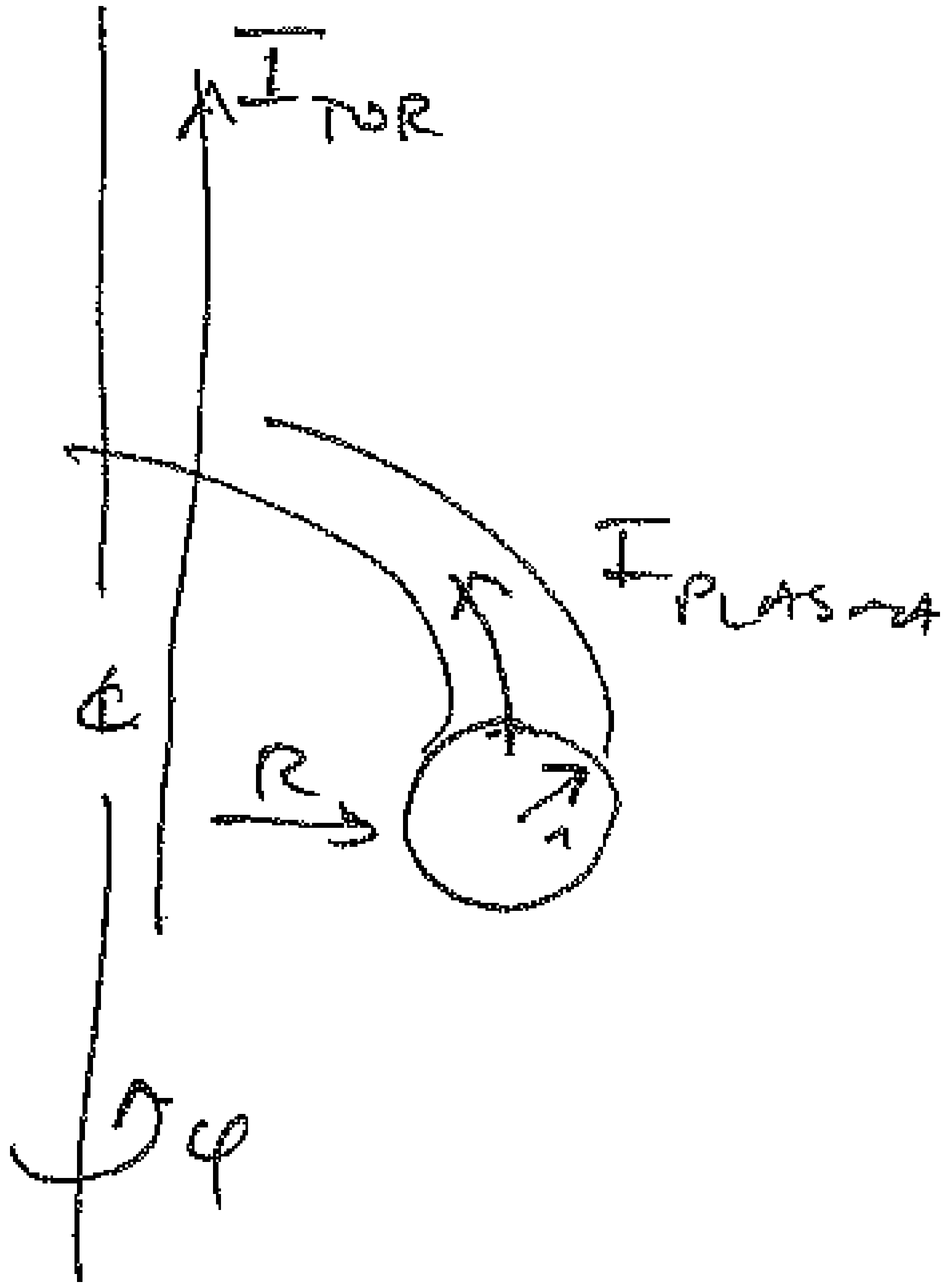
Banana ~
Poloidal Gyroradius

gyration :: bounce :: toroidal precession

$1 :: \rho/R :: (\rho/R)^2$

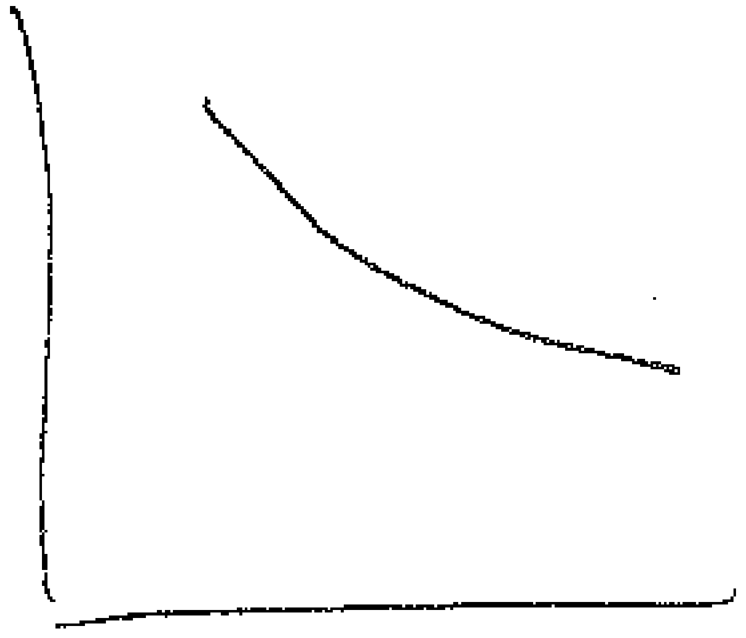
Fast :: Not so fast :: "Slow"

Toroidal Magnetic Field



$$\bar{B} = B_t \hat{\varphi} + \bar{B}_p$$

$$B_t = B_0 \left(\frac{R_0}{R} \right)$$



KRUSKAL - SHAFRANOV CRITERIA

STATES

$$\frac{r B_t}{R B_p} > 1$$

Toroidal Magnetic Field

|B| varies along the magnetic field

$$2\pi R B_t = \mu_0 I_{TOR}$$

$$2\pi a B_p = \mu_0 I_p$$

so $$\frac{a B_t}{R B_p} = \frac{\frac{a}{R} I_{TOR}}{\frac{R}{a} I_p} = \frac{I_{TOR}}{I_p}\left(\frac{a}{R}\right)^2$$

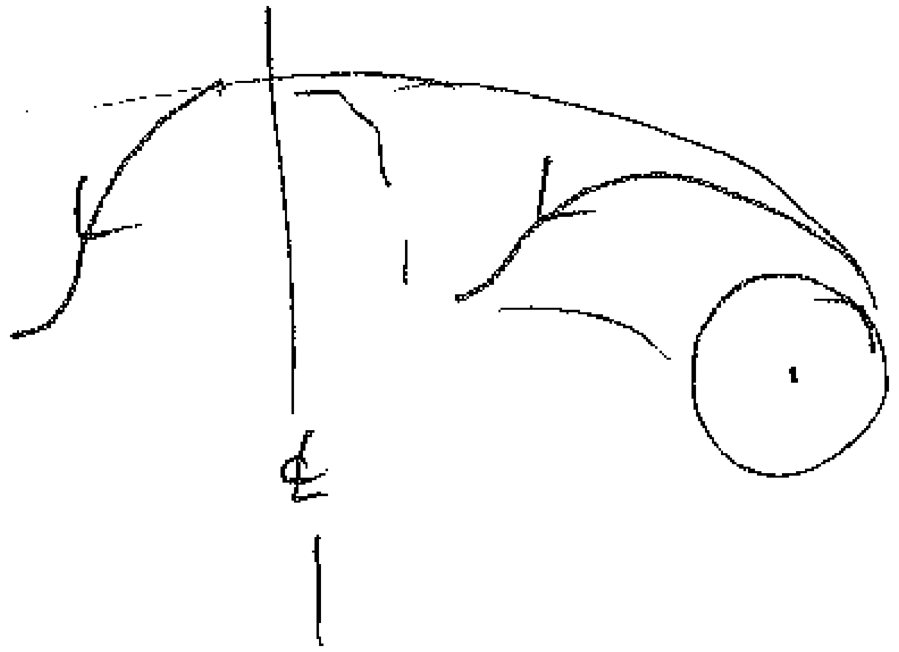
so $$\frac{I_{TOT}}{I_p} > \left(\frac{R}{a}\right)^2$$ VERY BIG

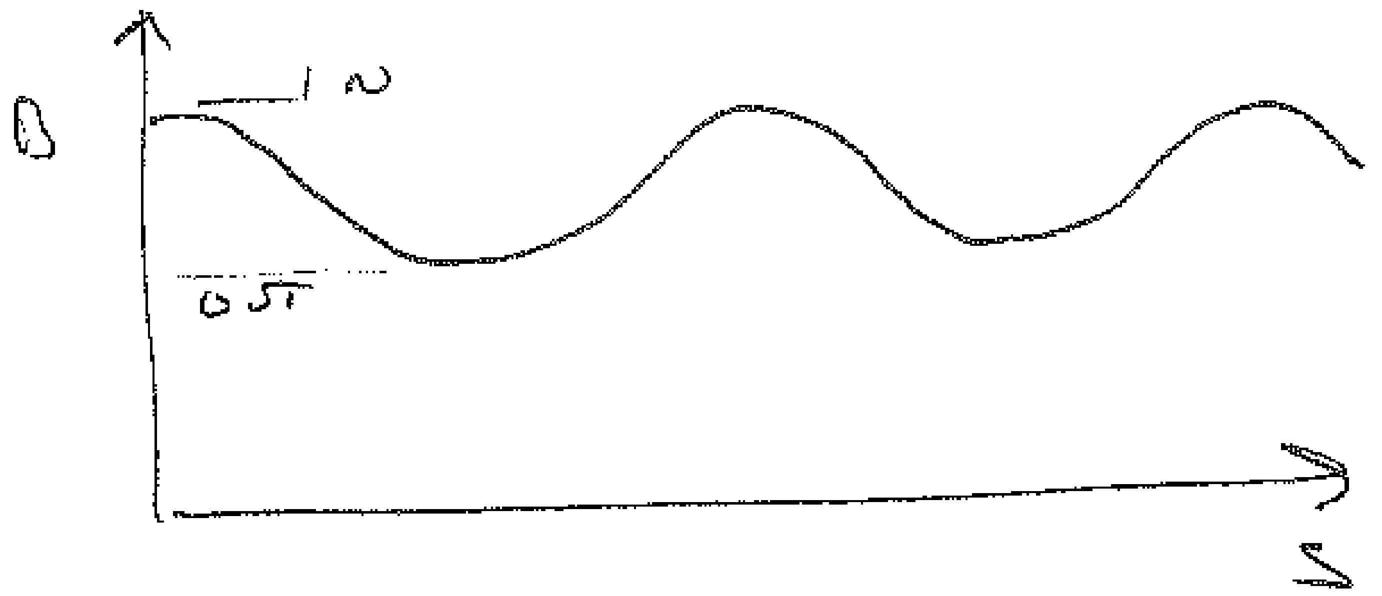
BECAUSE OF I_p,
THE FIELD LINES FORM
"NESTED" TORI → HELICAL
TRAJECTORIES

USUALLY $\frac{R}{a} > 3$

> 6 FOR COLUMBIA

$$\epsilon \equiv \frac{a}{R}$$





$$B_{OUT} = B_0 \frac{R_0}{R_0 + a} = B_0 \frac{1}{1+\epsilon}$$

$$B_{IN} = B_0 \frac{1}{1-\epsilon}$$

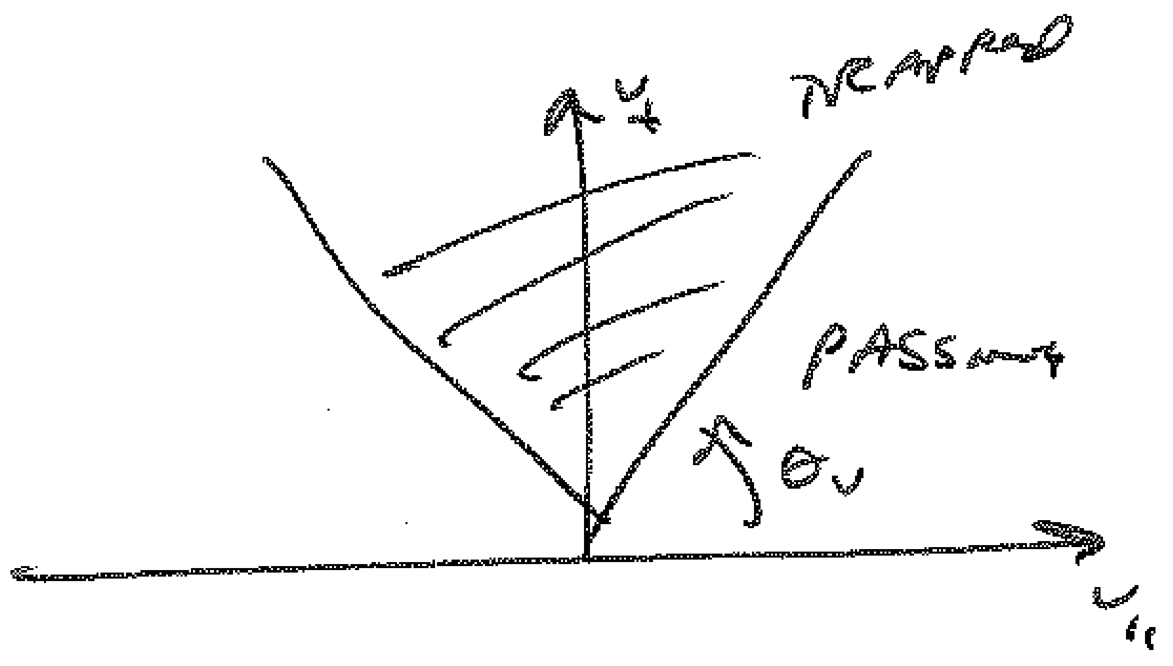
$$\frac{B_{MAX}}{B_{MIN}} = \frac{1+\epsilon}{1-\epsilon} \sim 1 + 2\epsilon$$

31

Toroidal Magnetic Field

Trapped and Passing Particles

Because $\mu =$ constant



@ outside equatorial midplane

Boundary btwn Trapped/Passing

"Mirror Trapping"

$$\tfrac{1}{2} v_{\parallel 0}^2 + \mu B_{out} = \mu B_{in}$$

$$\tfrac{1}{2} v_{\parallel 0}^2 + \tfrac{1}{2} v_{\perp 0}^2 = \tfrac{1}{2} v_{\perp 0}^2 \frac{B_{in}}{B_{out}}$$

$R =$ mirror ratio

$$= \frac{B_{in}}{B_{out}} \cong 1 + 2\epsilon$$

$$\therefore \quad v_{\parallel 0}^2 = v_{\perp 0}^2 (R-1)$$

$$= v_{\perp 0}^2 \, 2\epsilon$$

Toroidal Magnetic Field

How Many Trapped Particles?

How many trapped particles?

$$\% \text{ of passing particles} = \frac{\int_0^{\theta_v} 2\pi v^2 \sin\theta \, d\theta \, dv \, f}{\int_0^{\pi/2} 2\pi v^2 \sin\theta \, d\theta \, dv \, f}$$

$$\hat{=} \frac{\int_0^{\theta_v} \sin\theta \, d\theta}{\int_0^{2\pi} \sin\theta \, d\theta} = 1 - \cos\theta_v$$

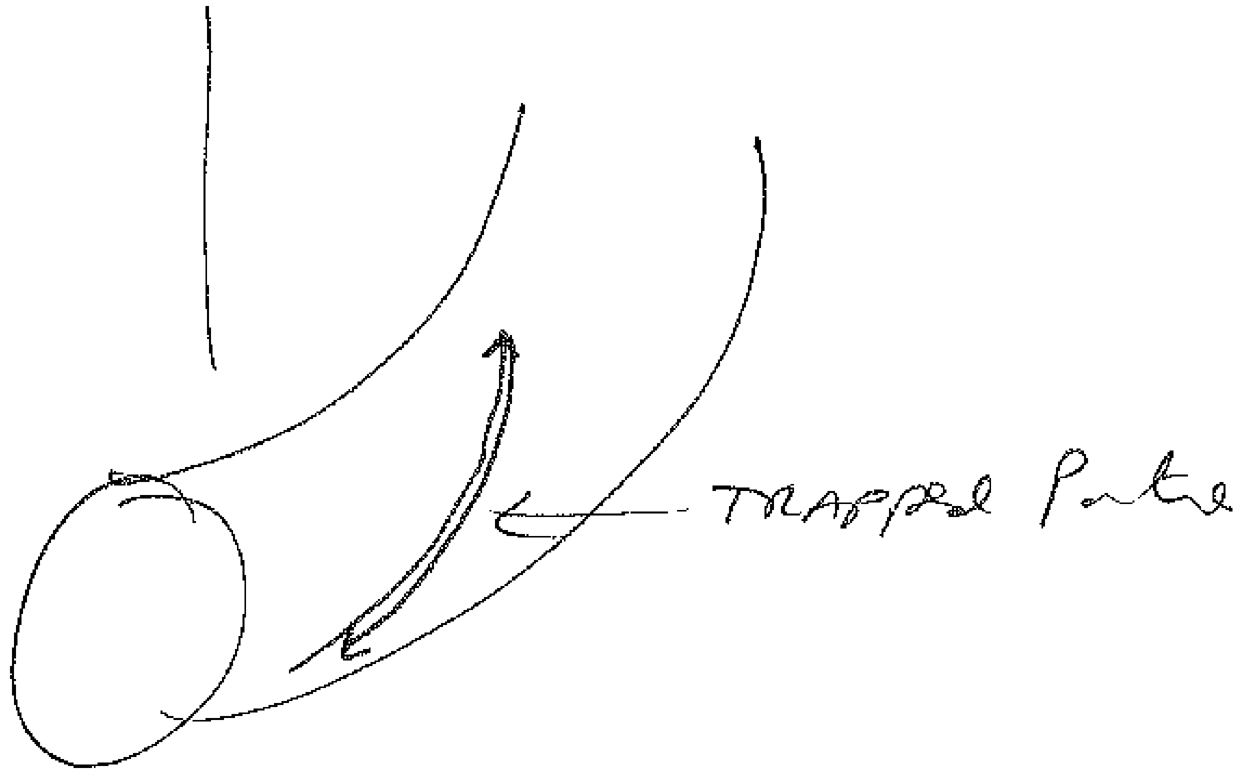
$$\% \text{ Trapped particles} = \cos\theta_v = \frac{v_\parallel}{v} = \frac{v_\perp \sqrt{2\epsilon}}{\sqrt{v_\perp^2 + v_\perp^2 2\epsilon}} \sim \underline{\sqrt{2\epsilon}}$$

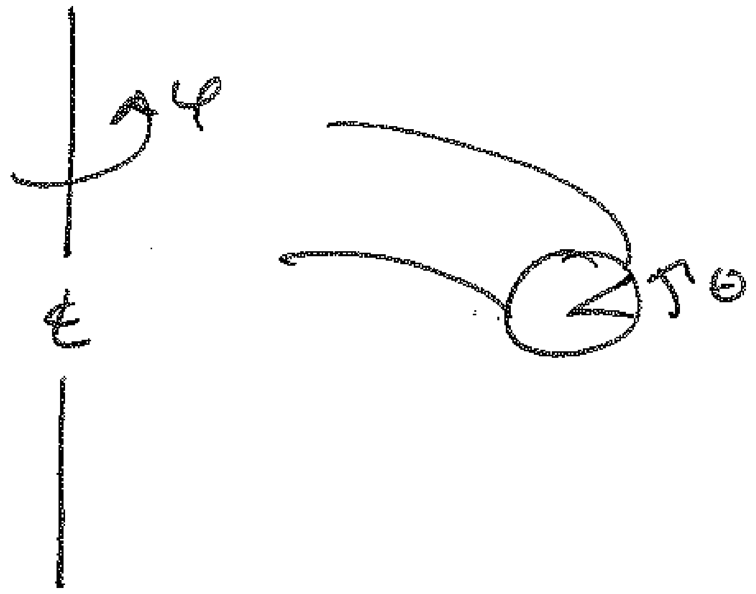
QUITE A FEW!

Toroidal Magnetic Field

Bounce Motion

How long & Bounce Frequency?

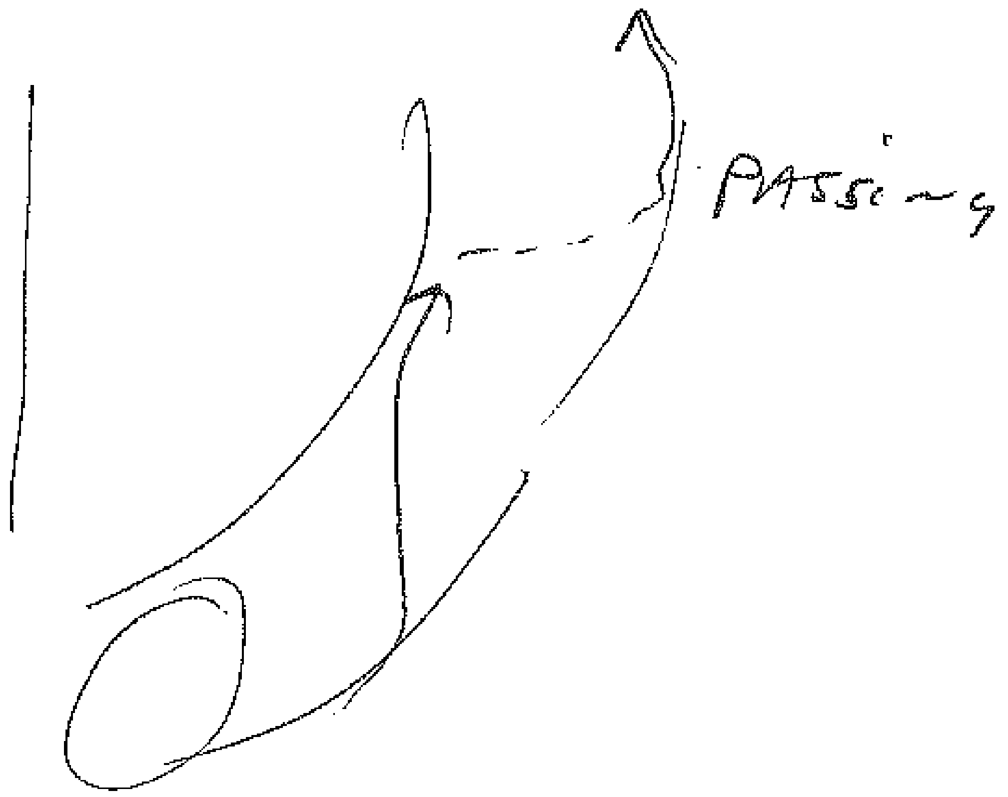


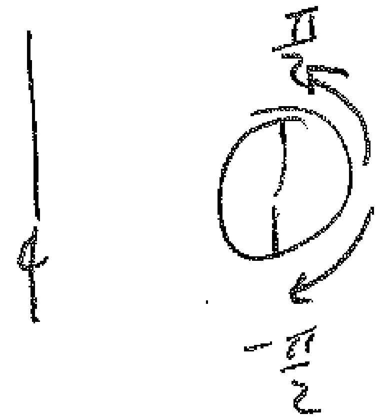


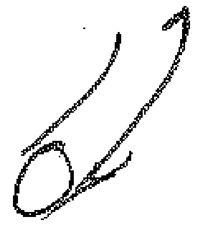
$$d\theta = q\, d\varphi$$

$$(ds)^2 = (r\, d\theta)^2 + (R\, d\varphi)^2$$
$$= (r\, d\theta)^2 + \left(\frac{R}{q}\, d\theta\right)^2$$
$$= \left(\frac{R}{q}\right)^2 \left(1 + \left(\frac{r}{R} q\right)^2\right) d\theta$$

$$ds \cong \frac{R}{q}\, d\theta$$







$$\int ds = \frac{R}{q}\pi = \frac{\pi R}{q}$$

$$\text{Bounce Frequency} \sim \frac{v_\parallel}{L} \sim \frac{v\, 2q}{\pi R}$$

$$\frac{\omega_B}{\omega_c} \sim \frac{2}{\pi}\, q \left(\frac{\rho}{R}\right) << 1$$

Toroidal Magnetic Field

Drift Motion

$$\bar{V}_D = \frac{\hat{b}\times}{\Omega}\left(\frac{\mu\nabla B}{m} + v_\parallel^2\,\bar{\kappa}\right)$$

$$\bar{B}\cdot\bar{B} = B_t^2 + B_p^2 \qquad \text{let} \quad q(r) \equiv \frac{r B_t}{R B_p}$$

$$= B_t^2\left(1 + \left(\frac{r}{R}\right)^2 \frac{1}{q^2}\right)$$

$$\nabla B \sim \nabla B_t \sim -\frac{\hat{r}}{R} B_t$$

$$\text{let} \quad \bar{B}_p \sim B_p\,\hat{\theta}$$

$$\bar{\kappa} = (\hat{b}\cdot\nabla)\hat{b} \sim -\frac{\hat{r}}{R} + \left(\frac{B_p}{B}\right)^2(\hat{\theta}\cdot\nabla)\hat{\theta} \qquad \hat{b} = \frac{B_t\hat{\varphi} + B_p\hat{\theta}}{B}$$

$$= -\frac{\hat{r}}{R} - \left(\frac{B_p}{B}\right)^2\frac{\hat{r}}{r} \qquad \simeq \hat{\varphi} + \frac{B_p}{B_t}\hat{\theta}$$

Toroidal curvature >> Poloidal Curvature

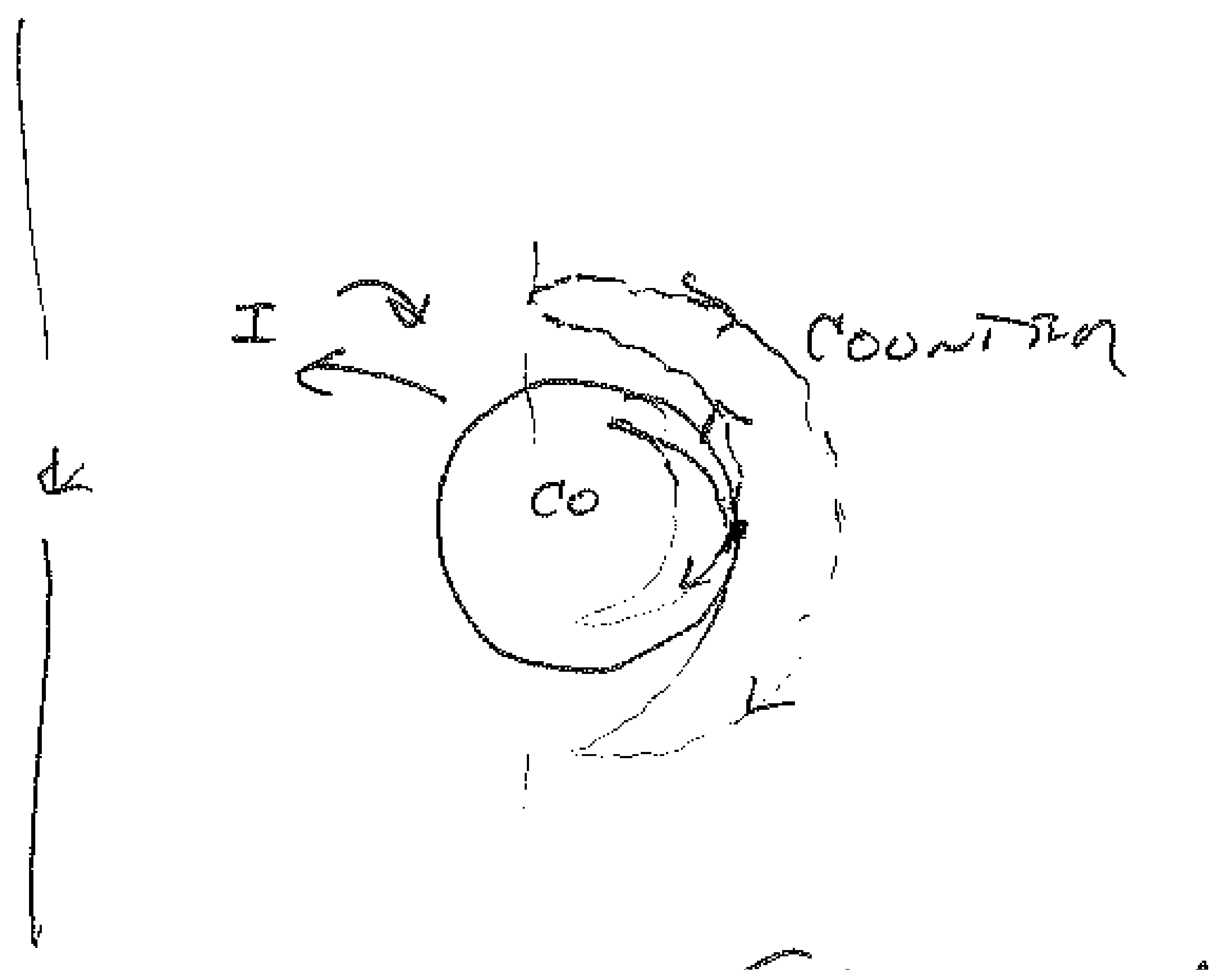
Toroidal Magnetic Field

Drift Motion

$$\vec{V}_D \sim -\frac{\hat{\varphi} \times \hat{r}}{\Omega R}\left(\mu B + v_\parallel^2\right)$$

$$\sim \frac{\hat{z}}{\Omega R}\left(\mu B + v_\parallel^2\right)$$

$$\sim \frac{\hat{z}}{R}\, v \rho \sim \hat{z}\, v\left(\frac{\rho}{R}\right)$$



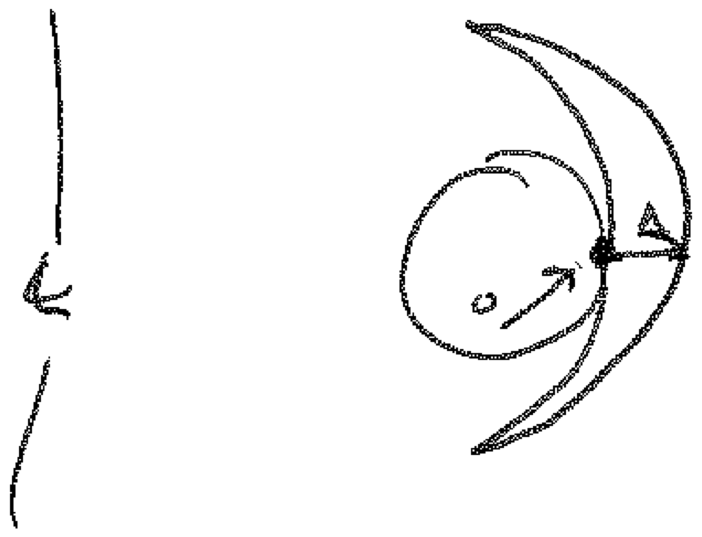
Toroidal Magnetic Field

How far off a flux surface?

$$P_\varphi \sim \text{const} = R\left[mV_\varphi + \frac{q}{c}A_\varphi\right]$$

Define $A_\varphi = \int_{r_0}^{r} B_p\, dr$

So $A_\varphi(r_0) = 0$ $\quad A_\varphi(r) = A_\varphi(r_0) + \Delta B_p$



$$R m V_\parallel \cong -R m V_\parallel + \frac{q}{c} R \Delta B_p \quad \text{so}$$

Thus $V_\parallel = \frac{1}{2}\frac{q B_p}{c\, m}\Delta$

Banana Orbits

$$I \simeq I = \oint \mathbf{p}\cdot\frac{\partial \mathbf{q}}{\partial \gamma}\, d\gamma. \qquad (2.75)$$

ere, I is an adiabatic invariant.

To evaluate I for a magnetized plasma recall that the canonical momentum for arged particles is (Jackson 1998)

$$\mathbf{p} = m\mathbf{v} + e\mathbf{A}, \qquad (2.76)$$

here $\mathbf{A}$ is the vector potential.

But if we call

$$B \cong B_0(1-\epsilon\cos\theta)$$

$$\mu B_{t0}(1-\epsilon\cos\theta_m) = \tfrac{1}{2}V_\parallel^2 + \mu B_t(1-\epsilon\cos\theta)$$

Then $V_\parallel^2 = 2\mu B_t\,\epsilon(\cos\theta - \cos\theta_m)$ $\quad$ Trapped Particles

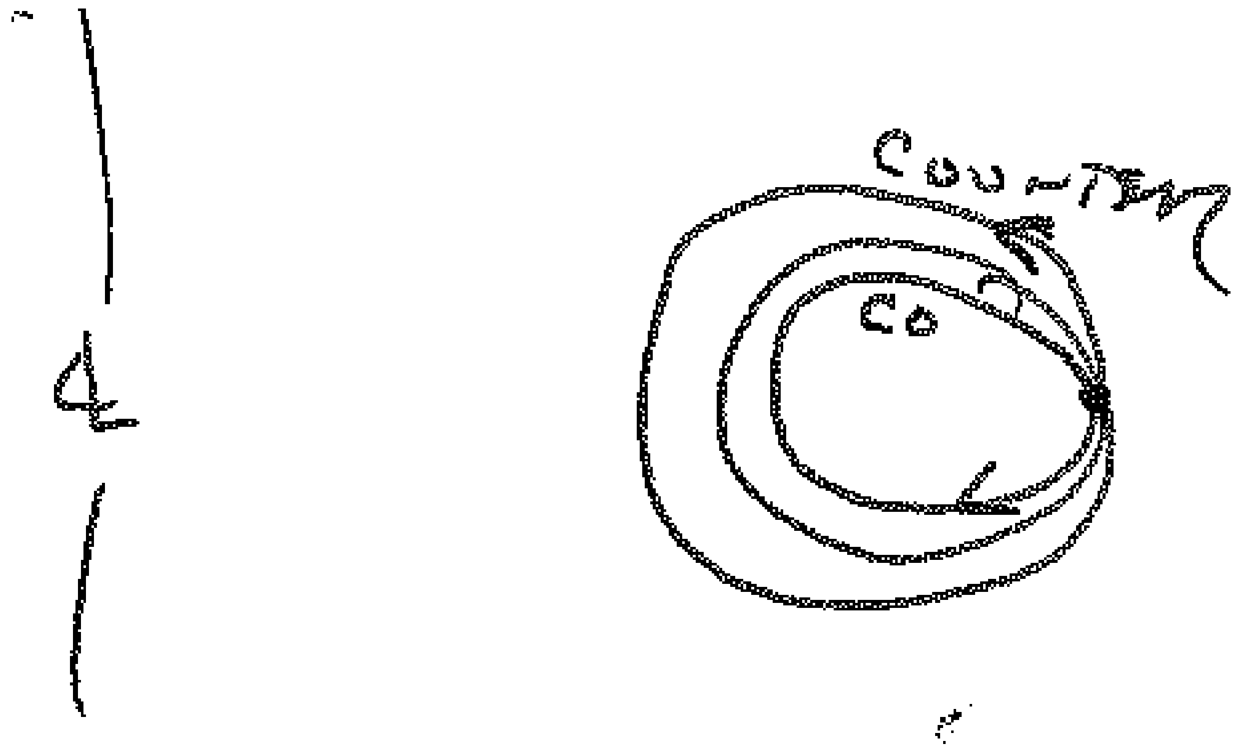
$V_\parallel \to 0$ as $\theta \to \theta_m$

$$\frac{\Delta}{2} = \frac{V_\parallel}{\omega_p} = \frac{\sqrt{2\mu B_t}}{\omega_p}\sqrt{\epsilon}\sqrt{\cos\theta - \cos\theta_m}$$

Poloidal Gyroradius

Toroidal Magnetic Field

Passing Orbits



$$R m v_\parallel^{out} = (R - r + \Delta) m v_\parallel^{in} + \frac{q}{c} R \Delta B_p$$

$$r v_\parallel = \Delta v_\parallel + \frac{q B_p}{c m} R \Delta$$

$$\frac{r}{R} v_\parallel / \omega_p = \Delta$$

$\sim$ smaller $\quad \epsilon \frac{v}{\omega_p} \sim \Delta$

ω_p = poloidal "cyclotron" frequency

Toroidal Magnetic Field

Toroidal Precession Frequency

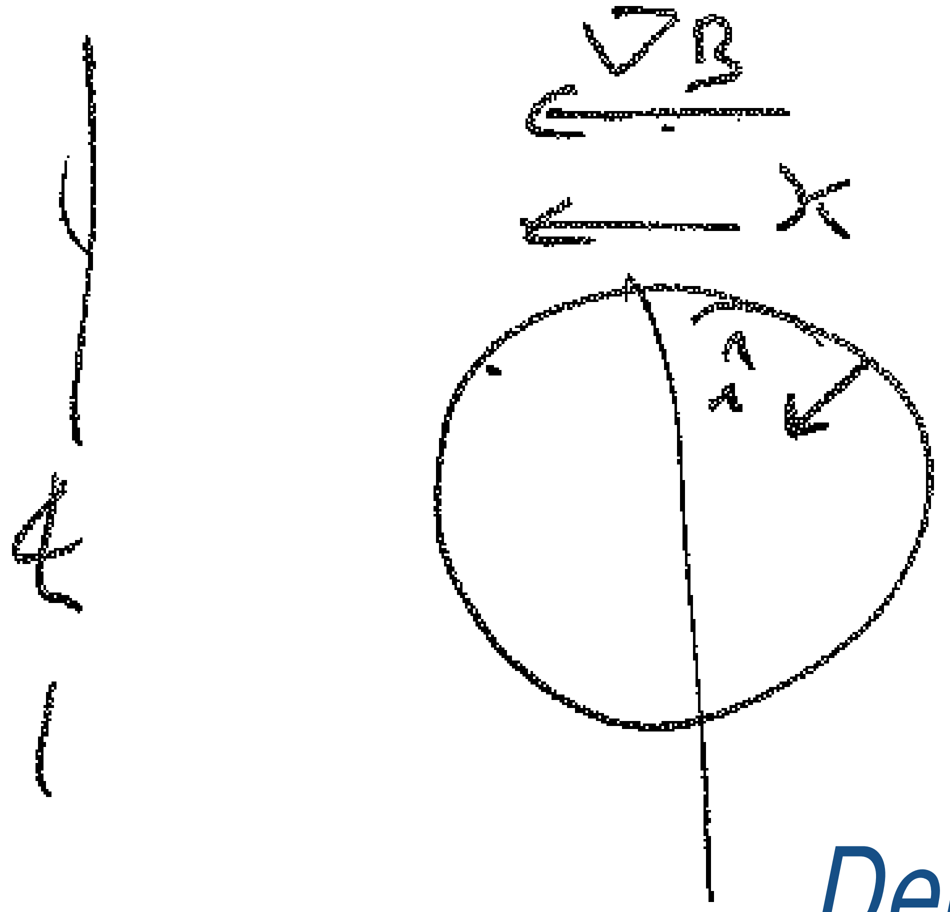
$$\hat\varphi \cdot \bar v_D = \frac{\hat\varphi \cdot \hat b \times}{\Omega}\left(\frac{\mu \nabla B}{m} + v_\parallel^2 \bar\kappa\right)$$

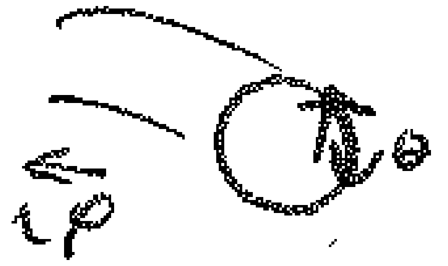
$$= \hat\varphi \times \hat b \cdot (\ \ )$$

$$= -\frac{B_p}{B}\frac{\hat r}{\Omega} \cdot \left(\frac{\mu \nabla B}{m} + v_\parallel^2 \bar\kappa\right)$$

$$A \cdot B \times C = -A \cdot C \times B$$
$$= -A \times C \cdot B$$

$$\hat\varphi \times \hat b = \hat\varphi \times (B_t \hat\varphi + \bar B_p)\frac{1}{B}$$
$$= \hat\varphi \times \bar B_p \frac{1}{B}$$
$$= \frac{B_p}{B}\underbrace{\hat\varphi \times \hat\theta}_{-\hat r}$$





$$\omega_D \sim \omega_\varphi \sim \frac{\hat\varphi \cdot v_D}{R} \sim \frac{B_p}{BR}\left(\frac{\mu B}{R} + \frac{v_\parallel^2}{R}\right)\frac{1}{\Omega_t}$$

$$\sim \frac{B_p}{BR^2}\,\omega_c \frac{v^2}{\omega_c^2} = \frac{B_p}{B}\,\omega_c\left(\frac{\rho}{R}\right)^2$$

SLIGHTLY SLOWER BY $\frac{B_p}{B}$!

Depending upon turning point, toroidal drift reverses!

Monday: Homework #3

- Fitzpatrick: Exercise #1 in Chapter 2
- Piel: All seven problems in Ch. 3 (answers in back of text)

Next Lecture

- Piel / Chapter 4: Stochastic Processes in Plasma
- Distribution function
- Collisions