Lecture4:

Plasma Physics 1

APPH E6101x
Columbia University

Outline

- Homework #2
- Charged Particle Drifts (Summary)
- Gyro-averaging
- More details from last week
- Adiabatic Invariants (Part 1)

- Examples (Part 2):
  - Mirror
  - Magnetosphere (dipole)
  - Tokamak

Gyro Motion (Summary)

$$\bar{r} = \bar{R} + \bar{\rho}$$
$$\bar{v} = \bar{V} + \bar{w}$$

(R = guiding center, V = guiding center velocity)

$$\bar{\rho} = \frac{w}{\Omega}\left[\hat{e}_1 \sin\theta + \hat{e}_2 \cos\theta\right] \qquad \theta(t) = \int_0^t dt'\,\Omega(t') + \phi$$ = gyro phase

$$\bar{w} = \frac{\partial \bar{\rho}}{\partial t} \qquad \frac{dv_\parallel}{dt} = -\frac{\mu}{m}\,\hat{b}\cdot\bar{\nabla}|B|$$

$$\mu = \frac{\frac{1}{2} m w^2}{B} = \text{const}$$ = (1st) adiabatic invariant

$$\bar{V} = \frac{\bar{a}\times\bar{\Omega}}{\Omega^2} + \frac{1}{\Omega^2}\frac{d}{dt}\bar{a} + \frac{\hat{b}\times}{\Omega}\left(\frac{\mu\bar{\nabla}|B|}{m} + v_\parallel^2\,\bar{\kappa}\right)$$
$$+ v_\parallel \hat{b}$$

and for trapped particles with ρ/L << 1,
(μ, J, Φ) ~ adiabatic invariants

Gyroaveraging: Piel Sec. 3.2

$$\mathbf{F} = q\mathbf{v} \times \mathbf{B}$$

Taylor expansion…

$$F_y = -qv_x B_z(y) \quad B_z(y) = B_0 + y(t)(\partial B_z/\partial y)$$

$$F_y = -qv_\perp \sin(\omega_c t)\left[B_0 \pm r_L \sin(\omega_c t)\frac{\partial B_z}{\partial y}\right]$$

Key "multiple timescale" approximation…

$$F_y = -qv_\perp \sin(\omega_c t) B_0 \left[1 \pm \frac{r_L(\partial B_z/\partial y)}{B_0}\sin(\omega_c t)\right]$$

$$\langle F_y \rangle = \mp q v_\perp r_L \frac{\partial B_z}{\partial y}\langle \sin^2(\omega_c t)\rangle = e v_\perp r_L \frac{\partial B_z}{\partial y}\frac{1}{2}$$

Average over gyro phase…

Gyroaveraging: Fitzpatrick Sec. 2.3

Multiple Timescales…

$$\frac{\partial \mathbf{z}}{\partial t} + \frac{1}{\epsilon}\frac{\partial \mathbf{z}}{\partial \tau} = \mathbf{f}(\mathbf{z}, t, \tau),$$

τ = fast
t = slow

$$\mathbf{z}(t, \tau) = \mathbf{Z}(t) + \epsilon\, \boldsymbol{\zeta}(\mathbf{Z}, t, \tau).$$

Average over gyro phase…

$$\langle \boldsymbol{\zeta}(\mathbf{Z}, t, \tau)\rangle \equiv \frac{1}{2\pi}\oint \boldsymbol{\zeta}(\mathbf{Z}, t, \tau)\, d\tau = 0,$$

"Multiple timescale" approximation…

$$\boldsymbol{\zeta} = \boldsymbol{\zeta}_0(\mathbf{Z}, t, \tau) + \epsilon\, \boldsymbol{\zeta}_1(\mathbf{Z}, t, \tau) + \epsilon^2\, \boldsymbol{\zeta}_2(\mathbf{Z}, t, \tau) + \cdots,$$

$$\frac{d\mathbf{Z}}{dt} = \mathbf{F}_0(\mathbf{Z}, t) + \epsilon\, \mathbf{F}_1(\mathbf{Z}, t) + \epsilon^2\, \mathbf{F}_2(\mathbf{Z}, t) + \cdots,$$

$$\frac{d\mathbf{Z}}{dt} = \langle \mathbf{f}\rangle(\mathbf{Z}, t) + \epsilon\, \langle \boldsymbol{\zeta}_0 \cdot \nabla \mathbf{f}\rangle(\mathbf{Z}, t) + O(\epsilon^2).$$

Evidently, the secular motion of the "guiding center" position $\mathbf{Z}$ is determined to lowest order by the average of the "force" $\mathbf{f}$, and to next order by the correlation between the oscillation in the "position" $\mathbf{z}$ and the oscillation in the spatial gradient of the "force."

Separating the Vector E.O.M.

PARALLEL:

$$\hat{b} \cdot \left[\frac{d^2 \bar{r}}{dt^2} = \bar{a} - \bar{\Omega} \times \frac{d\bar{r}}{dt} \right]$$

$$\frac{dv_{\parallel}}{dt} = \hat{b} \cdot \bar{a}$$

$$V_{\parallel}(t) = \hat{b} \cdot \bar{a}\, t + v_{\parallel}(0)$$

PERP:

$$\hat{b} \times (\cdots \times \hat{b}) = \text{PERPENDICULAR COMPONENT}$$

$$\hat{b} \times (\bar{A} \times \hat{b}) = \bar{A} - \hat{b}\,(\hat{b} \cdot \bar{A}) = \bar{A}_{\perp}$$

THEN

1ST

$$\frac{d^2 \bar{R}_{\perp}}{dt^2} + \frac{d^2 \bar{\rho}}{dt^2} = \bar{a}_{\perp} - \bar{\Omega} \times \bar{W} - \bar{\Omega} \times \bar{V}$$

Fast Gyromotion

(Separately)

$$\frac{d^2\bar{\rho}}{dt^2} = \frac{d\bar{w}}{dt} = -\bar{\Omega}\times\bar{w}$$

$$\bar{w} = -\bar{\Omega}\times\bar{\rho} = \frac{d\bar{\rho}}{dt}$$

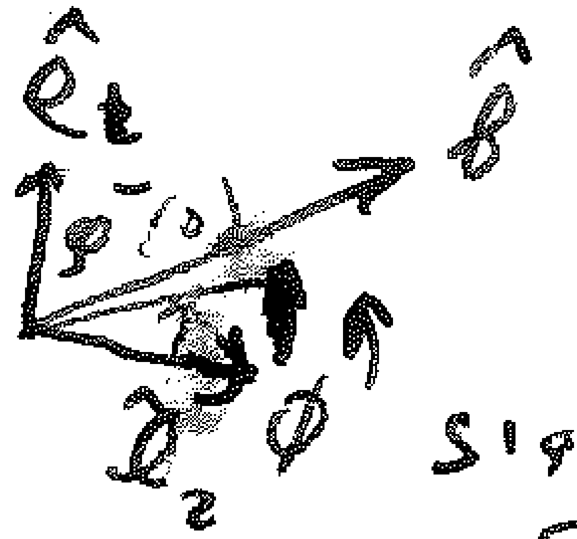
$$\frac{d^2\bar{\rho}}{dt^2} = +\bar{\Omega}\times(\bar{\Omega}\times\bar{\rho}) = -|\Omega|^2\bar{\rho}$$

$$\bar{\rho} = \frac{w}{\Omega}\left[\hat{e}_1\sin(\Omega t+\phi) + \hat{e}_2\cos(\Omega t+\phi)\right]$$

gyrophase

$$\bar{w} = w\left[\hat{e}_1\cos(\Omega t+\phi) - \hat{e}_2\sin(\Omega t+\phi)\right]$$

$$\Omega = \hat{b}\cdot\bar{\Omega}$$



$\phi = 0$

SIGN RIGHT FOR IONS

Slow Drift

(Separately)

LET $$\frac{d\bar{R}_\perp}{dt} = \bar{V}_\perp$$

$$\frac{d\bar{V}_\perp}{dt} = \bar{a}_\perp - \bar{\Omega} \times \bar{V}_\perp$$

MULTIPLY BY $\bar{\Omega} \times (\,\cdots\,)$

$$\bar{\Omega} \times \frac{d\bar{V}_\perp}{dt} = \bar{\Omega} \times \bar{a}_\perp + \Omega^2 \bar{V}_\perp$$

OR $$\bar{V}_\perp = \frac{\bar{a}_\perp \times \bar{\Omega}}{\Omega^2} + \frac{\bar{\Omega} \times \frac{d\bar{V}_\perp}{dt}}{\Omega^2}$$

$a(t)$ slowly…

- GRYOMOTION $\Rightarrow$ EXACTLY THE SAME.
- DRIFT / GUIDING CENTER MOTION

$$\bar{V}_{\perp} = \frac{\bar{a}_{\perp} \times \bar{\Omega}}{\Omega^2} + \frac{\bar{\Omega} \times \frac{dV_{\perp}}{dt}}{\Omega^2}$$

NOW IF $\frac{1}{\Omega}\frac{dV_{\perp}}{dt} \ll 1$ THEN $\bar{a}$ IS SLOWLY VARYING ....

SO LET'S MAKE AN EXPANSION

$$V_{\perp}(t) = V_{\perp}\big|_0 + V_{\perp}\big|_1 + V_{\perp}\big|_2 + \cdots$$

$$\uparrow \quad O^{\text{TH}} \quad \cdots \quad \frac{1}{\Omega}\frac{dV_{\perp}}{dt} \quad \cdots \quad \left(\frac{1}{\Omega}\frac{dV_{\perp}}{dt}\right)^2$$

$$\bar{V}_{\perp}\big|_0 = \frac{\bar{a}_{\perp} \times \bar{\Omega}}{\Omega^2}$$

$$\bar{V}_{\perp}\big|_1 = \frac{\Omega \times \frac{d}{dt}\left(V_{\perp}\big|_0\right)}{\Omega^2}$$

a(t) slowly…

SO LET'S MAKE AN EXPANSION

$$V_\perp(t) = V_\perp\big|_0 + V_\perp\big|_1 + V_\perp\big|_2 + \cdots$$

$$\uparrow$$

0^{TH} … $\frac{1}{\Omega}\frac{dV_\perp}{dt}$ … $\left(\frac{1}{\Omega}\frac{d}{dt}\right)^2$

$$\bar V_\perp\big|_0 = \frac{\bar a_\perp \times \bar\Omega}{\Omega^2}$$

$$\bar V_\perp\big|_1 = \frac{\Omega \times \frac{d}{dt}\left(V_\perp\big|_0\right)}{\Omega^2}$$

ETC…

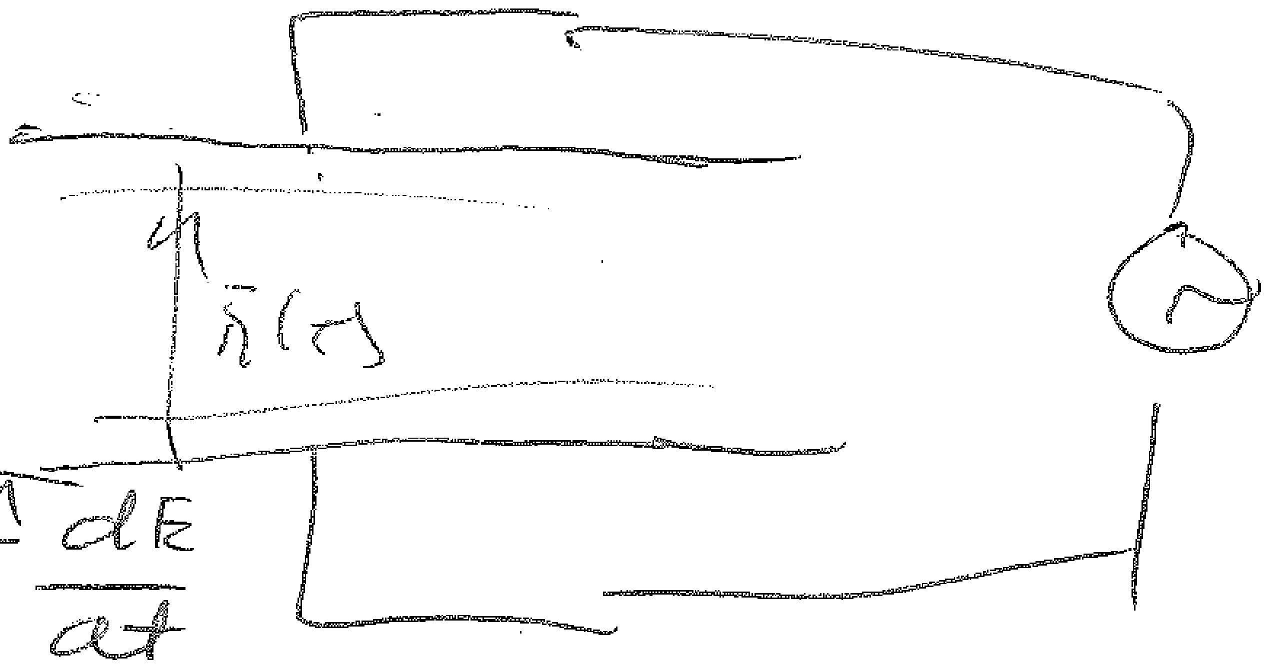
$$\bar V_\perp(t) \simeq \frac{\bar a(t)\times\bar\Omega}{\Omega^2} + \frac{1}{\Omega^2}\,\bar\Omega\times\frac{d}{dt}\left(\frac{\bar a(t)\times\bar\Omega}{\Omega^2} + \cdots\right)$$

$$\simeq \underbrace{\frac{\bar a(t)\times\bar\Omega}{\Omega^2}}_{\uparrow\ E\times B} + \underbrace{\frac{1}{\Omega^2}\frac{d}{dt}\bar a_\perp(t)}_{\text{POLARIZATION DRIFT}} + \cdots$$

WITH $\bar a = \frac{e}{m}\bar E$ $\xrightarrow{\text{THEN}}$

Polarization Drift: Plasma Capacitor

$$J_{POL} = \sum_s q_s n_s V_{POL}$$



$$V_{POL} = \frac{m^2}{q^2 B^2} \frac{q}{m} \frac{dE}{dt}\, q\, n \sim \frac{nm}{B^2} \frac{dE}{dt}$$

$$\nabla \cdot J = -\frac{\partial \rho}{\partial t}$$

$$(d/dt) \left[\nabla \cdot \left(\frac{nM}{B^2} \right) \bar{E} = -\rho \right]$$

$$\epsilon E = D \qquad \epsilon = \frac{nm}{B^2} \sim \left(\frac{e^2 n}{m \epsilon_0} \right) \epsilon_0 \frac{m^2}{e^2 B^2}$$

$$\sim \epsilon_0 \left(\frac{\omega_{pi}^2}{\omega_{ci}^2} \right) \gg 1$$

$$B = 1\,\text{kG} \qquad \omega_{ci} = 2\pi\, \underbrace{10^4 \cdot 10^3}_{10\,\text{MHz}}$$

$$n \sim 10^{10} \qquad \omega_{pi} \underset{2\pi}{\sim} 10^3\, 10^5 \sim 100\,\text{MHz}$$

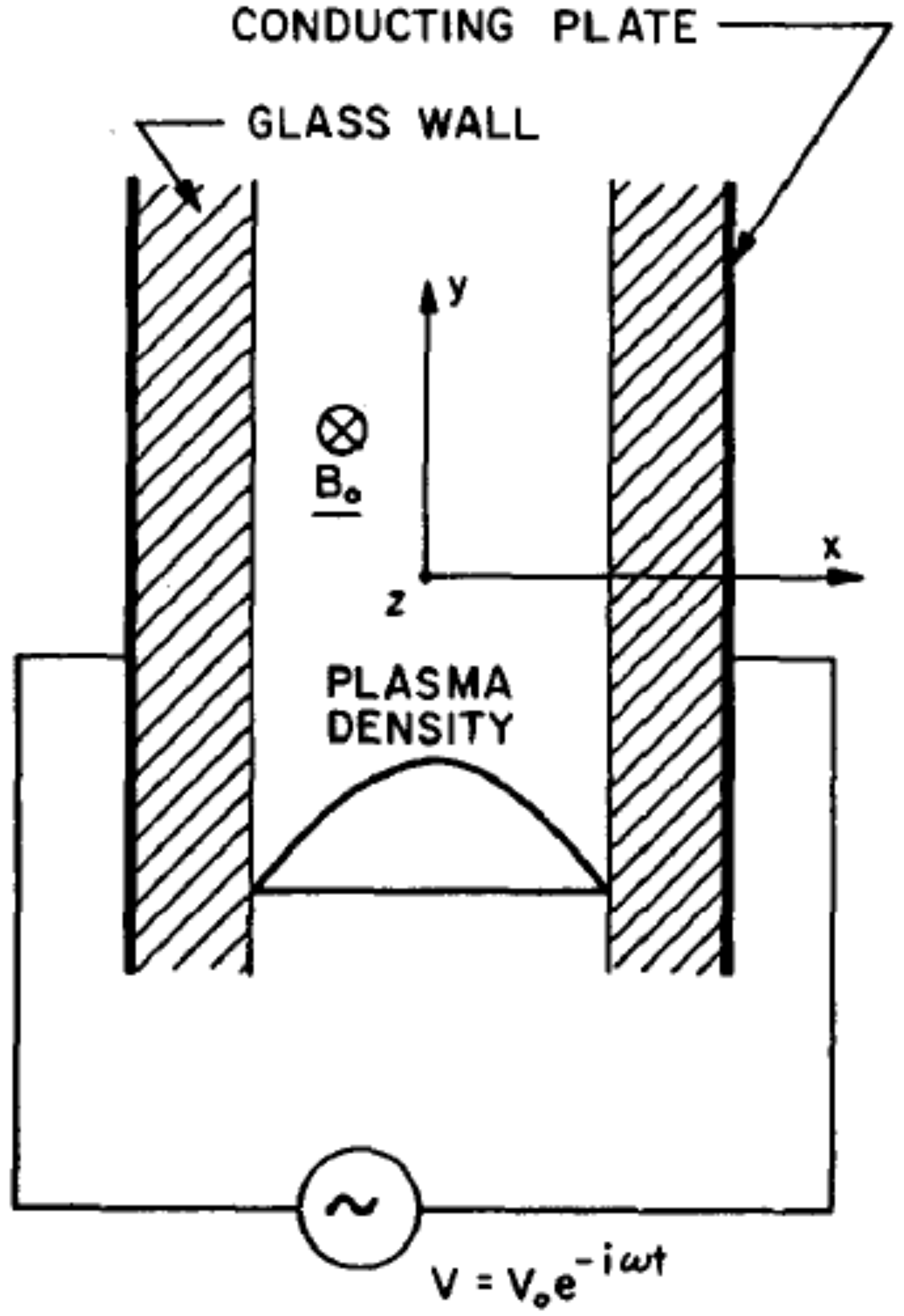
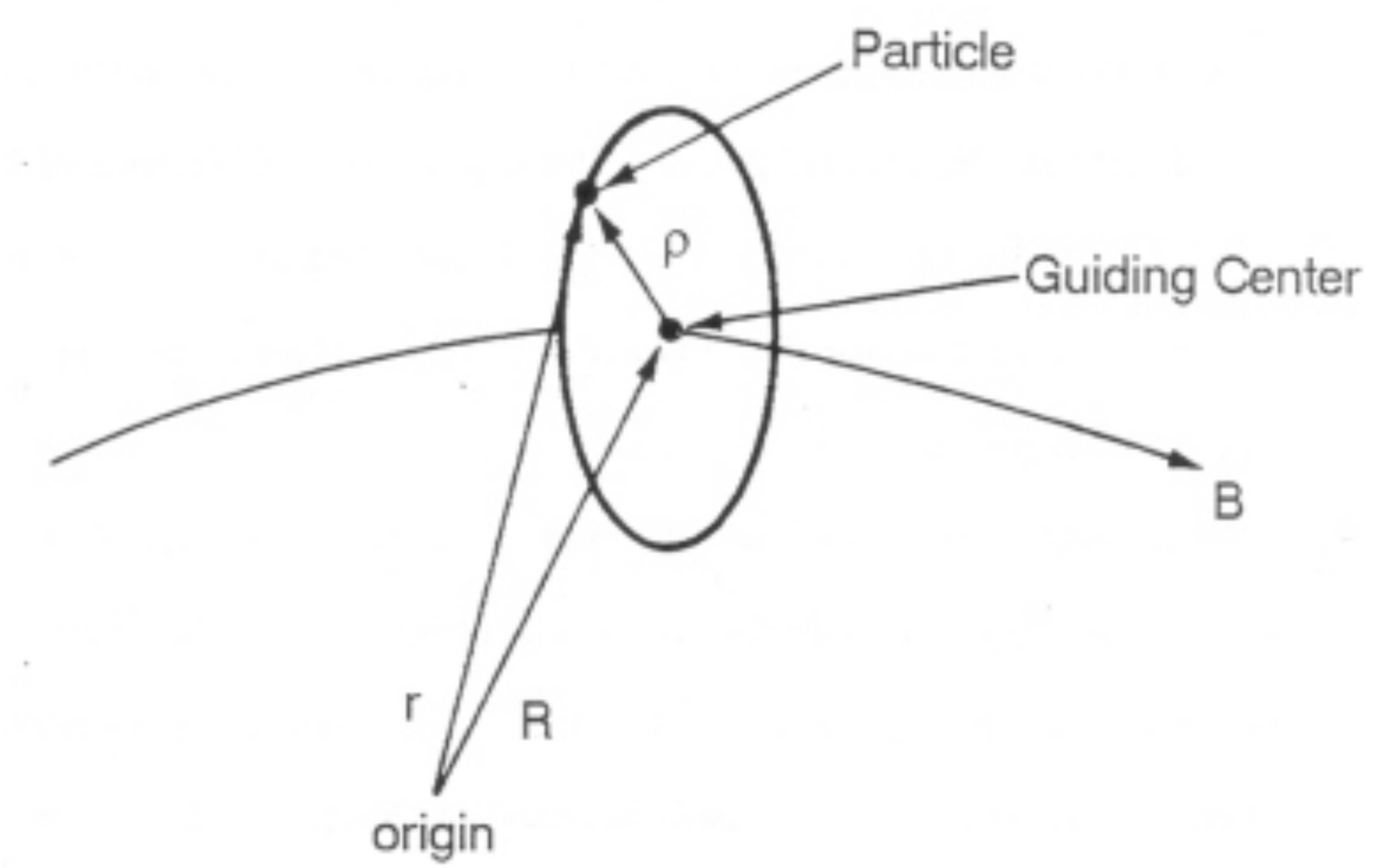


FIG. 1. Geometry of the plasma capacitor. The density variation is shown on the figure.

See: A. L. Peratt, H. H. Kuehl; Plasma Capacitor in a Magnetic Field. *Phys. Fluids* (1972); 15 (6): 1117–1127; https://doi.org/10.1063/1.1694037

Motion with a *slowly* varying B…

NOW, BOTH GYROMOTION AND DRIFT WILL CHANGE (SINCE $\bar{r}$ CHANGES) BUT TO ANALYZE THIS REQUIRES CAREFUL ALGEBRA… WE'LL DO THAT IN THE NEXT STEP.



Use Bernstein's notation…

(See "handout": http://sites.apam.columbia.edu/courses/apph6101x/Plasma1-Adiabatic-Handout.pdf)

$B(x)$ changing slowly…

NOW, BOTH GYROMOTION AND DRIFT WILL CHANGE (SINCE $\bar{\Omega}$ CHANGES) BUT TO ANALYZE THIS REQUIRES CAREFUL ALGEBRA… WE'LL DO THAT IN THE NEXT STEP.

HERE, WE START WITH…

$$\frac{d\bar{\eta}}{dt} = \dot{\bar{\eta}} \times \Big(\underbrace{\bar{\Omega} + (\bar{\rho}\cdot\bar{\nabla})\bar{\Omega} + \cdots}_{\text{TAYLOR EXPAND ABOUT GYRO CENTER}}\Big)$$

$$\bar{\Omega}(\bar{\eta}) = \bar{\Omega}(\bar{R} + \bar{\rho})$$
$$\approx \bar{\Omega}(\bar{R}) + (\bar{\rho}\cdot\bar{\nabla})\bar{\Omega} + \cdots$$

$$\frac{d\bar{\eta}}{dt} \approx \dot{\bar{\eta}} \times \bar{\Omega} + \dot{\bar{\eta}} \times (\bar{\rho}\cdot\bar{\nabla})\bar{\Omega} + \cdots$$

(1) (2) (3)

Drifts (separately)

NOTE THAT THERE ARE OSCILLATING (GYRATING) AND NON-OSCILLATING TERMS.

SO LET'S LOOK ONLY AT THE <u>DRIFTS</u>. SO WE LOOK AT THE <u>GYRO-AVERAGED</u> DRIFT MOTION

1ST TWO ARE EASY..

$$\left\langle \frac{d\dot{\bar{r}}}{dt} \right\rangle_\theta = \frac{d\bar{V}}{dt} \quad ; \quad \left\langle \dot{\bar{r}} \times \bar{r} \right\rangle_\theta = \bar{V} \times \bar{r}$$

Drifts (separately)

LAST IS THE PRODUCT OF TWO OSCILLATING TERMS...

$$\frac{1}{2\pi}\int_0^{2\pi} d\theta \; \dot{\bar{r}} \times (\bar{\rho}\cdot\bar{\nabla})\,\bar{r} = \frac{1}{2\pi}\int_0^{2\pi} d\theta \; \bar{w} \times (\bar{\rho}\cdot\bar{\nabla})\,\bar{r}$$

WHICH GIVES SOMETHING LIKE A CONSTANT PLUS SOMETHING THAT GOES LIKE $2\omega_c$! WE WANT THE CONSTANT PART...

NOW, I WANT TO AVERAGE THE

$\bar{w}\,\bar{\rho}$ TERMS

SO I GOING TO DO THE FOLLOWING

$$\bar{w} \times (\bar{\rho}\cdot\bar{\nabla})\,\bar{r} = \bar{w}\,\underbrace{(\bar{\rho}\cdot\bar{\nabla})}\times\bar{r}$$

THIS IS JUST AN OPERATOR WHICH ACTS ON $\bar{r}$

(IT'S THE SAME THING)

Drifts (separately)

THEN USING

$$\bar{\omega} = \omega\left[\hat{e}_1 \cos\theta - \hat{e}_2 \sin\theta\right]$$
$$\bar{\rho} = \frac{w}{\Omega}\left[\hat{e}_1 \sin\theta + \hat{e}_2 \cos\theta\right]$$

$$\langle \bar{w}\bar{\rho}\rangle_\theta = \frac{w^2}{\Omega}\frac{1}{2\pi}\int_0^{2\pi} d\theta\,\left[\hat{e}_1\cos\theta - \hat{e}_2 \sin\theta\right]\left[\hat{e}_1 \sin\theta + \hat{e}_2 \cos\theta\right]$$
$$\to \frac{w^2}{\Omega}\frac{1}{2\pi}\int_0^{2\pi} d\theta\,\left[\hat{e}_1\hat{e}_2\cos^2\theta - \hat{e}_2\hat{e}_1 \sin^2\theta\right]$$

or

$$\langle \bar{w}\,\bar{\rho}\rangle_\theta = \frac{w^2}{2\Omega}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\langle \dot{\hat{\imath}} \times (\bar{\rho}\cdot\bar{\nabla})\bar{\Omega} \rangle_\theta = \frac{\omega^2}{2\Omega}\left[\hat{e}_1\hat{e}_2 - \hat{e}_2\hat{e}_1\right]\cdot\bar{\nabla}\times\bar{\Omega}$$

$$= -\frac{\omega^2}{2\Omega}(\hat{b}\times\bar{\nabla})\times\bar{\Omega}$$

since $\hat{b}\times\bar{\nabla} = (\hat{e}_1\times\hat{e}_2)\times\bar{\nabla}$ OK

$$= \hat{e}_2\,\hat{e}_1\cdot\bar{\nabla} - \hat{e}_1\,\hat{e}_2\cdot\bar{\nabla}$$

$$\frac{d\bar{v}}{dt} = \bar{v}\times\bar{\Omega} - \frac{\omega^2}{2\Omega}(\hat{b}\times\bar{\nabla})\times\bar{\Omega}$$

$$= \bar{v}\times\bar{\Omega} - \frac{\omega^2}{2\Omega}\left[\bar{\nabla}(\bar{\Omega}\cdot\hat{b}) - \hat{b}\,\underbrace{\bar{\nabla}\cdot\bar{\Omega}}_{\to\, 0}\right]$$

$\nabla\cdot\bar{B} = 0$

now defining $\boxed{\mu \equiv \frac{\frac{1}{2}m\omega^2}{B}}$

$$\boxed{\frac{d\bar{v}}{dt} = \bar{v}\times\bar{\Omega} - \frac{\mu}{m}\bar{\nabla}|B|}$$

!!

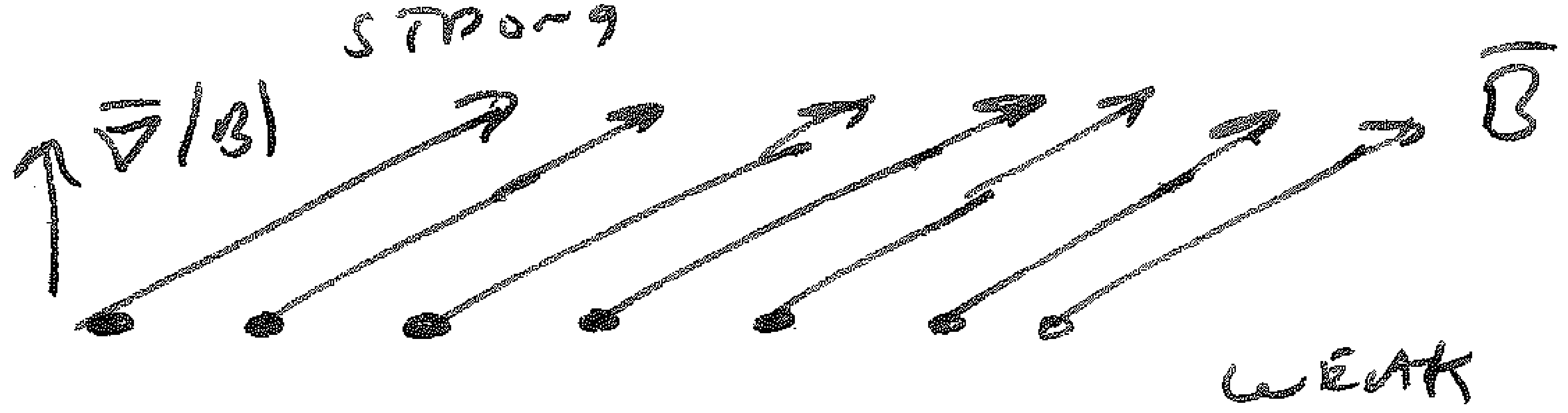
Gradient-B Drift

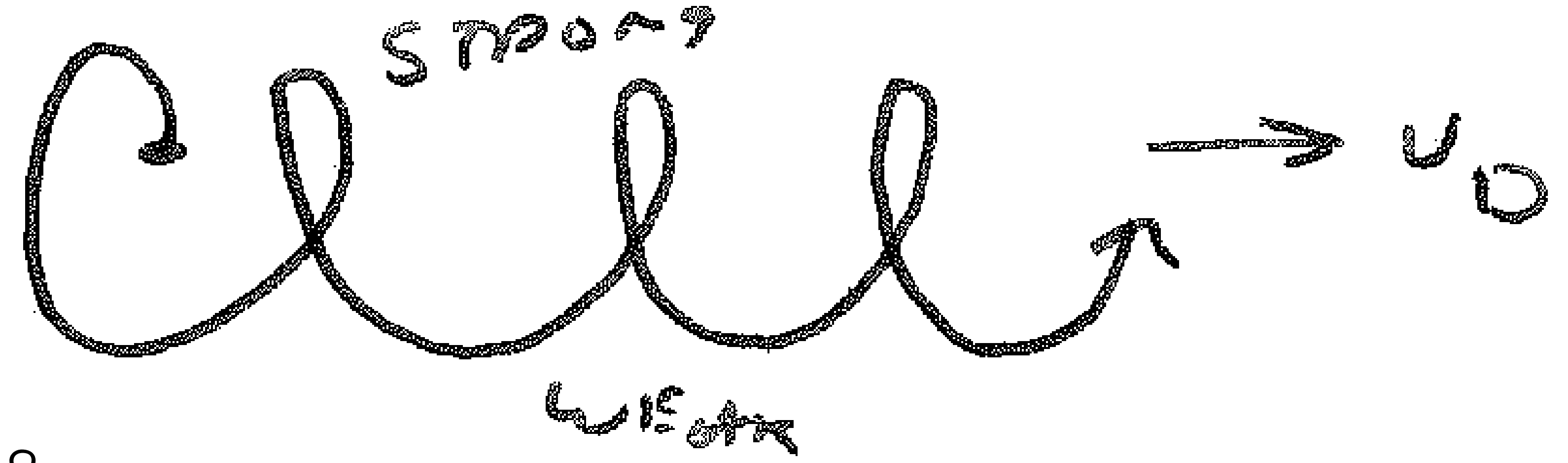
IF $\vec{\nabla}|B|$, μ AND $\vec{\Omega}(R)$ ARE CONSTANT, THEN

AND $\hat{b}\cdot\vec{\nabla}|B| = 0$

$$\vec{V}\,\Omega^2 = +\frac{\mu}{m}\,\vec{\Omega}\times\vec{\nabla}|B|$$

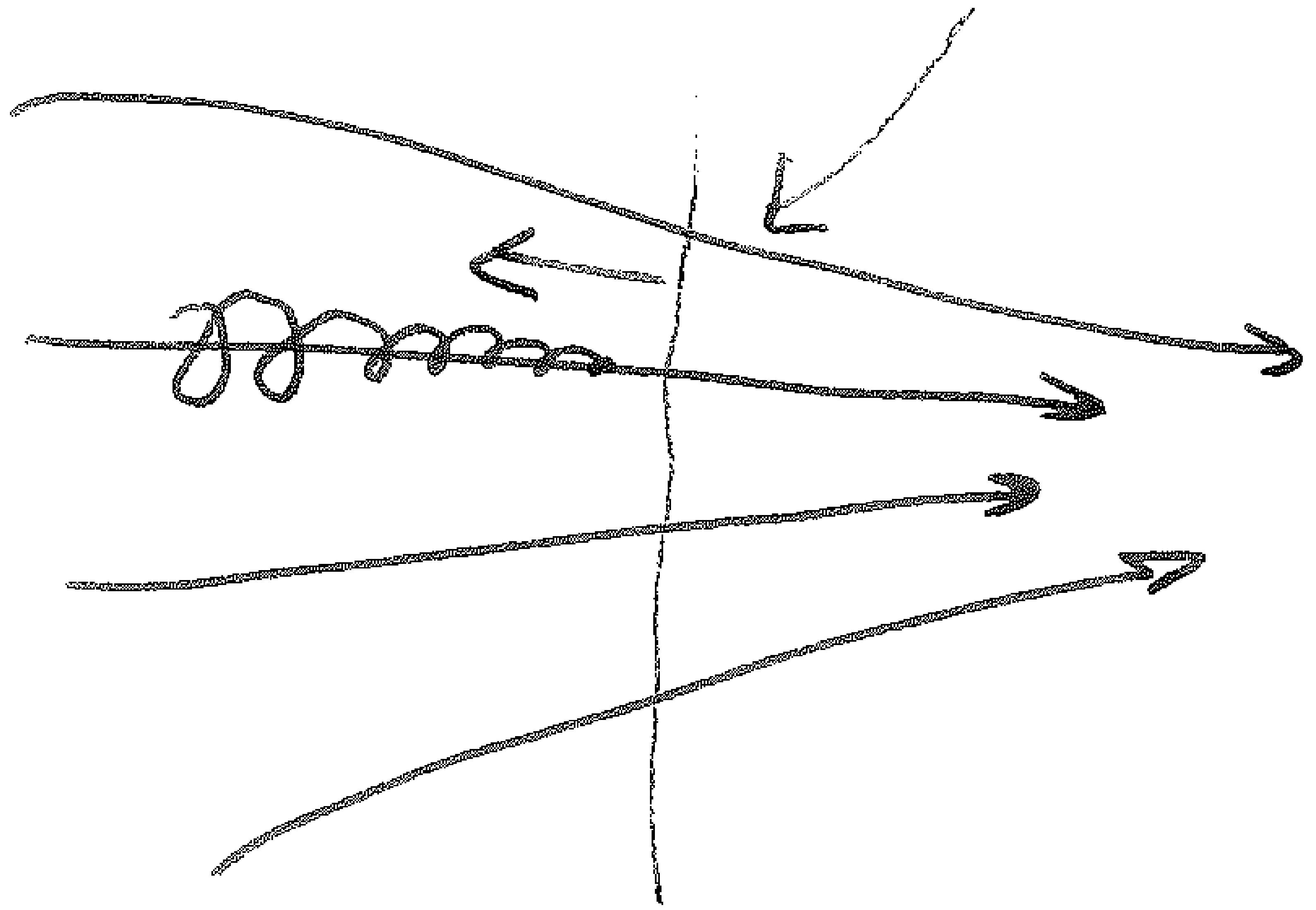
$$\vec{V} = +\frac{\mu}{m}\,\frac{\vec{\Omega}\times\vec{\nabla}|B|}{\Omega^2} \quad !!$$





(Parallel to B) Mirroring…

$$\hat{b}\cdot\frac{d\bar{v}}{dt} = -\underbrace{\frac{\mu}{m}\,\hat{b}\cdot\nabla|B|}_{\mu\cdot\text{GRAD } B \text{ FORCE}}$$!!



$$\frac{d\hat{b}}{dt} = V_\parallel\,\bar{\kappa} = V_\parallel\,(\hat{b}\cdot\bar{\nabla})\hat{b}$$

$$\frac{d}{dt}\hat{b}\cdot\bar{V} - \bar{V}\cdot\frac{d\hat{b}}{dt} = \frac{d}{dt}V_\parallel - V_\parallel\left(\bar{V}_\perp\cdot\bar{\kappa}\right)$$ ← VERY SMALL SINCE $V_\perp \sim \mathcal{O}(\kappa)$

$$\vec r \times \frac{d\vec v}{dt} = r^2 \vec v_\perp - \frac{\mu}{m}\, \vec r \times \vec\nabla |B|$$

$$\vec V_\perp = \frac{\mu}{m}\frac{1}{r}\,\hat b \times \vec\nabla|B| + \frac{\hat b \times \frac{d\vec v}{dt}}{r}$$

NOTICE TOTAL $\vec V$

NOW LET'S SOLVE THIS BY ITERATION...

(Perpendicular to B) Curvature Drift

$$\bar V \simeq \underbrace{\frac{\mu}{m}\frac{1}{r}\,\hat b \times \vec\nabla|B|}_{\text{THIS IS A DRIFT}} + \underbrace{v_\parallel \hat b}_{\text{THIS IS A REAL FAST SPEED}}$$

SLOW

$V\left(\frac{\rho}{L_B}\right)$

$$\bar V \simeq v_\parallel \hat b$$

!!

$$V_\perp \simeq \frac{\mu}{m}\frac{1}{r}\,\hat b \times \vec\nabla|B| + \frac{\hat b \times \frac{d}{dt}\left(v_\parallel \hat b\right)}{r}$$

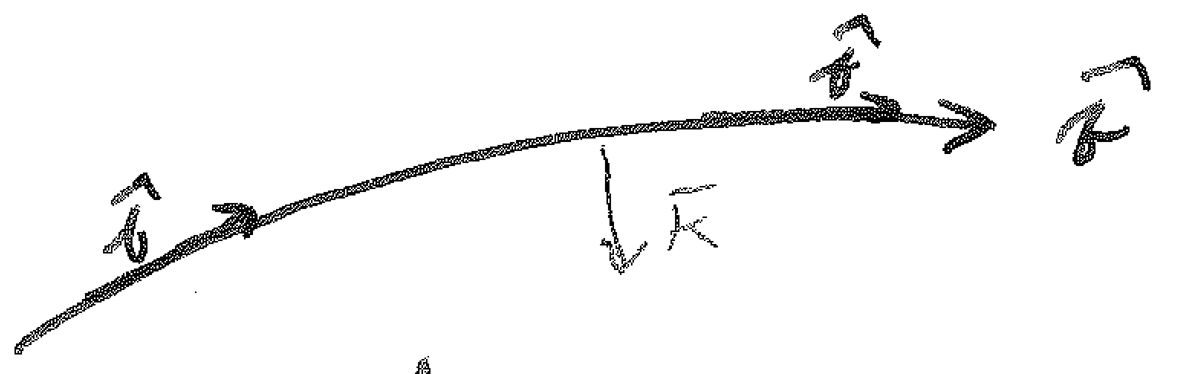
Curvature Drift

$$\bar{V} \approx \underbrace{\frac{\mu}{m}\frac{1}{\Omega}\hat{b}\times\nabla|B|}_{\substack{\text{THIS IS}\\ \text{SLOW A DRIFT}\\ v\left(\frac{\rho}{L_B}\right)}} + \underbrace{v_\parallel \hat{b}}_{\substack{\text{THIS is}\\ \text{A REAL FAST}\\ \text{SPEED}}}$$

$$\bar{V} \approx v_\parallel \hat{b}$$

$$V_\perp \approx \frac{\mu}{m}\frac{1}{\Omega}\hat{b}\times\nabla|B| + \frac{\hat{b}\times\frac{d}{dt}(v_\parallel \hat{b})}{\Omega}$$

$$\frac{d}{dt}(v_\parallel \hat{b}) = \hat{b}\frac{dv_\parallel}{dt} + v_\parallel \frac{d}{dt}\hat{b}$$



$$= \underbrace{\hat{b}\frac{dv_\parallel}{dt}}_{\substack{\text{THIS is}\\ \parallel \text{ to } B}} + v_\parallel^2 \underbrace{\hat{b}\cdot\nabla\hat{b}}_{\kappa\,=\,\text{CURVATURE!!}}$$

$$\frac{d}{dt} \approx v_\parallel \hat{b}\cdot\nabla$$

$$\kappa = \frac{1}{R_c}$$

Adiabatic Invariants (Part 1)

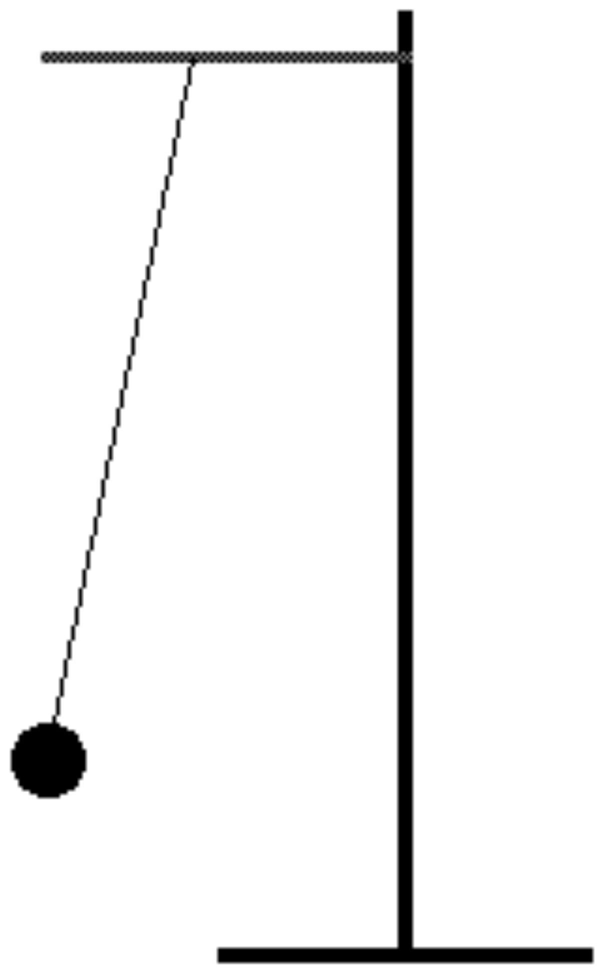
$$(\mu,\ J,\ \psi)$$

See Fitzpatrick Sec. 2.8

Harmonic Oscillator

$$\frac{d^2 x}{dt^2} + \omega^2 x = 0$$

- Separate descriptions of perpendicular and parallel motion
- Fast gyration around B
- Slow perpendicular drift of gyro center



IF ω = CONSTANT

$$x(t) = x_0 \sin(\omega t + \phi)$$

$$\frac{dx}{dt} = \omega x_0 \cos(\omega t + \phi)$$

KINETIC ENERGY PER UNIT MASS $= \frac{1}{2}\left(\frac{dx}{dt}\right)^2$

FORCE PER UNIT MASS $= -\nabla V = -\frac{\partial V}{\partial x}$

V = POTENTIAL PER UNIT MASS

SO $V(x) = \frac{1}{2}\omega^2 x^2$

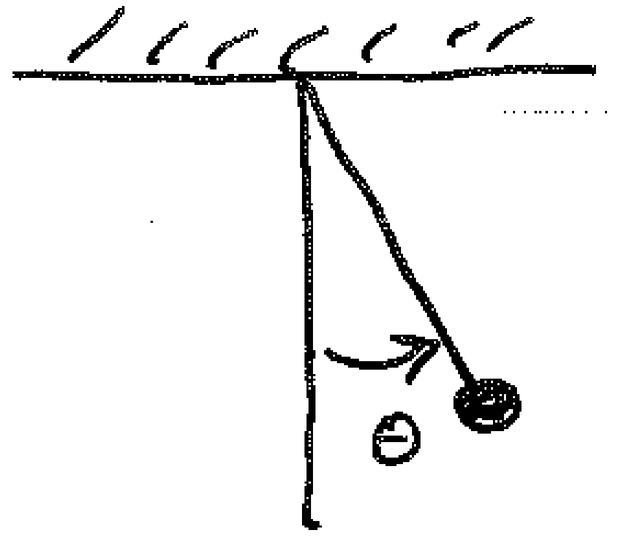
TOTAL ENERGY PER UNIT MASS

$$W = \frac{1}{2}\left(\frac{dx}{dt}\right)^2 + V(x)$$

$$= \frac{1}{2}\omega^2 x_0^2$$

Hamiltonian for an Oscillator

CANNONICAL TRANSFORMATION



PENDULUM

$$(P_\theta, \theta) \Longrightarrow (J, \varphi)$$

ACTION-ANGLE COORDINATES

REAL (PHYSICAL) COORDINATES

WE USE A "GENERATING FUNCTION" WELL-KNOWN IN CLASSICAL, HAMILTONIAN MECHANICS

$$H_{OLD} = P_\theta \dot{\theta} - L = \frac{P_\theta^2}{2m} + V(\theta)$$

$$P_\theta = m\dot{\theta}$$

$$V(\theta) = -\frac{mg}{\ell}\cos\theta$$

Hamiltonian for an Oscillator

DURING THE CANNONICAL TRANSFORMATION, WE REQUIRE

$$\delta \oint L\,dt = \delta \oint (P_\theta \dot{\theta} - H_{OLD})\,dt = \delta \oint (P_J \dot{\varphi} - H_{NEW})\,dt$$

AND THESE TWO TERMS MUST DIFFER ONLY IN A TOTAL DERIVATIVE OF THE GENERATING FUNCTION, F

SINCE $\dfrac{dF}{dt} = \dot{\theta}\dfrac{\partial F}{\partial \theta} + \dot{\varphi}\dfrac{\partial F}{\partial \varphi} + \dfrac{\partial F}{\partial t}$, WE MUST HAVE

$$H_{NEW} = H_{OLD} + \dot{\varphi}\left(P_J + \frac{\partial F}{\partial \varphi}\right) - \dot{\theta}\left(P_\theta - \frac{\partial F}{\partial \theta}\right) + \frac{\partial F}{\partial t}$$

WE ARE FREE TO CHOOSE $F(\theta, \varphi, t)$ AS WE WISH PROVIDED THAT IT SATISFIES OUR REQUIREMENTS.

Hamiltonian for an Oscillator

WE CHOOSE $P_\theta = \frac{\partial F}{\partial \theta}$

$P_J = -\frac{\partial F}{\partial \varphi}$

THEN, $H_{NEW} = H_{OLD} + \frac{\partial F}{\partial t}$ (VERY NICE)

FOR A PENDULUM, IF $F(\theta, \varphi, t) = \frac{1}{2} m \omega_B(t)\, \theta^2 \frac{\cos\varphi}{\sin\varphi}$

THEN

$$P_\theta = m\dot{\theta} = \frac{\partial F}{\partial \theta} = m \omega_B \theta \frac{\cos\varphi}{\sin\varphi} \quad \text{BUT} \quad \theta = \theta_m \sin\varphi$$

$$= m \omega_B \theta_m \cos\varphi \quad \leftarrow \text{SO}$$

$$P_J = -\frac{\partial F}{\partial \varphi} = \frac{1}{2} m \omega_B \theta_m^2 = J$$

THUS

$$\boxed{H_{NEW} = \omega_B J + \frac{1}{2} J \underbrace{\sin 2\varphi}_{\substack{\text{HARMONIC} \\ \text{OF OSCILLATION} \\ \text{FREQUENCY}}} \underbrace{\frac{1}{\omega_B} \frac{\partial \omega_B}{\partial t}}_{\text{SMALL TERM}}}$$

IF $\dot{\omega}_B = 0$, THEN

$\dot{J} = 0$

$\dot{\varphi} = \omega_B$

Very nice!

Adiabatic Invariants (Part 1)

$$(\mu,\ J,\ \psi)$$

See Fitzpatrick Sec. 2.8

What happens when ω(t) changes slowly with time?

- Separate descriptions of perpendicular and parallel motion
- Fast gyration around B
- Slow perpendicular drift of gyro center

LET'S SAY

AND WE DEMAND $x(\theta)$ TO BE PERIODIC

$$x(t) = x(t, \theta) \qquad \frac{d\theta}{dt} = \nu(t)$$

$$\frac{dx}{dt} = \frac{\partial x}{\partial t} + \frac{d\theta}{dt}\frac{\partial x}{\partial \theta} = \frac{\partial x}{\partial t} + \nu \frac{\partial x}{\partial \theta}$$

$$\frac{d^2x}{dt^2} = \frac{\partial^2 x}{\partial t^2} + \nu \frac{\partial^2 x}{\partial \theta \partial t} + \frac{\partial}{\partial t}\left(\nu \frac{\partial x}{\partial \theta}\right) + \nu \frac{\partial}{\partial \theta}\left(\nu \frac{\partial x}{\partial \theta}\right)$$

$$= \frac{\partial^2 x}{\partial t^2} + 2\nu \frac{\partial^2 x}{\partial \theta \partial t} + \frac{d\nu}{dt}\frac{\partial x}{\partial \theta} + \nu^2 \frac{\partial^2 x}{\partial \theta^2}$$

Order by Order with $\dot{\nu}/\nu \ll 1$

PUTTING THIS INTO OUR EQUATION

$$\underbrace{\nu^2 \frac{\partial^2 x}{\partial \theta^2} + \omega^2 x}_{0^{TH}} = \underbrace{-\frac{\partial^2 x}{\partial t^2}}_{2^{ND}} \underbrace{- 2\nu \frac{\partial^2 x}{\partial \theta \partial t} - \frac{\partial \nu}{dt} \frac{\partial x}{\partial \theta}}_{1^{ST}}$$

$\omega(t)!$

NOW, WE'RE GOING TO SOLVE THIS AS IF

$\frac{d\nu}{dt}$ IS VERY SLOW $\quad \left|\frac{\dot{\nu}}{\nu}\right| \ll \nu$

$\frac{\partial x}{\partial t}$ IS VERY SLOW $\quad \frac{|\dot{x}|}{|x|} \ll \nu$

MOST RAPID MOTION IS IN THE OSCILLATION.

Order by Order with $\dot{\nu}/\nu \ll 1$

THEN, TO LEADING ORDER...

$$\nu^2 \frac{\partial^2 x}{\partial \theta^2} + \omega^2 x = 0 \quad \text{or} \quad \frac{d\theta}{dt} = \nu = \omega$$

WKB ✓

$$x = x_0(t) \sin(\theta(t)) \; ; \quad \theta(t) = \int_0^t dt' \omega(t') + \phi$$

but, how does energy and amplitude change with frequency?

Understanding the Energy/Amplitude

LET'S MULTIPLY BY $\frac{\partial x}{\partial \theta}$

$$\frac{\partial}{\partial \theta}\left(\frac{1}{2}\gamma^2\left(\frac{\partial x}{\partial \theta}\right)^2\right) + \frac{\partial}{\partial \theta}\left(\frac{1}{2}\omega^2 x^2\right)$$

$$= -\frac{\partial x}{\partial \theta}\frac{\partial^2 x}{\partial t^2} - \gamma\frac{\partial}{\partial t}\left(\left(\frac{\partial x}{\partial \theta}\right)^2\right) - \frac{dv}{dt}\left(\frac{\partial x}{\partial \theta}\right)^2$$

$$\frac{\partial}{\partial \theta}\left(\frac{1}{2}\gamma^2\left(\frac{\partial x}{\partial \theta}\right)^2 + \frac{1}{2}\omega^2 x^2\right) = -\frac{\partial x}{\partial \theta}\frac{\partial^2 x}{\partial t^2} - \frac{\partial}{\partial t}\left(\gamma\left(\frac{\partial x}{\partial \theta}\right)^2\right)$$

NOW, THE TERM $\frac{\partial x}{\partial \theta}\frac{\partial^2 x}{\partial t^2}$ IS VERY VERY SMALL.

Understanding the Energy/Amplitude

NOW, THE TERM $\frac{\partial x}{\partial \theta} \frac{\partial^2 x}{\partial t^2}$ IS <u>VERY</u> <u>VERY</u> <u>SMALL</u>.

SO IN THE SPIRIT OF OUR PERTURBATION ANALYSIS, WE'RE GOING TO IGNORE THIS TERM.

THEN

$$\frac{\partial}{\partial t}\left(\nu \left(\frac{\partial x}{\partial \theta}\right)^2\right) = -\frac{\partial}{\partial \theta}\left(\frac{1}{2}\nu^2\left(\frac{\partial x}{\partial \theta}\right)^2 + \frac{1}{2}\omega^2 x^2\right)$$

BUT $x(\theta)$ MUST BE <u>PERIODIC</u>.

So we can integrate over θ $\qquad \frac{1}{2}\nu^2\left(\frac{\partial x}{\partial \theta}\right)^2 + \frac{1}{2}\omega^2 x^2 = \text{PERIODIC}$

THUS! $\qquad \boxed{\left\langle \nu \left(\frac{\partial x}{\partial \theta}\right)^2 \right\rangle_\theta \cong \text{CONSTANT}}$

d/dt (…) = 0

Understanding the Energy/Amplitude

$$\left\langle \nu \left(\frac{\partial x}{\partial \theta}\right)^2 \right\rangle_\theta = \frac{\nu}{2\pi}\int_0^{2\pi} d\theta \; x_0^2 \cos^2\theta$$

$$= \frac{1}{2}\,\nu\, x_0^2 = \underline{\underline{\frac{1}{2}\,\omega x_0^2}}$$

$$\oint \nu \frac{\partial x}{\partial \theta}\frac{\partial x}{\partial \theta}\frac{d\theta}{2\pi}$$

$$p = \frac{dx}{dt} \sim \nu \frac{\partial x}{\partial \theta}$$

$$dq = \frac{\partial x}{\partial \theta} d\theta$$

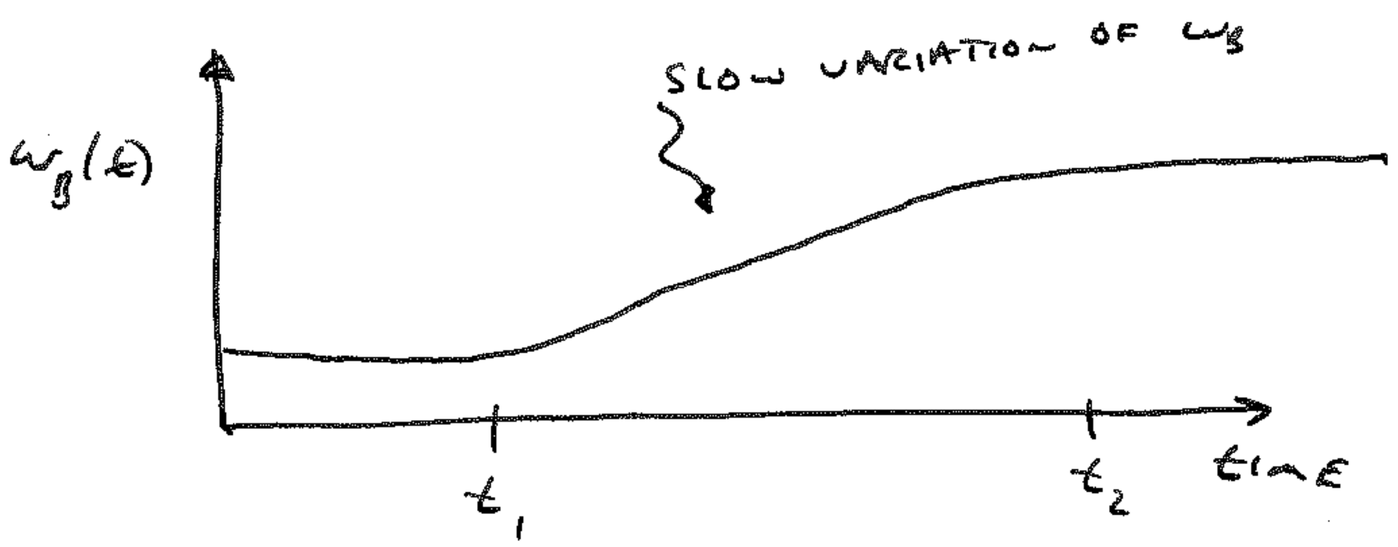
$$\text{ACTION} = \oint_{0 \le \theta \le 2\pi} P\,dq = \boxed{\frac{\text{ENERGY}}{\text{Frequency}}}.$$

How Good are Adiabatic Invariants?

EXACT EQUATIONS OF MOTION

$$\begin{cases} \dot{J} = -\dfrac{\partial H_{NEW}}{\partial \varphi} = -J\cos 2\varphi \, \dfrac{1}{\omega_B} \dfrac{\partial \omega_B}{\partial t} \\ \dot{\varphi} = \omega_B + \dfrac{1}{2} \sin 2\varphi \, \dfrac{1}{\omega_B} \dfrac{\partial \omega_B}{\partial t} \end{cases}$$

FROM <u>MECHANICS</u>, LANDAU + LIFSHITZ . . .



$$\Delta J = J(t_2) - J(t_1) = + \int_{t_1}^{t_2} J \cos 2\varphi \, \frac{1}{\omega_B} \frac{\partial \omega_B}{\partial t} \, dt$$

BUT $$\frac{J\cos 2\varphi}{\omega_B} = \sum_{p=-\infty}^{\infty} \Lambda_p e^{ip\varphi} \quad !$$

MUST BE A PERIODIC FUNCTION OF φ

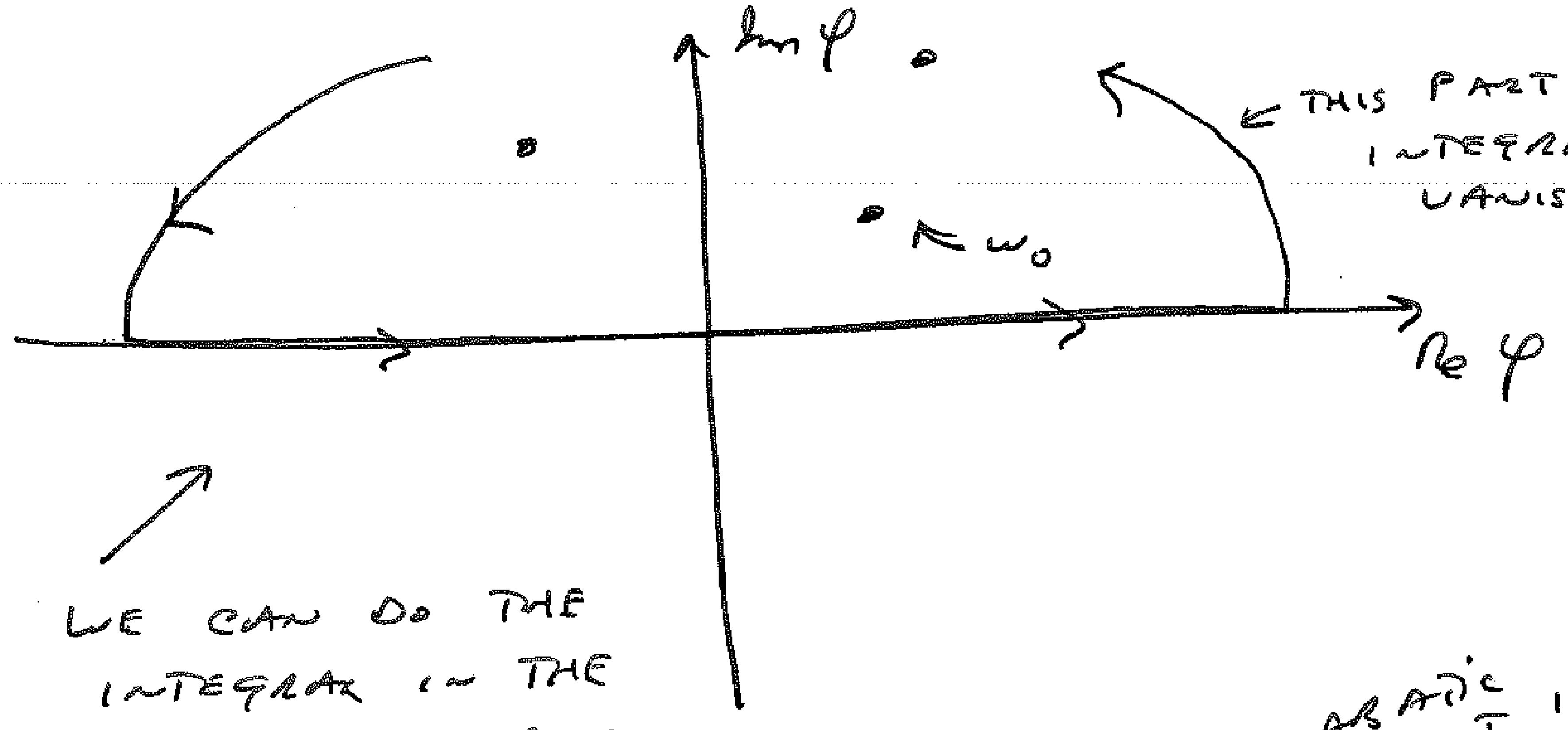
$$= 2\,\mathrm{Re}\left\{ \sum_{p=1}^{\infty} e^{ip\varphi} \Lambda_p \right\}$$

How Good are Adiabatic Invariants?

Answer:
Exponentially good

$$\Delta J = 2\,\mathrm{Re}\left\{ \int_{-\infty}^{\infty} dt\, \Lambda_p\, e^{ip\varphi} \frac{\partial \omega_B}{\partial t} \right\}$$

$$= 2\,\mathrm{Re}\left\{ \int_{-\infty}^{\infty} d\varphi\, \Lambda_p\, e^{ip\varphi} \frac{\partial \omega_B}{\partial \varphi} \right\} \qquad \left(dt = \frac{d\varphi}{\omega_B}\right)$$



THIS PART OF INTEGRAL VANISHES

WE CAN DO THE INTEGRAL IN THE COMPLEX φ-PLANE

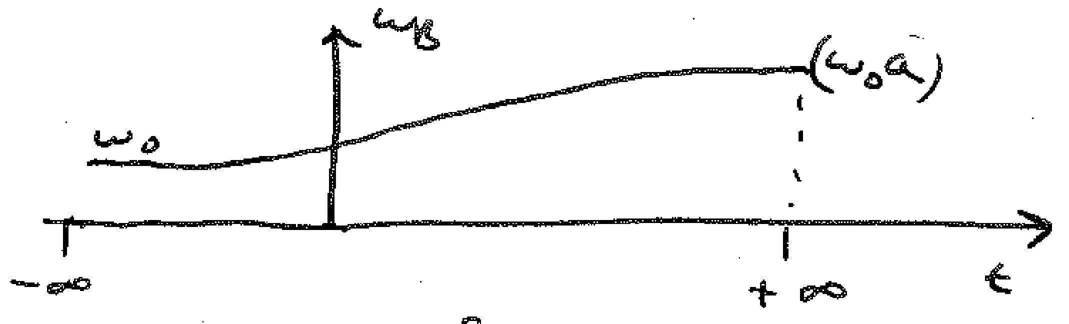
$$\Delta J \sim e^{ip\omega_0}$$

AN ADIABATIC INVARIANT IS EXP. GOOD

WHERE ω_0 = LOWEST SINGULARITY IN UPPER HALF PLANE

ω_0 IS SINGULARITY OF $\omega_B(t)$

EXAMPLE: $\omega_B(t) = \omega_0^2 \dfrac{1 + a^2 e^{\alpha t}}{1 + e^{\alpha t}}$ $\qquad \alpha \ll \omega_0$, $a > 0$



THEN $\dfrac{d\omega_B}{dt} = \dfrac{1}{2}\alpha\,\omega_B \left(\dfrac{a^2}{e^{-\alpha t} + a^2} - \dfrac{1}{e^{-\alpha t} + 1} \right)$ $\qquad \mathrm{Im}\,\omega_0 = \dfrac{\pi \omega_0 a}{\alpha}$

Nonadiabaticity in mirror machines

Phys. Fluids 21(4), April 1978

Ronald H. Cohen, George Rowlands,[a] and James H. Foote

Lawrence Livermore Laboratory, University of California, Livermore, California 94550

(Received 14 February 1977; final manuscript received 25 October 1977)

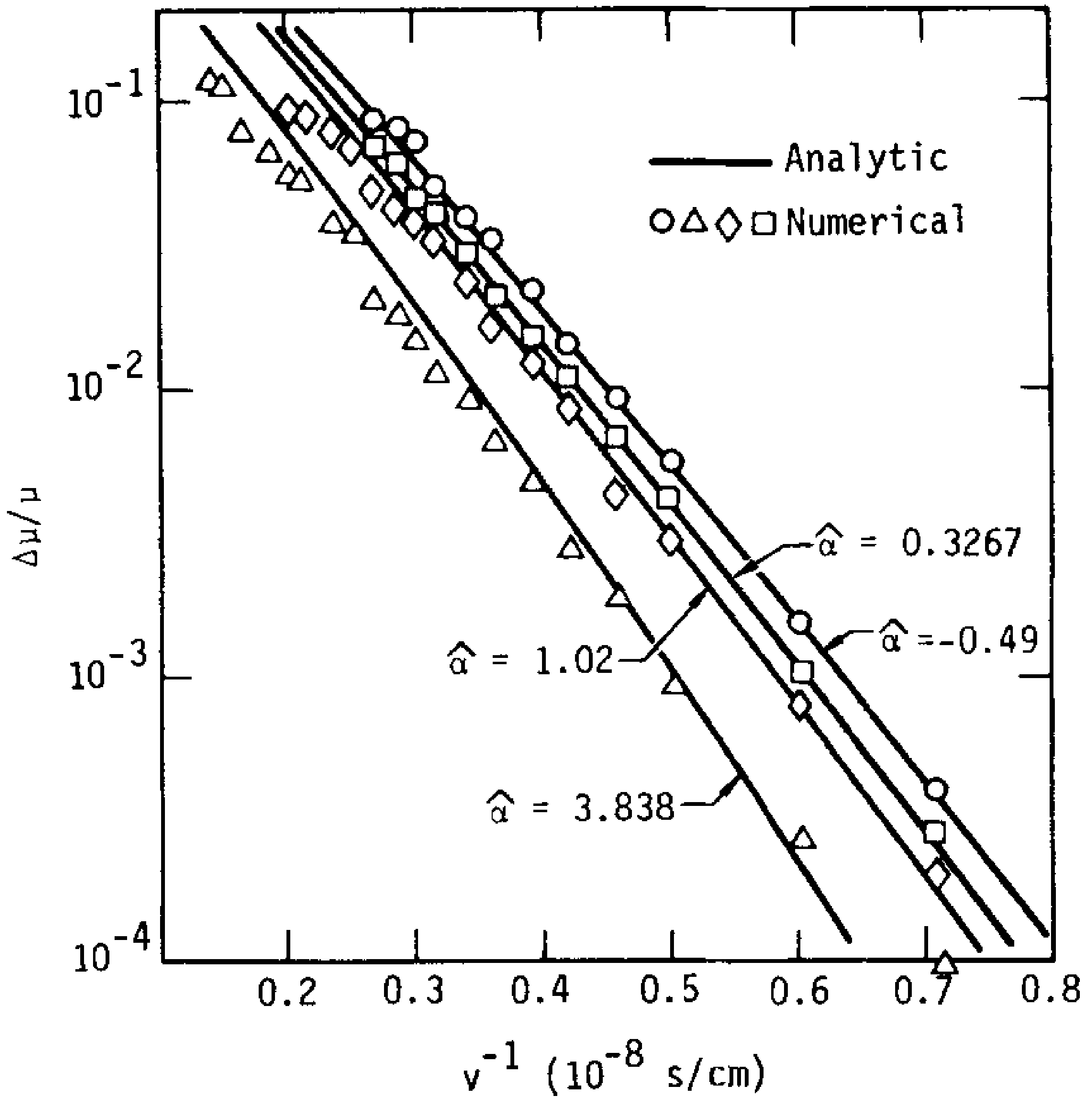


FIG. 1. $\Delta\mu/\mu$ versus v^{-1} for the model field defined by Eqs. (21) and (29) with $L = 20.2$ cm, $B_{00} = 10$ kG. Protons are started at $z = 0$, $r_0 = 3$ cm with $\mathbf{v}_\perp$ radial, $v_\perp/v = 0.544$.

Drift Hamiltonian

$$H(\rho_\parallel, \alpha, \psi, \chi) = \frac{1}{2}\rho_\parallel^2 \frac{eB^2}{mc} + \frac{\mu c}{e} B + c\Phi;$$

$$\mathbf{B} = \nabla\alpha \times \nabla\psi,$$

$$\mathbf{B} = \nabla\chi + \beta\nabla\psi + \gamma\nabla\alpha.$$

$$\nabla\chi \cdot (\nabla\alpha \times \nabla\psi) = B^2$$

then

$$\frac{d\chi}{dt} = \frac{\partial H}{\partial \rho_\parallel}, \quad \frac{d\rho_\parallel}{dt} = -\frac{\partial H}{\partial \chi}, \quad \frac{d\psi}{dt} = \frac{\partial H}{\partial \alpha}, \quad \frac{d\alpha}{dt} = -\frac{\partial H}{\partial \psi}.$$

The adiabatic invariance of J,

$$J = \frac{e}{c}\int \rho_\parallel \, d\chi = \int m v_\parallel \, dl,$$

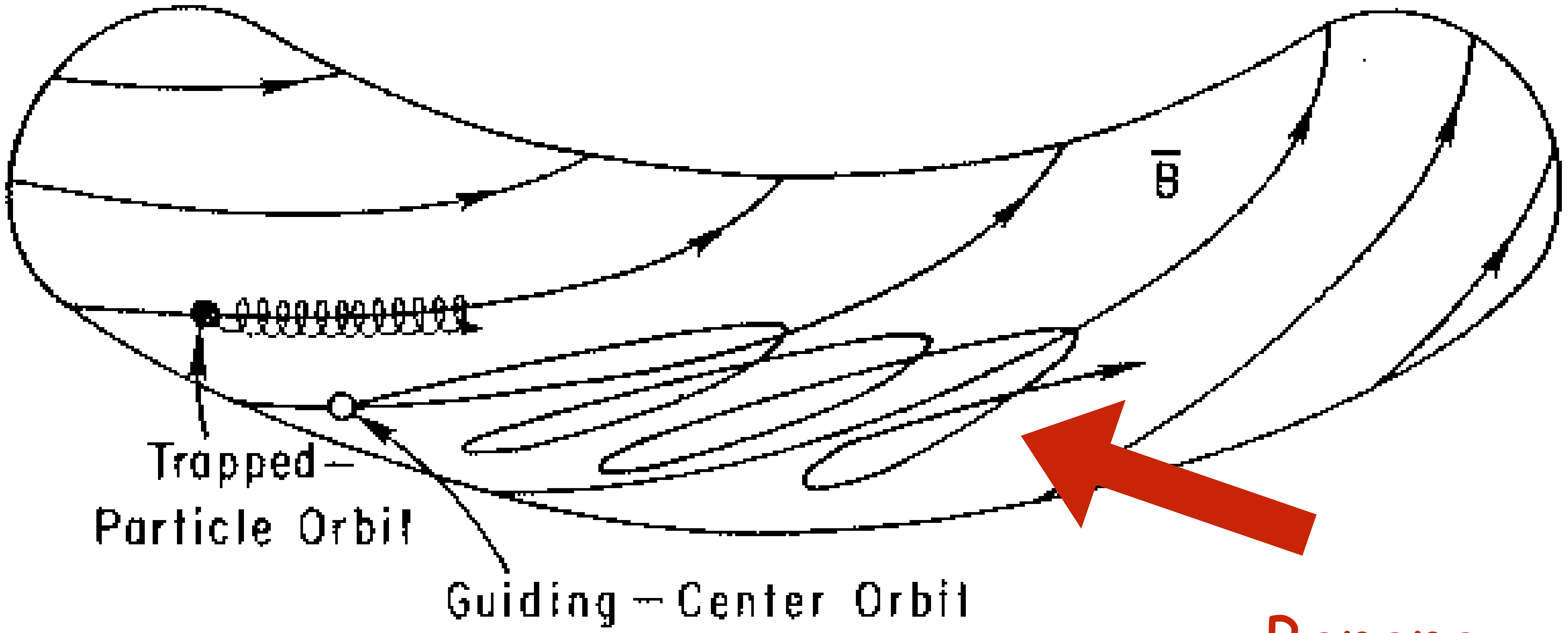
then follows from the standard classical mechanics[4] treatment. The Hamiltonian is just the energy E times c/e and it is conserved.

Guiding center drift equations

Allen H. Boozer

Phys. Fluids 23(5), May 1980

Examples of Confined Orbits (Part 2)

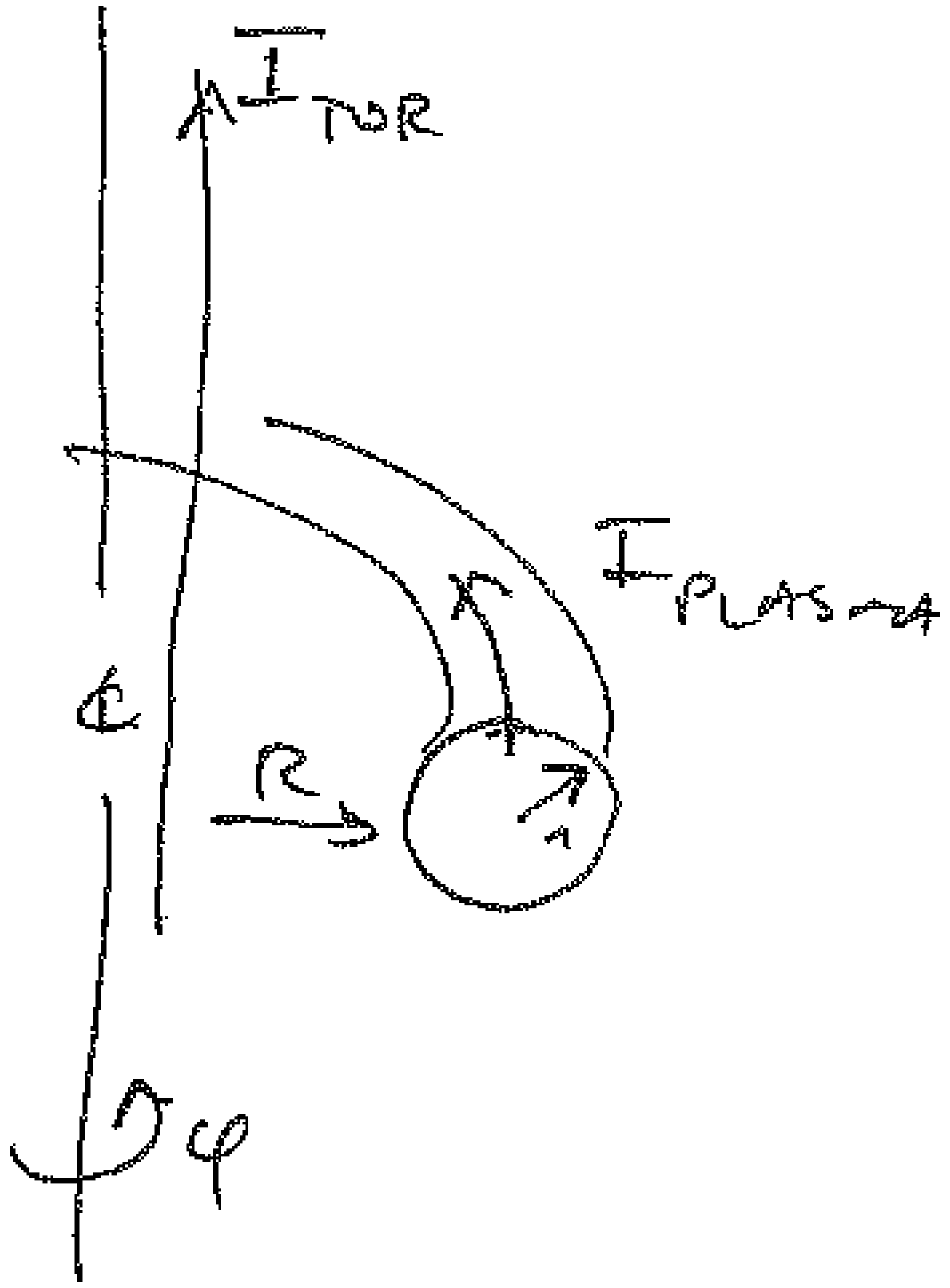


gyration :: bounce :: toroidal precession

1 :: ρ/R :: (ρ/R)^2

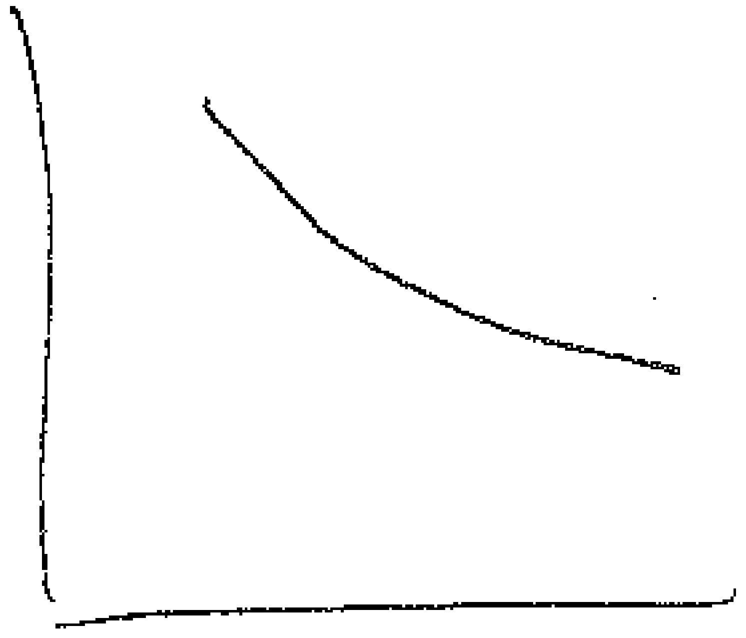
Fast :: Not so fast :: “Slow”

Toroidal Magnetic Field



$$\bar{B} = B_t \hat{\varphi} + \bar{B}_p$$

$$B_t = B_0 \left(\frac{R_0}{R} \right)$$



KRUSKAL – SHAFRANOV CRITERIA

STATES

$$\frac{r B_t}{R B_p} > 1$$

Toroidal Magnetic Field

|B| varies along the magnetic field

$$2\pi R B_t = \mu_0 I_{TOR} \qquad \text{so} \qquad \frac{r B_t}{R B_p} = \frac{\frac{1}{R} I_{TOR}}{\frac{R}{r} I_p} = \frac{I_{TOR}}{I_p}\left(\frac{r}{R}\right)^2$$

$$2\pi r B_p = \mu_0 I_p$$

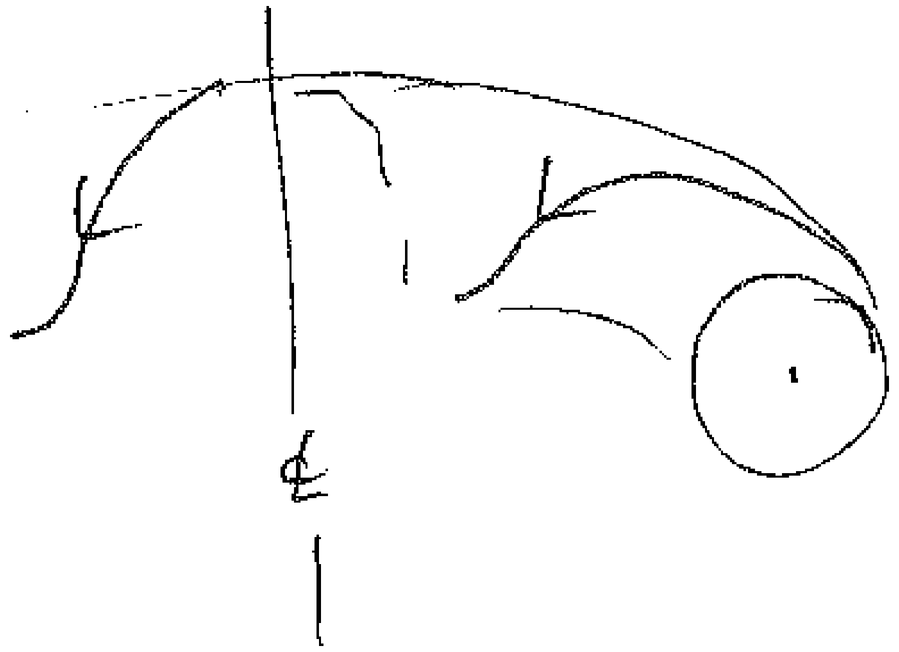
So $\frac{I_{TOT}}{I_p} > \left(\frac{R}{r}\right)^2$ VERY BIG

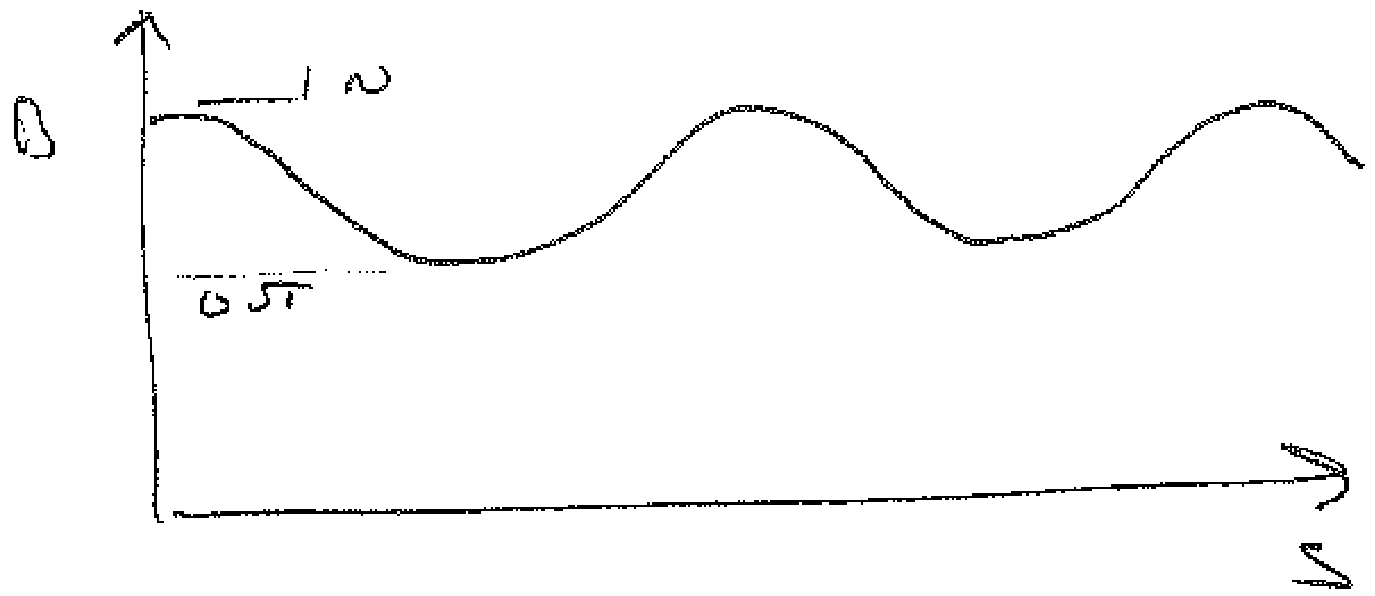
BECAUSE OF I_p, THE FIELD LINES FORM "NESTED" TORI → HELICAL TRAJECTORIES

USUALLY $\frac{R}{r} > 3$

$\epsilon \equiv \frac{a}{R}$

> 6 FOR COLUMBIA





$$B_{OUT} = B_0 \frac{R_0}{R_0 + r} = B_0 \frac{1}{1+\epsilon}$$

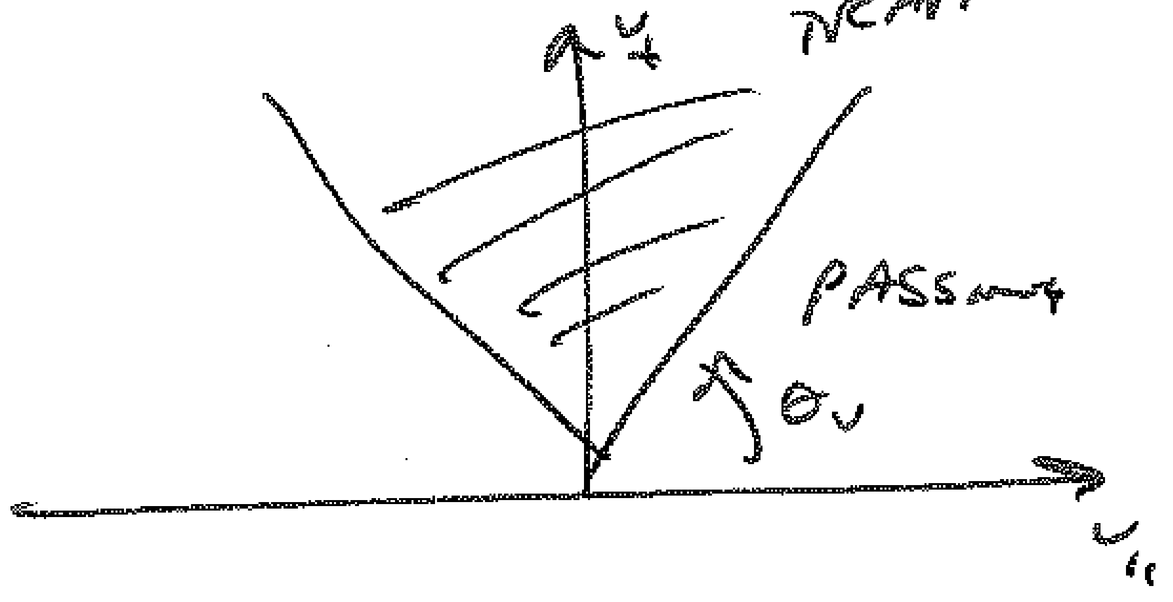
$$B_{IN} = B_0 \frac{1}{1-\epsilon}$$

$$\frac{B_{MAX}}{B_{MIN}} = \frac{1+\epsilon}{1-\epsilon} \sim 1 + 2\epsilon$$

Toroidal Magnetic Field

Trapped and Passing Particles

BECAUSE μ = CONSTANT



@ OUTSIDE EQUATORIAL MIDPLANE

BOUNDARY BTWN TRAPPED/PASSING

$$\tfrac{1}{2}v_{\parallel 0}^2 + \mu B_{OUT} = \mu B_{IN}$$

$$\tfrac{1}{2}v_{\parallel 0}^2 + \tfrac{1}{2}v_{\perp 0}^2 = \tfrac{1}{2}v_{\perp 0}^2 \frac{B_{IN}}{B_{OUT}}$$

R = MIRROR RATIO

$$= \frac{B_{IN}}{B_{OUT}} \cong 1 + 2\epsilon$$

$$\therefore \quad v_{\parallel 0}^2 = v_{\perp 0}^2 (R-1)$$
$$= v_{\perp 0}^2 \, 2\epsilon$$

Toroidal Magnetic Field

How Many Trapped Particles?

How many trapped particles?

$$\% \text{ of passing particles} = \frac{\int_0^{\theta_v} 2\pi v^2 \sin\theta \, d\theta \, dv \, f}{\int_0^{\pi/2} 2\pi v^2 \sin\theta \, d\theta \, dv \, f}$$

$$\cong \frac{\int_0^{\theta_v} \sin\theta \, d\theta}{\int_0^{2\pi} \sin\theta \, d\theta} = 1 - \cos\theta_v$$

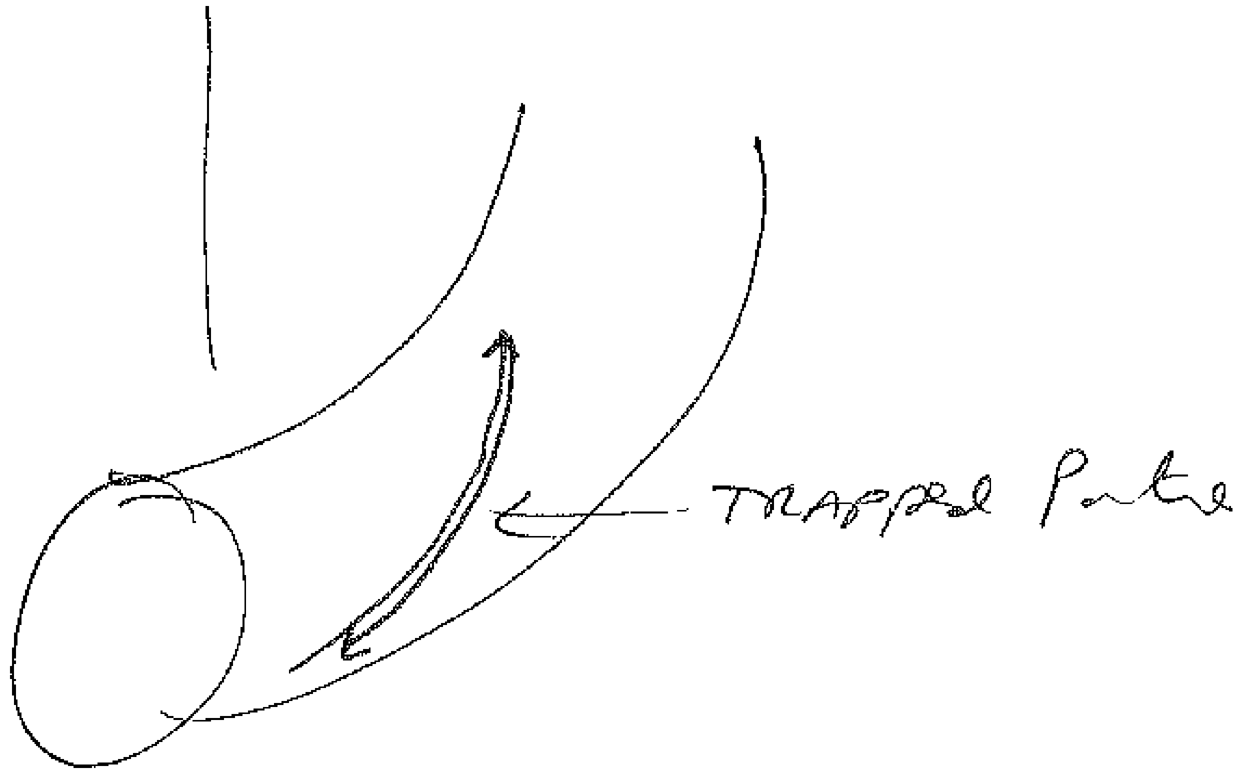
$$\% \text{ Trapped particles} = \cos\theta_v = \frac{v_\parallel}{v} = \frac{v_\perp \sqrt{2\epsilon}}{\sqrt{v_\perp^2 + v_\perp^2 2\epsilon}} \sim \sqrt{2\epsilon}$$

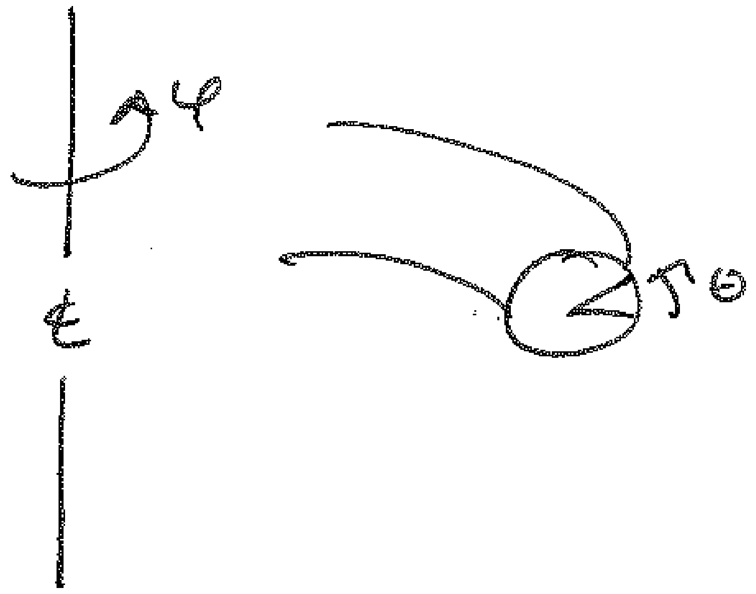
Quite a few!

Toroidal Magnetic Field

Bounce Motion

How long 4 Bounce Frequency?

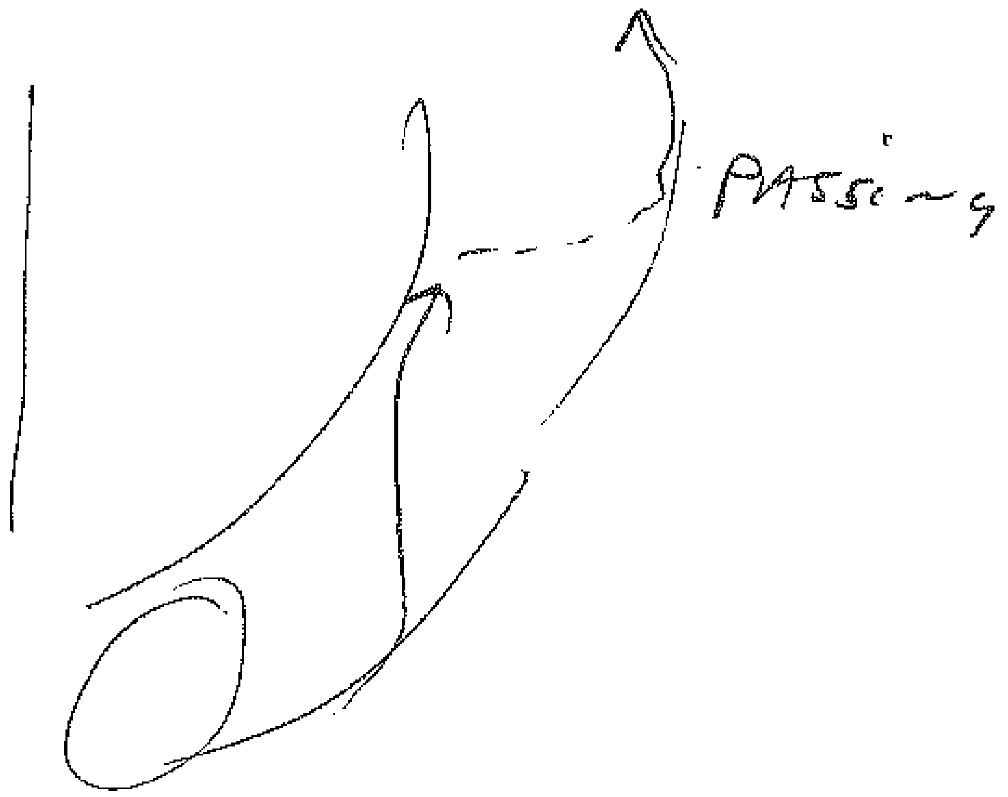


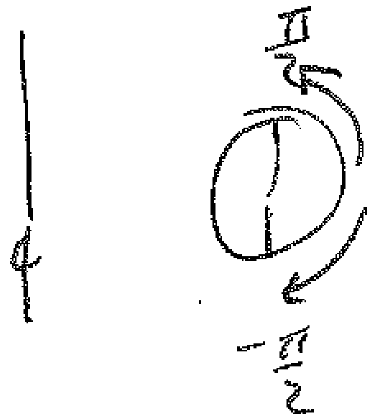


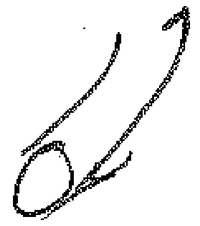
$$d\theta = q\, d\varphi$$

$$(ds)^2 = (r\, d\theta)^2 + (R\, d\varphi)^2$$
$$= (r\, d\theta)^2 + \left(\frac{R}{q}\, d\theta\right)^2$$
$$= \left(\frac{R}{q}\right)^2 \left(1 + \left(\frac{r}{R} q\right)^2\right) d\theta$$

$$ds \cong \frac{R}{q}\, d\theta$$







$$\int ds = \frac{R}{q}\pi = \frac{\pi R}{q}$$

Bounce Frequency $\sim \frac{v_\parallel}{L} \sim \frac{v\, 2q}{\pi R}$

$$\frac{\omega_B}{\omega_c} \sim \frac{2}{\pi} q \left(\frac{\rho}{R}\right) << 1$$

Toroidal Magnetic Field

Drift Motion

$$\bar{V}_D = \frac{\hat{b}\times}{\Omega}\left(\frac{\mu \nabla B}{m} + v_\parallel^2 \bar{\kappa}\right)$$

$$\bar{B}\cdot\bar{B} = B_t^2 + B_p^2 \qquad \text{let } q(r) \equiv \frac{r B_t}{R B_p}$$

$$= B_t^2\left(1 + \left(\frac{r}{R}\right)^2 \frac{1}{q^2}\right)$$

$$\nabla B \sim \nabla B_t \sim -\frac{\hat{r}}{R} B_t$$

$$\text{let } \bar{B}_p \sim B_p \hat{\theta}$$

$$\bar{\kappa} = (\hat{b}\cdot\nabla)\hat{b} \sim -\frac{\hat{r}}{R} + \left(\frac{B_p}{B}\right)^2 (\hat{\theta}\cdot\nabla)\hat{\theta} \qquad \hat{b} = \frac{B_t\hat{\varphi} + B_p\hat{\theta}}{B}$$

$$= -\frac{\hat{r}}{R} - \left(\frac{B_p}{B}\right)^2 \frac{\hat{r}}{r} \qquad \approx \hat{\varphi} + \frac{B_p}{B_t}\hat{\theta}$$

So major effect is <u>TOROIDAL</u> curvature!

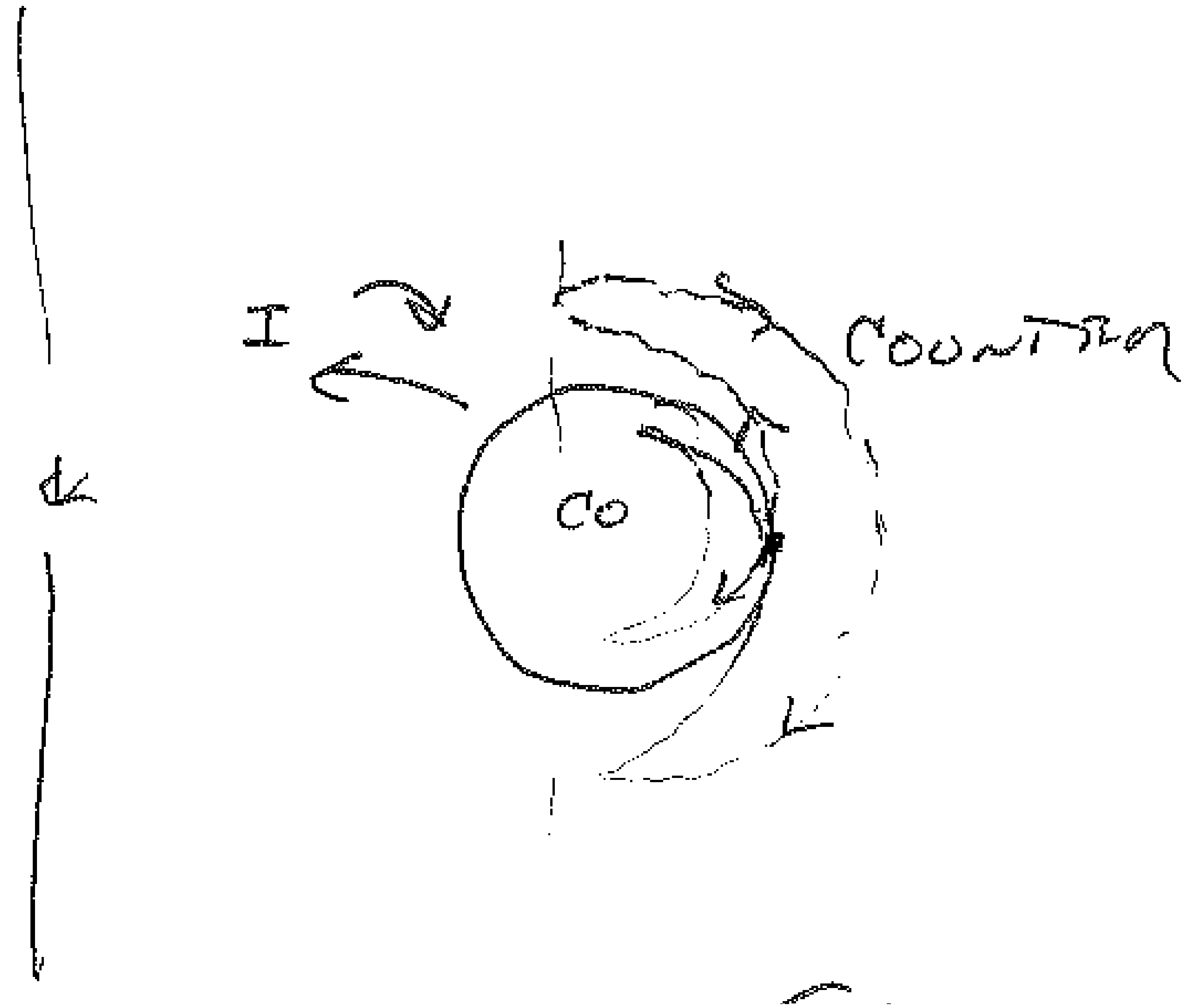
Toroidal Magnetic Field

Drift Motion

$$\vec{V}_D \sim -\frac{\hat{\varphi}\times\hat{r}}{\Omega R}\left(\mu B + v_\parallel^2\right)$$

$$\sim \frac{\hat{z}}{\Omega R}\left(\mu B + v_\parallel^2\right)$$

$$\sim \frac{\hat{z}}{R}\, v\rho \sim \hat{z}\, v\left(\frac{\rho}{R}\right)$$



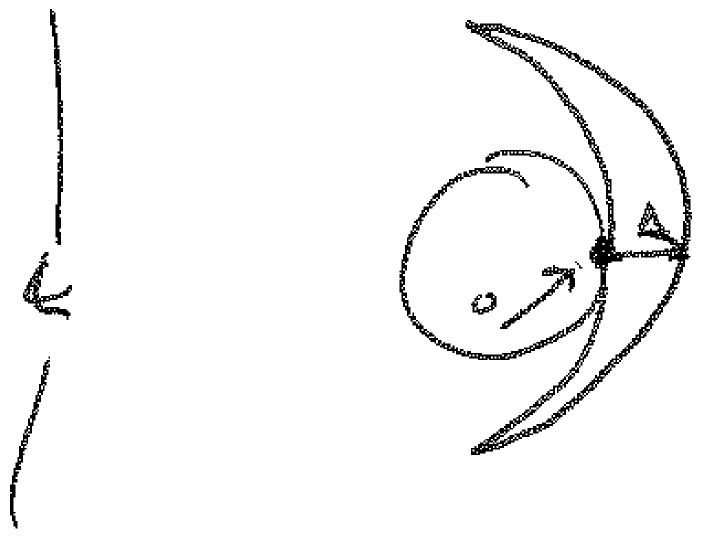
Toroidal Magnetic Field

How far off a flux surface?

$$P_\varphi \sim \text{constant} = R\left[mV_\varphi + \frac{q}{c}A_\varphi\right]$$

Define $\quad A_\varphi = \int_{r_0}^{r} B_p \, dr$

So $\quad A_\varphi(r_0) = 0 \qquad A_\varphi(r) = A_\varphi(r_0) + \Delta B_p$



$$R m V_\parallel \cong -R m V_\parallel + \frac{q}{c} R \Delta B_p$$

Thus $\quad V_\parallel = \frac{1}{2}\frac{q B_p}{c m}\Delta$

But if we recall

$$B \cong B_{t0}(1 - \epsilon\cos\theta)$$

$$\mu B_{t0}(1 - \epsilon\cos\theta_m) = \frac{1}{2}v_\parallel^2 + \mu B_t(1 - \epsilon\cos\theta)$$

Then $\quad v_\parallel^2 = 2\mu B_t\,\epsilon(\cos\theta - \cos\theta_m)$

Trapped Particles

$v_\parallel \to 0$ as $\theta \to \theta_m$

So

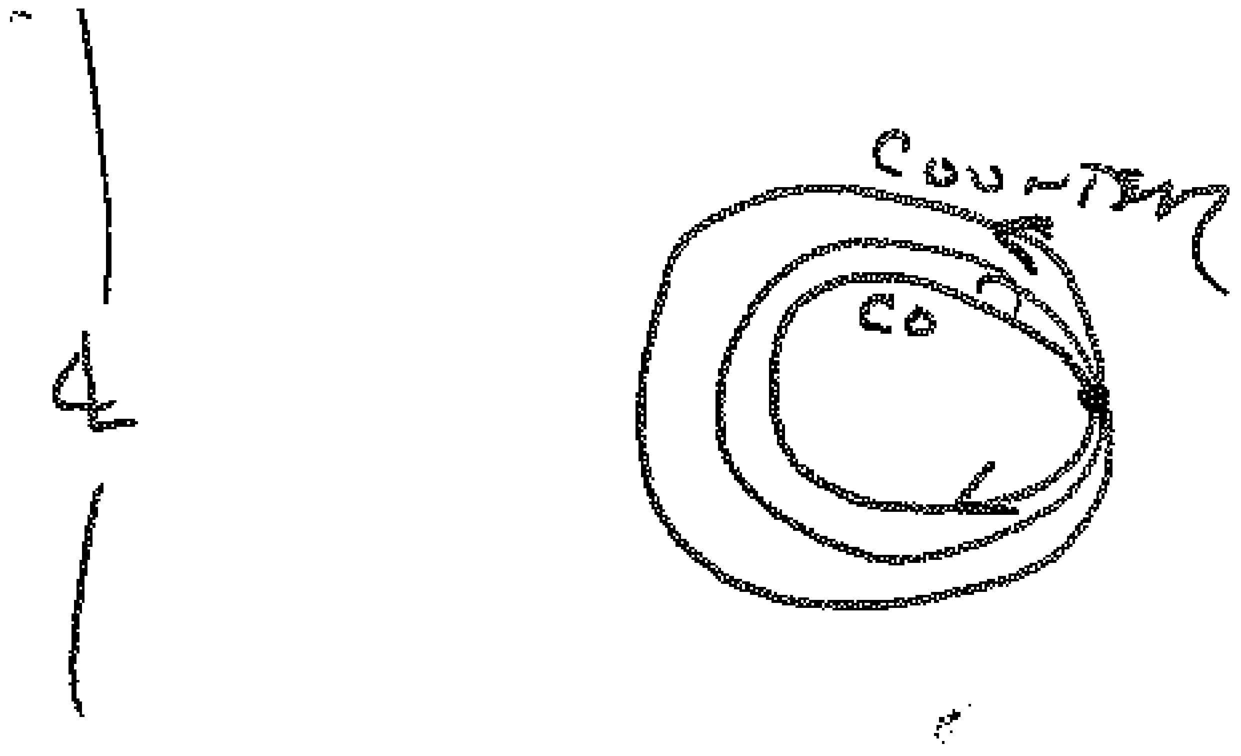
$$\frac{\Delta}{2} = \frac{v_\parallel}{\omega_p} = \frac{\sqrt{2\mu B_t}}{\omega_p}\sqrt{\epsilon}\sqrt{\cos\theta - \cos\theta_m}$$

Poloidal Gyroradius

Banana Orbits

Toroidal Magnetic Field

Passing Orbits



$$R m v_\parallel^{out} = (R - r + \Delta) m v_\parallel^{in} + \frac{q}{c} R \Delta B_p$$

$$r v_\parallel = \Delta v_\parallel + \frac{q B_p}{c m} R \Delta$$

$$\frac{r}{R} v_\parallel / \omega_p = \Delta$$

ω_p = poloidal “cyclotron” frequency

$\sim$ smaller $\epsilon \frac{v}{\omega_p} \sim \Delta$

Toroidal Magnetic Field

Toroidal Precession Frequency

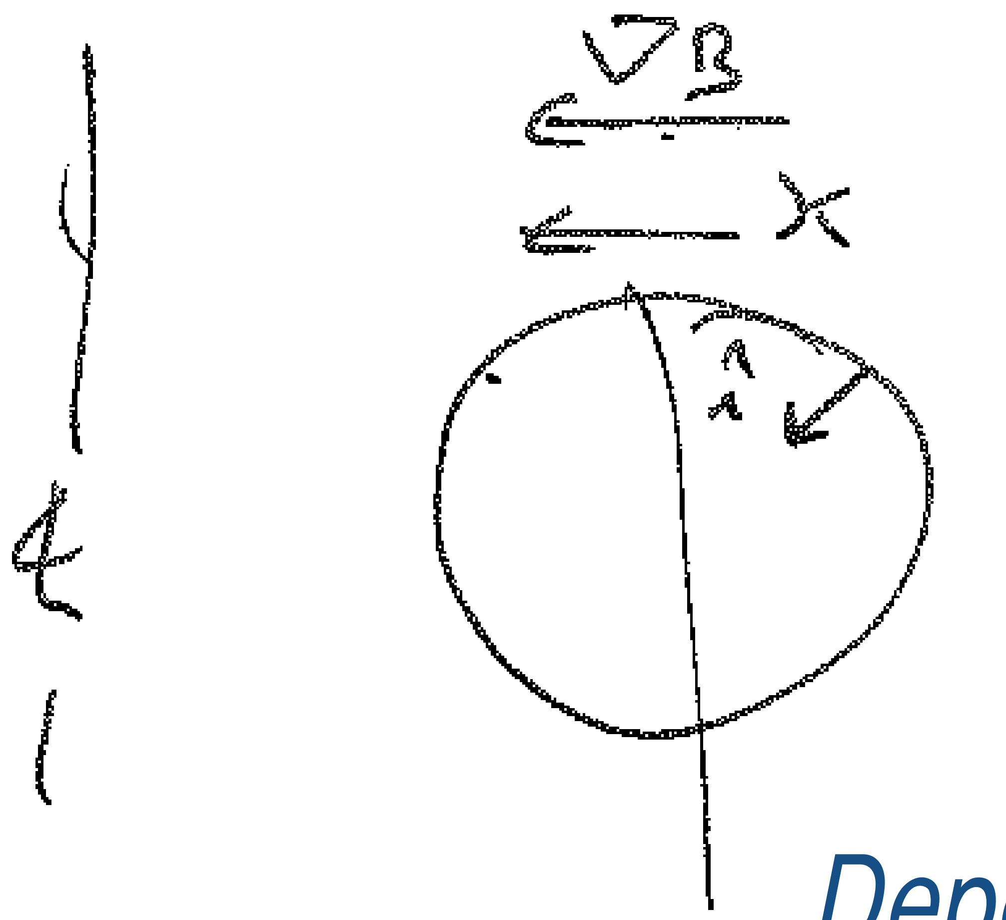
$$\hat{\varphi} \cdot \bar{v}_D = \frac{\hat{\varphi} \cdot \hat{b} \times}{\Omega} \left(\frac{\mu \nabla B}{m} + v_\parallel^2 \bar{\kappa} \right)$$

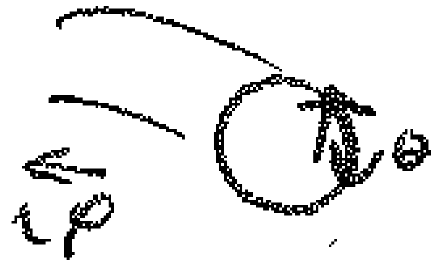
$$= \frac{\hat{\varphi} \times \hat{b}}{\Omega} \cdot (\ \ )$$

$$= -\frac{B_p}{B} \frac{\hat{r}}{\Omega} \cdot \left(\frac{\mu \nabla B}{m} + v_\parallel^2 \bar{\kappa} \right)$$

$$A \cdot B \times C = -A \cdot C \times B$$
$$= -A \times C \cdot B$$

$$\hat{\varphi} \times \hat{b} = \hat{\varphi} \times (B_t \hat{\varphi} + \bar{B}_p) \frac{1}{B}$$
$$= \hat{\varphi} \times \bar{B}_p \frac{1}{B}$$
$$= \frac{B_p}{B} \underbrace{\hat{\varphi} \times \hat{\theta}}_{-\hat{r}}$$





$$\omega_D \sim \omega_\varphi \sim \frac{\hat{\varphi} \cdot v_D}{R} \sim \frac{B_p}{BR} \left(\frac{\mu B}{R} + \frac{v_\parallel^2}{R} \right) \frac{1}{\Omega_t}$$

$$\sim \frac{B_p}{BR^2} \omega_c \frac{v^2}{\omega_c^2} = \frac{B_p}{B} \omega_c \left(\frac{\rho}{R} \right)^2$$

SLIGHTLY SLOWER BY $\frac{B_p}{B}$!

Depending upon turning point, toroidal drift reverses!

Homework #3

- Fitzpatrick: Exercise #1 in Chapter 2
- Piel: All seven problems in Ch. 3 (answers in back of text)

Next Lecture

- Piel / Chapter 4: Stochastic Processes in Plasma
- Distribution function
- Collisions