Lecture3:

Plasma Physics 1

APPH E6101x
Columbia University

Outline

- Motion induced by electric oscillations *without* a magnetic field
- Motion induced by electric oscillations *with* a magnetic field
- Charged Particle Drifts in a *uniform* magnetic field
- Charged Particle Drifts in a *slowly* varying magnetic field
- Adiabatic Invariants

Nobel Prize in Physics 2018 half jointly to Gérard Mourou and Donna Strickland
"for their method of generating high-intensity, ultra-short optical pulses"

https://www.nobelprize.org/prizes/physics/2018/summary/

Oscillations *without* Magnetic Field

$$\frac{d\bar v}{dt} = -\frac{e\bar E(t)}{m_e}$$

$$E(t) = \mathrm{Re}\{\hat E_0 e^{-j\omega t}\} = \tilde E_0 \cos(\omega t)$$

$$v(t) = \mathrm{Re}\{\hat v e^{-j\omega t}\}$$

So $-j\omega\tilde v = -\dfrac{e\tilde E_0}{m_e}$

$$\tilde v = -j\frac{e\hat E_0}{m_0\omega} = \underbrace{-j}_{\text{PHASE!}}\ \underbrace{\frac{e\lambda}{2\pi m_e c}\tilde E}_{\text{AMPLITUDE}}$$

$$= \frac{e\lambda}{2\pi m_0 c}\bar E_0 \sin(\omega t)$$

ISOTROPIC TOO

$$\tilde v = \begin{pmatrix} \tilde v & 0 & 0 \\ 0 & \tilde v & 0 \\ 0 & 0 & \tilde v \end{pmatrix} \cdot \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}$$

LIGHT INTENSITY

$$I = \frac{1}{2}\epsilon_0 c|\hat E_0|^2 \Rightarrow |E_0| \sim \sqrt{\frac{2I}{\epsilon_0 c}}$$

$$\therefore \frac{|\tilde v|}{c} \sim \frac{e\lambda}{2\pi m_0 c^2}|E| \doteq 850\,\lambda_\mu\sqrt{I_{22}}$$

INTENSITY UNITS 10^{22} W/cm^2

WAVELENGTH μm

Highest laser intensity in lab is now 10^{23} W/cm^2 (!)

Hui Chen and Frederico Fiuza, *Phys. Plasmas 1* February 2023; 30 (2): 020601. https://doi.org/10.1063/5.0134819

Oscillations *with* Static Magnetic Field

$$\frac{d\bar{v}}{dt} = -\frac{e}{m_e}\bar{E} - \frac{e}{m_0}\bar{v}\times\bar{B}_0 \qquad \text{LET } \bar{B}_0 = \hat{z} B_0$$

$$\hat{E} = \mathrm{Re}\left\{\tilde{E}\, e^{-j\omega t}\right\} \qquad \hat{v} = \mathrm{Re}\left\{\tilde{v}\, e^{-j\omega t}\right\}$$

VECTORS

THEN

$$\boxed{-j\omega\hat{v} + \omega_{co}\bar{v}\times\hat{z} = -\frac{e}{m_0}\hat{E}}$$

BUT $\quad \bar{v}\times\hat{z} - j\frac{\omega_c}{\omega}\hat{v}_\perp = -j\frac{e}{m_0\omega}\bar{E}\times\hat{z} \quad$ SO THAT

$$\boxed{\hat{v} - \left(\frac{\omega_c}{\omega}\right)^2\hat{v}_\perp + \frac{e}{m_0\omega}\frac{\omega_c}{\omega}\hat{E}\times\hat{z} = -j\frac{e}{m_0\omega}\bar{E}}$$

TWO COMPONENTS

$$\hat{v}_\parallel = -j\frac{e}{m_0\omega}\tilde{E}_\parallel \quad \text{AND} \quad \hat{v}_\perp\left(1-\frac{\omega_c^2}{\omega^2}\right) = -j\frac{e}{m_0\omega}\bar{E}_\perp + \frac{e}{m_0\omega}\frac{\omega_c}{\omega}\hat{z}\times\hat{E}$$

$\neq$

NOT ISOTROPIC

$$\hat{V} = -j\frac{e}{m_0\omega}\begin{pmatrix} \frac{1}{(1-\omega_c^2/\omega^2)} & -\frac{\omega_c}{\omega}\frac{j}{(1-\frac{\omega_c^2}{\omega^2})} & 0 \\ +\frac{\omega_c}{\omega}\frac{j}{1-\omega_c^2/\omega^2} & \frac{1}{(1-\omega_c^2/\omega^2)} & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}$$

Charged particle motion in uniform/inhomogeneous, *static and slowly-varying* electric and magnetic fields

Part 1: Gyromotion in Uniform Field

$$m\dot{\mathbf{v}} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \,, \tag{3.1}$$

$$\begin{aligned} \dot{v}_x &= \frac{q}{m}\left(E_x + v_y B_z\right) \\ \dot{v}_y &= \frac{q}{m}\left(-v_x B_z\right) \\ \dot{v}_z &= \frac{q}{m} E_z \,. \end{aligned} \tag{3.7}$$

$$\begin{aligned} \ddot{v}_x &= -\omega_c^2 v_x \\ \ddot{v}_y &= -\omega_c^2 (v_y + E_x/B_z) \,. \end{aligned} \tag{3.8}$$

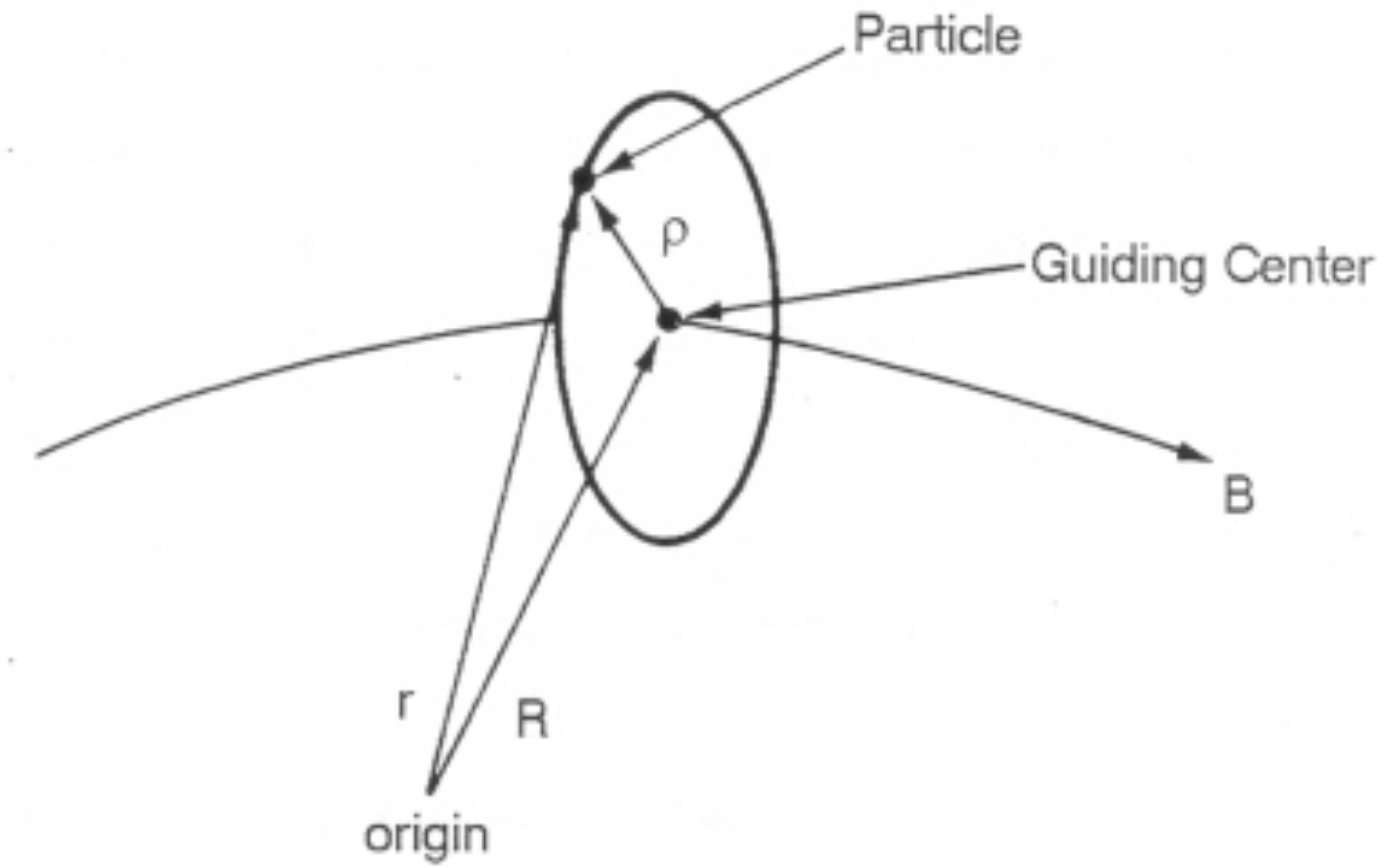
What drifts result from a static force that has the same sign for ions and electrons (like gravity)?

Key Dynamics

- Separate descriptions of perpendicular and parallel motion
- Fast gyration around B
- Slow perpendicular drift of gyro center
- Adiabatic motion with μ conservation

Guiding Center Motion in a Strong Magnetic Field

Part 2: Guiding Center Motion



References:

I.B. Bernstein, “The Motion of a Charged Particle in a Strong Magnetic Field,” *Advances in Plasma Physics* **4**, 311 (1971).

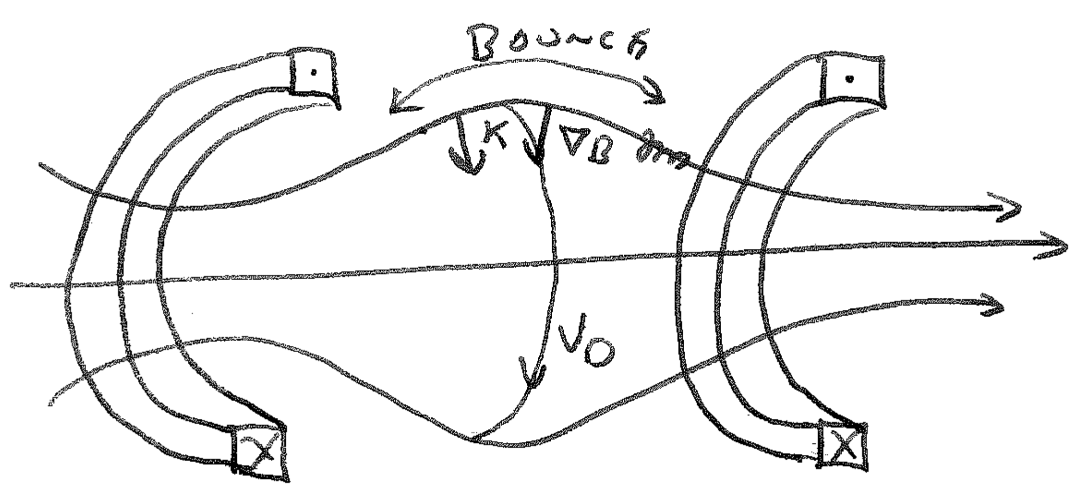
Theodore G. Northrop and Edward Teller, "Stability of the Adiabatic Motion of Charged Particles in the Earth's Field," *Phys. Rev.* 117, 215 (1960); https://doi.org/10.1103/PhysRev.117.215

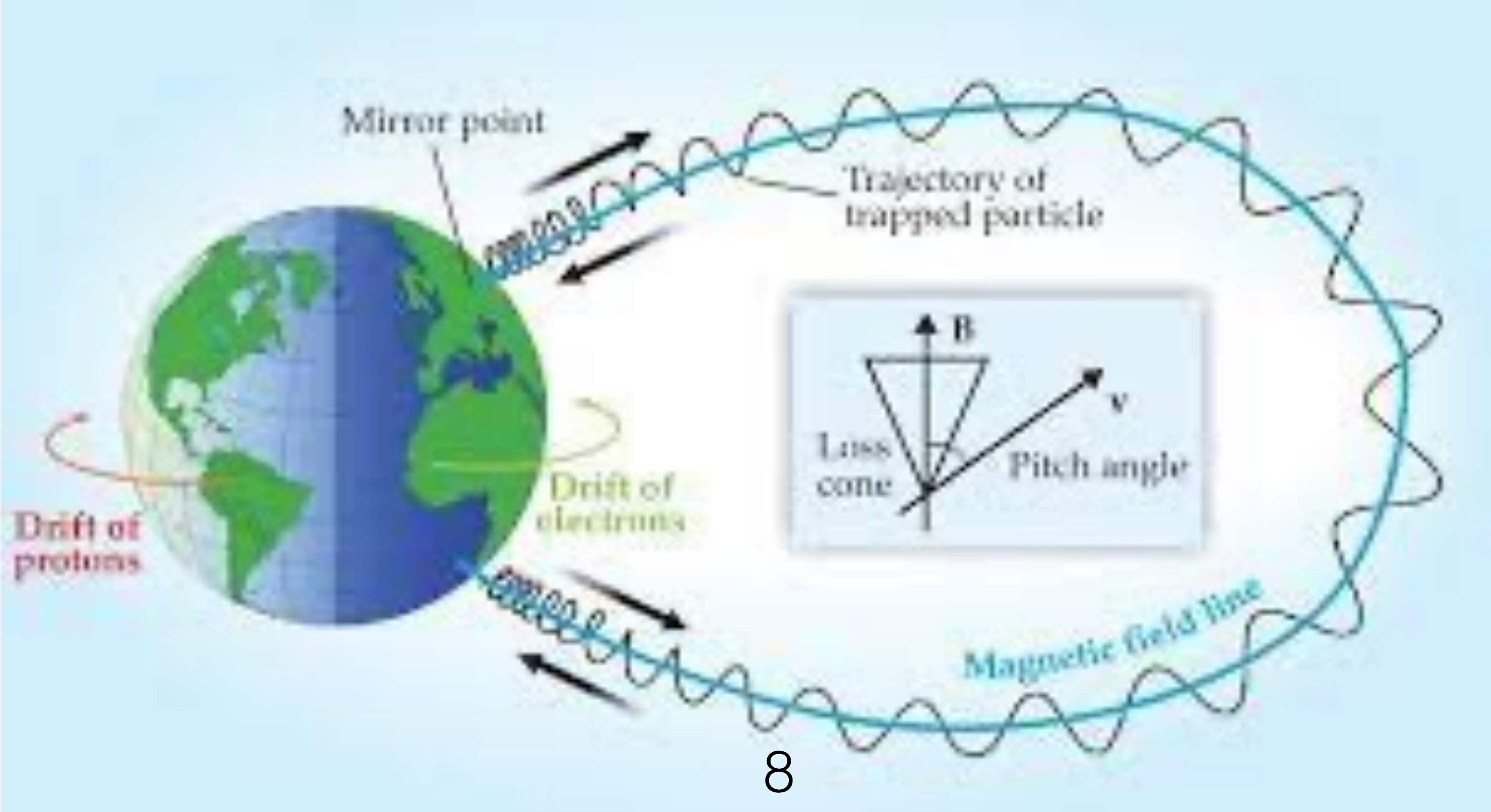
Summary

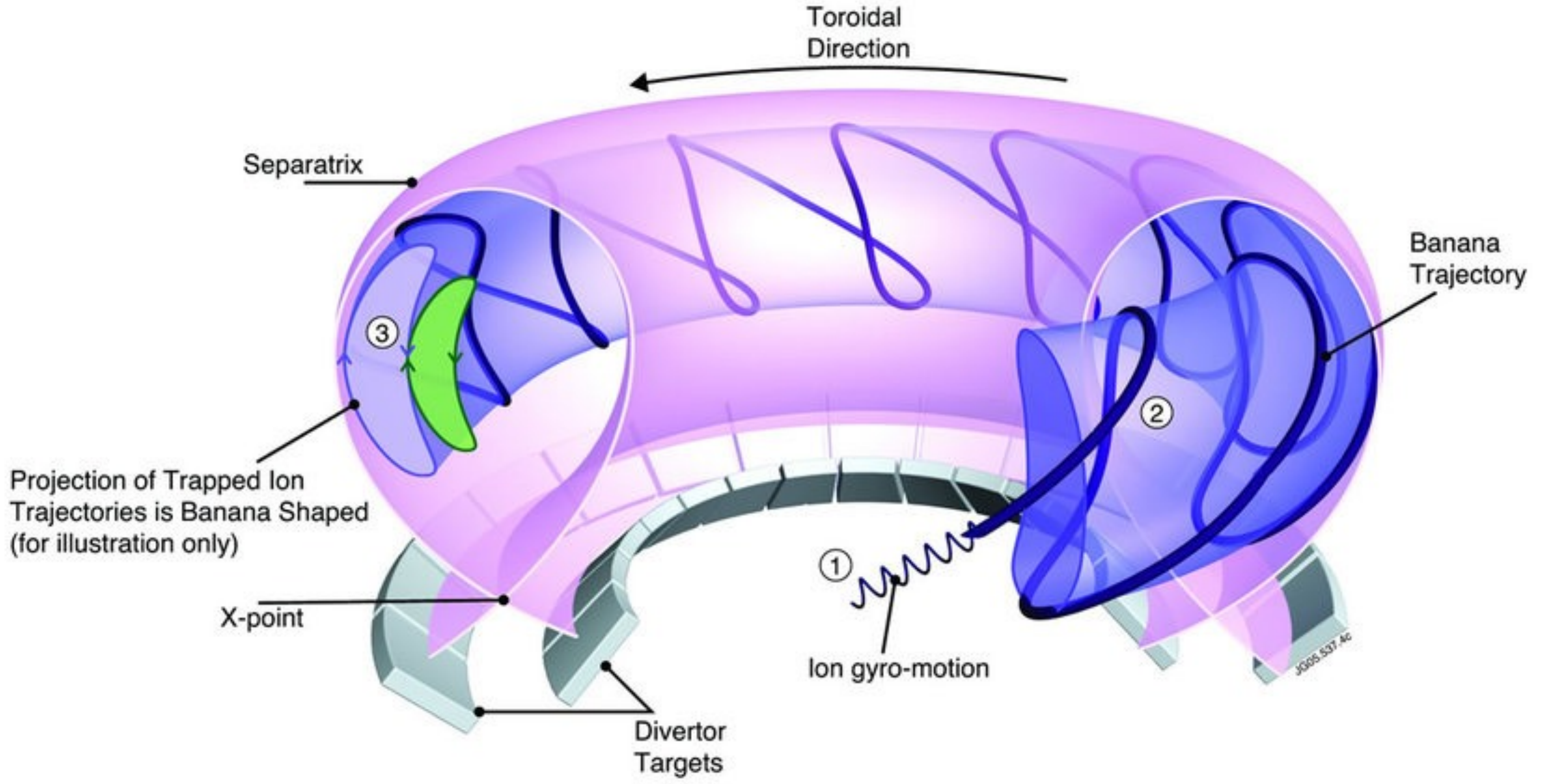
$$\bar{V} = v_\parallel \hat{b} + \frac{\hat{b}}{\Omega} \times \left(\frac{\mu}{m} \bar{\nabla}|B| + v_\parallel^2 \bar{\kappa} \right)$$

$$\frac{dv_\parallel}{dt} = -\frac{\mu}{m} \hat{b} \cdot \bar{\nabla}|B|$$

$\rho =$
$\omega =$ SAME AS BEFORE (but Ω changes…)



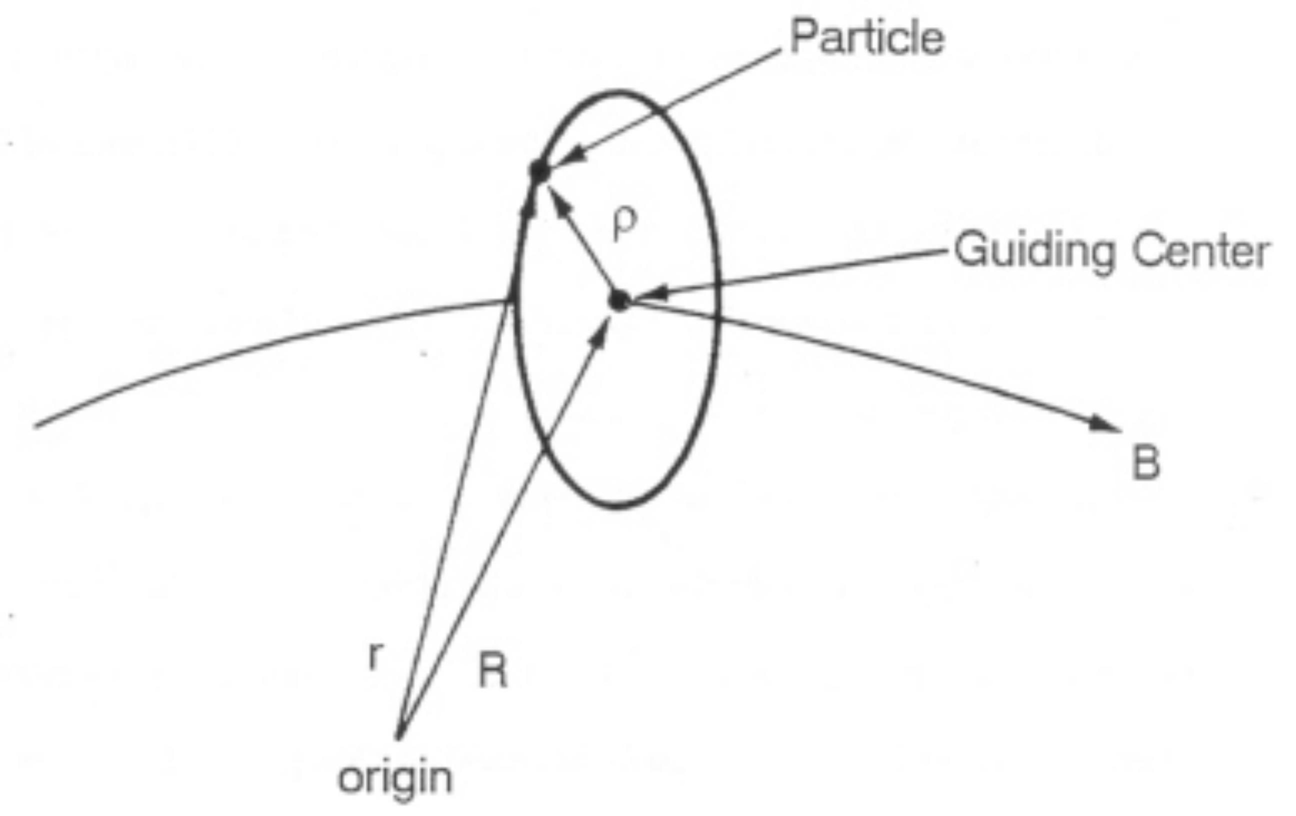




Some Definitions & Cases

$$m\dot{\mathbf{v}} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

- 3 equations, 2nd order, 6 initial conditions
- Constant **E** and **B**
- Slowly varying **E** (with uniform **B**)
- Let **E** $\longrightarrow$ 0, with **B** *slowly varying in space*
- …



Use Bernstein's notation…

(*See "handout": http://sites.apam.columbia.edu/courses/apph6101x/Plasma1-Adiabatic-Handout.pdf*)

Just Mechanics

NOW WE'RE GOING TO SOLVE THE EQS OF MOTION FOR A SINGLE PARTICLE IN A STRONG MAGNETIC FIELD — WHERE THE EFFECT OF THE CHARGE → WHICH SHOULD PERTURB THE FIELD — CAN BE IGNORED. THIS IS A "LINEAR" PROBLEM IN MECHANICS.

BUT IT IS STILL VERY COMPLICATED WHEN WE HAVE SPATIALLY OR TEMPORAL VARYING FIELDS.

WE START WITH THE LORENTZ FORCE

$$\frac{d^2\bar{r}}{dt^2} = \bar{a} - \bar{\Omega} \times \frac{d\bar{r}}{dt}$$

$\bar{a} = \frac{q}{m}\bar{E}$ + OTHER FORCES (ACCELERATION)

$\bar{\Omega} = \frac{q}{mc}\bar{B}$ (CYCLOTRON FREQUENCY) ROTATION FREQUENCY

NOTE SIGN OF "q"

("notation")

Simple Note 1

$$\frac{d\bar{v}}{dt} = \bar{a} - \bar{\Omega} \times \bar{v}$$

$$\bar{v} \cdot \frac{d\bar{v}}{dt} = \bar{v} \cdot (\bar{a} - \bar{\Omega} \times \bar{v})$$

so $$\frac{d}{dt}\left(\frac{1}{2} v^2\right) = \bar{v} \cdot \bar{a}$$

Simple Note 2

$$\hat{b} \cdot \frac{d\bar{v}}{dt} = \hat{b} \cdot \bar{a}$$

$$\bar{v}_\perp = \hat{b} \times (\bar{v} \times \hat{b})$$

*so we can **decouple** parallel and perpendicular motion*

Initial Value Case 1

$\bar{v} = v_{\parallel}\hat{b}$ at $t=0$ $\quad \bar{E} = E_{\parallel}\hat{b}$ $\quad B, E =$ constant

$$\bar{v}(t) = v_{\parallel}\hat{b} + \frac{q}{m}E_{\parallel}t\hat{b}$$

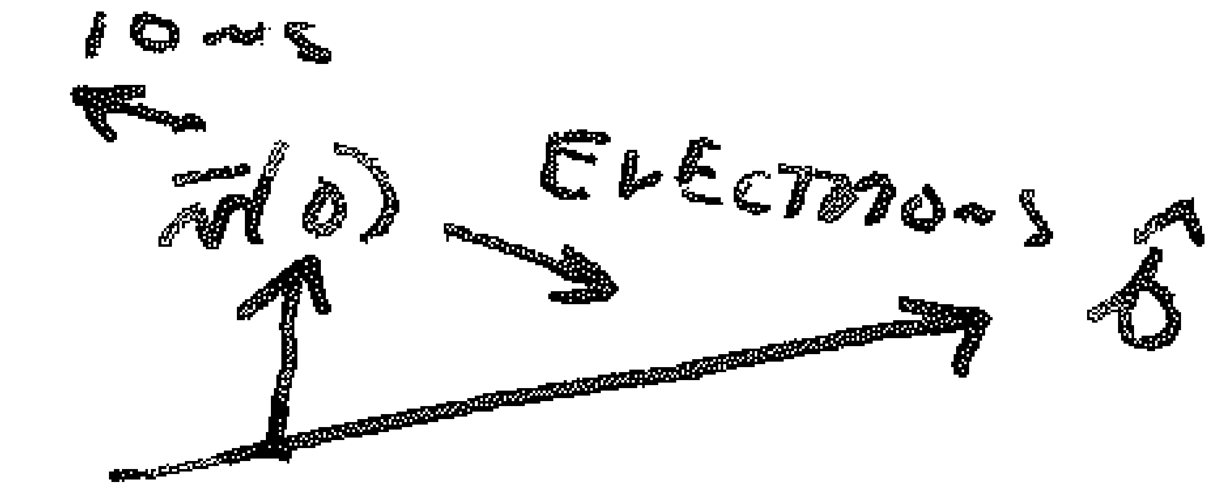
$$\bar{r}(t) = \bar{R}(0) + \hat{b}\left(v_{\parallel}t + \frac{1}{2}\frac{q}{m}E_{\parallel}t^2\right)$$

Initial Value Case 2

$\bar{v} = w\,\hat{e}_1$ AT $t = 0$

$\vec{E} = 0$

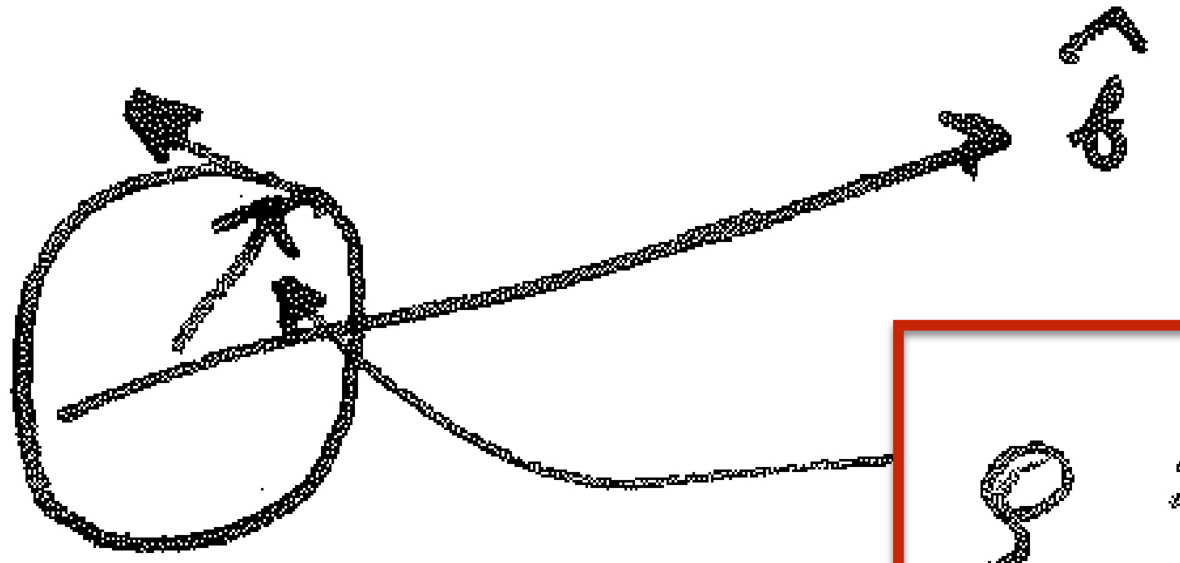
$\bar{B} = $ CONSTANT



$$\frac{d}{dt}\bar{v} = -\bar{\Omega} \times \bar{v}$$

$$\bar{v} \cdot \frac{d}{dt}\bar{v} = \frac{1}{2}\frac{d}{dt}|v^2| = 0$$

magnetic field does no work



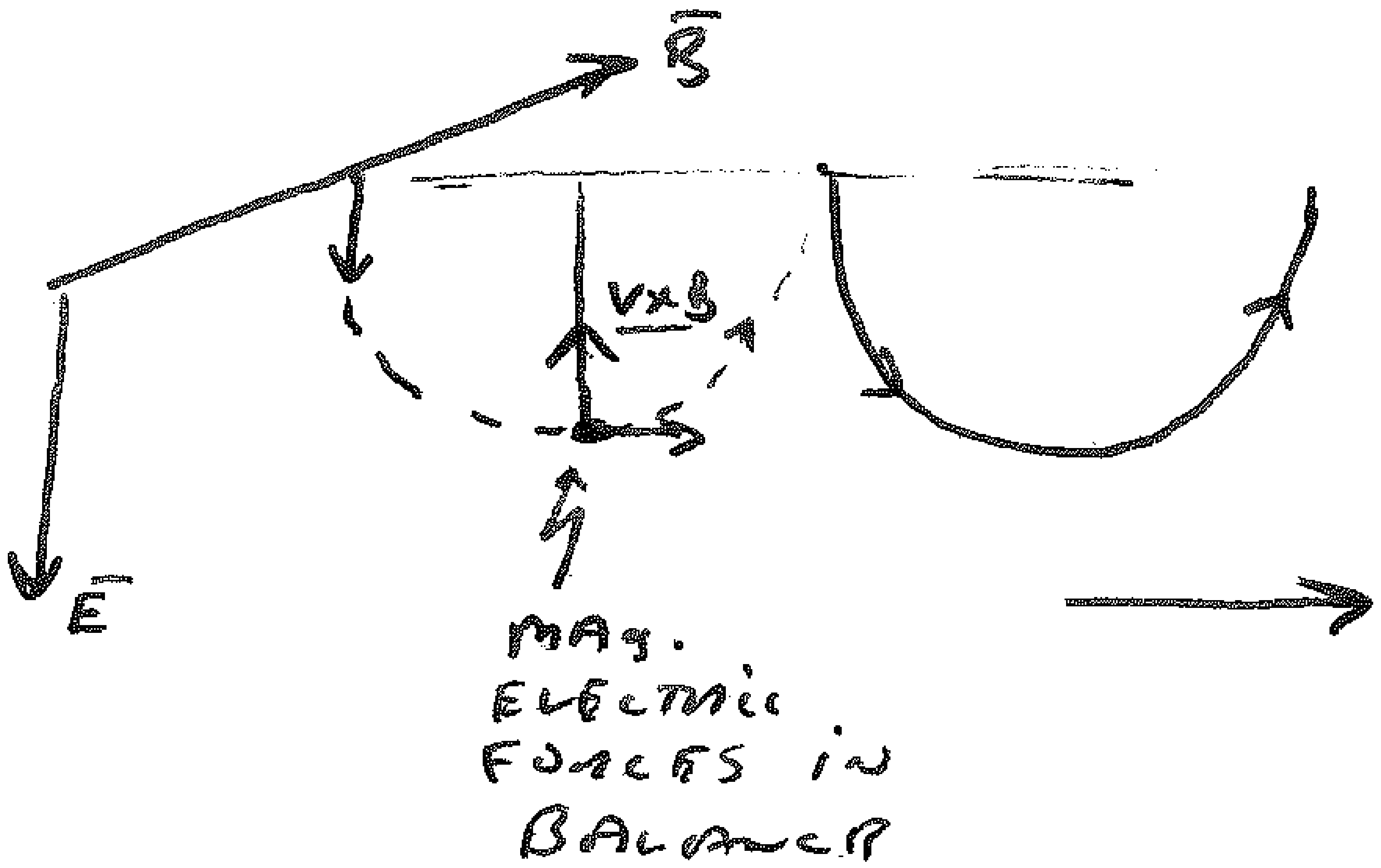
$$\boxed{\rho = \frac{w}{|\Omega|}}$$

$$\frac{d\bar{r}}{dt} = -\Omega \times \bar{r} = \bar{v}$$

Initial Value Case 3

$\boxed{\bar{v}(0) = 0}$ $\quad \bar{B} = \text{CONSTANT}$, $\bar{E} = \text{CONSTANT}$

WITH $E \perp B$



$\longrightarrow$ $V_D \sim E \times B$

DRIFT MOTION

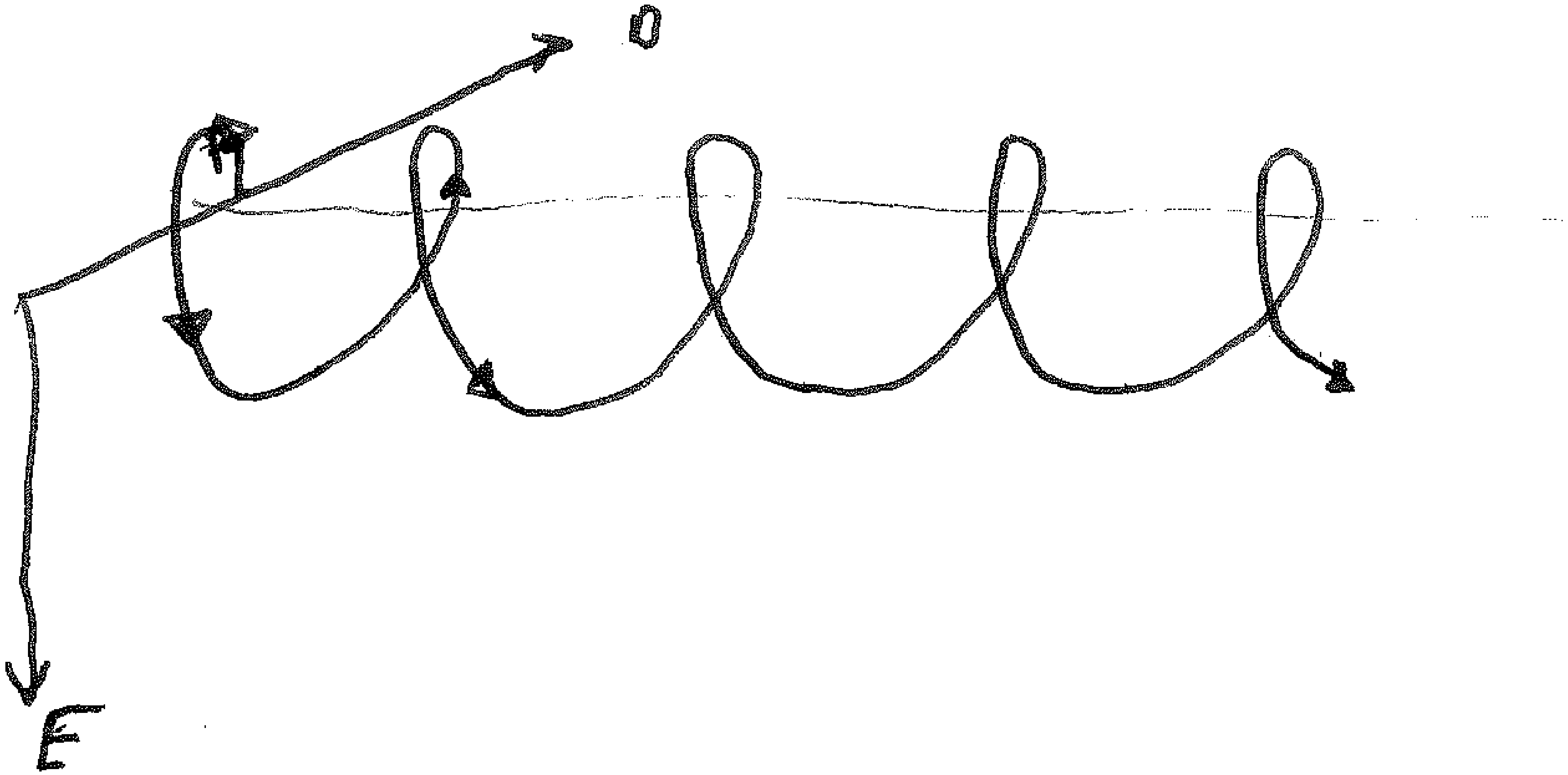
TOTAL = SUPERPOSITION OF

DRIFT PLUS ROTATION

Initial Value Case 4

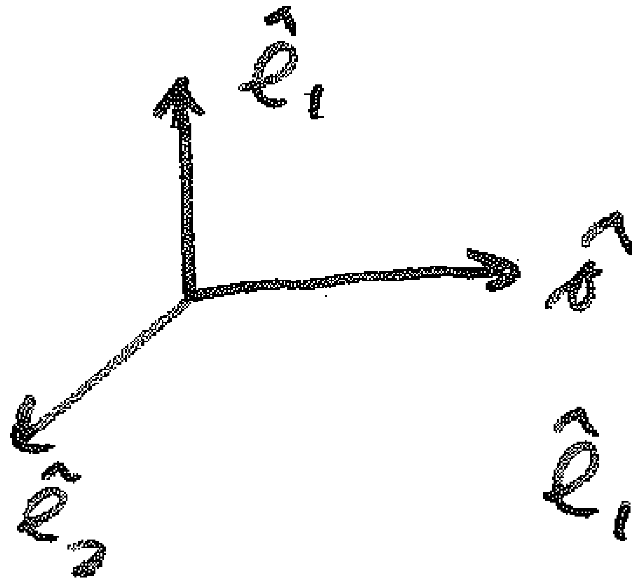
$\boxed{\vec{v}(0) \sim \omega \; \hat{e}_1}$ $\qquad \vec{B} \sim$ CONSTANT, $\quad E \sim$ CONSTANT

with $\underline{E \perp B}$



Defining Gyro-Averaging

SOME DEFINITIONS...



$$\bar{r} = \bar{R} + \bar{\rho} \qquad (\bar{\rho} \cdot \hat{b} = 0)$$

$$\dot{\bar{r}} = \bar{V} + \bar{w}$$

$$\hat{e}_1 \times \hat{e}_2 = \hat{b}$$

$$\bar{B} = B\hat{b}$$

$$\left(\frac{d\bar{r}}{dt} = \dot{\bar{r}} \right)$$

$$\bar{w} \cdot \hat{b} = 0$$

$$\bar{V} \cdot \hat{b} = v_{\parallel}$$

See: Sec. 3.2 *Piel* and
Secs. 2.3 and 2.4 in *Fitzpatrick*

THEN CALL θ = GYROANGLE

THEN $$\langle \cdots \rangle_\theta \equiv \frac{1}{2\pi} \int_0^{2\pi} d\theta \, (\cdots)$$

$$\langle \bar{r} \rangle_\theta = \bar{R}$$

$$\langle \dot{\bar{r}} \rangle_\theta = \bar{V}_\perp + v_\parallel \hat{b}$$

A physical solution to the E.O.M. in terms of gyration and drift

Separating the Vector E.O.M.

PARALLEL:

$$\hat{b}\cdot\left[\frac{d^2\bar{r}}{dt^2} = \bar{a} - \bar{\Omega}\times\frac{d\bar{r}}{dt}\right]$$

$$\frac{dv_\parallel}{dt} = \hat{b}\cdot\bar{a}$$

$$V_\parallel(t) = \hat{b}\cdot\bar{a}\,t + v_\parallel(0)$$

PERP:

$$\hat{b}\times(\cdots\times\hat{b}) = \text{PERPENDICULAR COMPONENT}$$

$$\hat{b}\times(\bar{A}\times\hat{b}) = \bar{A} - \hat{b}(\hat{b}\cdot\bar{A}) = \bar{A}_\perp$$

1ST

THEN

$$\frac{d^2\bar{R}_\perp}{dt^2} + \frac{d^2\bar{\rho}}{dt^2} = \bar{a}_\perp - \bar{\Omega}\times\bar{W} - \bar{\Omega}\times\bar{V}$$

Fast Gyromotion

(Separately)

$$\frac{d^2\bar{\rho}}{dt^2} = \frac{d\bar{\omega}}{dt} = -\bar{\Omega}\times\bar{\omega}$$

$$\bar{\omega} = -\bar{\Omega}\times\bar{\rho} = \frac{d\bar{\rho}}{dt}$$

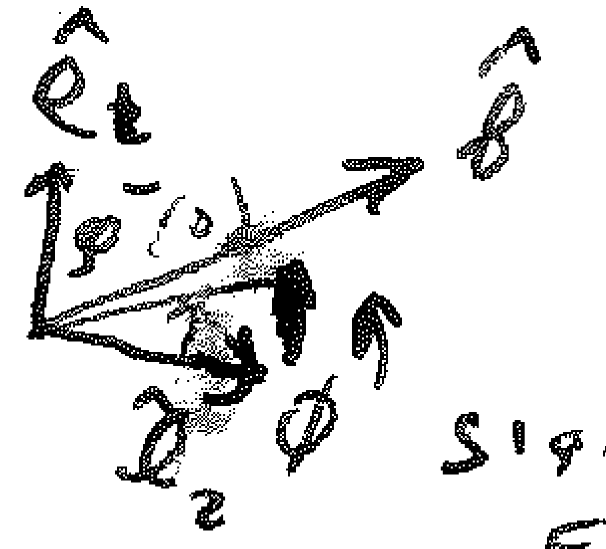
$$\frac{d^2\bar{\rho}}{dt^2} = +\bar{\Omega}\times(\bar{\Omega}\times\bar{\rho}) = -|\Omega|^2\bar{\rho}$$

$$\bar{\rho} = \frac{w}{\Omega}\left[\hat{e}_1 \sin(\Omega t+\phi) + \hat{e}_2 \cos(\Omega t+\phi)\right]$$

gyrophase

$$\bar{\omega} = \omega\left[\hat{e}_1 \cos(\Omega t+\phi) - \hat{e}_2 \sin(\Omega t+\phi)\right]$$

$$\Omega = \hat{b}\cdot\bar{\Omega}$$



$\phi = 0$

SIGN RIGHT FOR IONS

Slow Drift

(Separately)

LET $$\frac{d\bar{R}_\perp}{dt} = \bar{V}_\perp$$

$$\frac{d\bar{V}_\perp}{dt} = \bar{a}_\perp - \bar{\Omega} \times \bar{V}_\perp$$

MULTIPLY BY $\bar{\Omega} \times (\cdots)$

$$\bar{\Omega} \times \frac{d\bar{V}_\perp}{dt} = \bar{\Omega} \times \bar{a}_\perp + \Omega^2 \bar{V}_\perp$$

OR $$\bar{V}_\perp = \frac{\bar{a}_\perp \times \bar{\Omega}}{\Omega^2} + \frac{\bar{\Omega} \times \frac{d\bar{V}_\perp}{dt}}{\Omega^2}$$

$a(t)$ slowly…

- GRYOMOTION ⟹ EXACTLY THE SAME.
- DRIFT / GUIDING CENTER MOTION

$$\bar{V}_\perp = \frac{\bar{a}_\perp \times \bar{\Omega}}{\Omega^2} + \frac{\bar{\Omega} \times \frac{dV_\perp}{dt}}{\Omega^2}$$

NOW IF $\frac{1}{\Omega}\frac{dV_\perp}{dt} \ll 1$ THEN $\bar{a}$ IS SLOWLY VARYING ....

SO LET'S MAKE AN EXPANSION

$$V_\perp(t) = V_\perp\big|_0 + V_\perp\big|_1 + V_\perp\big|_2 + \cdots$$

$$\uparrow \quad O^{\text{TH}} \quad \cdots \quad \frac{1}{\Omega}\frac{dV_\perp}{dt} \quad \cdots \quad \left(\frac{1}{\Omega}\frac{dV_\perp}{dt}\right)^2$$

$$\bar{V}_\perp\big|_0 = \frac{\bar{a}_\perp \times \bar{\Omega}}{\Omega^2}$$

$$\bar{V}_\perp\big|_1 = \frac{\bar{\Omega} \times \frac{d}{dt}\left(V_\perp\big|_0\right)}{\Omega^2}$$

a(t) slowly...

SO LET'S MAKE AN EXPANSION

$$V_\perp(t) = V_\perp\big|_0 + V_\perp\big|_1 + V_\perp\big|_2 + \cdots$$

0^{TH} .. $\frac{1}{r}\frac{dV_\perp}{dt}$ .. $\left(\frac{1}{r}\frac{dV_\perp}{dt}\right)^2$

$$\bar{V}_\perp\big|_0 = \frac{\bar{a}_\perp \times \bar{r}}{r^2}$$

$$\bar{V}_\perp\big|_1 = \frac{r \times \frac{d}{dt}\left(V_\perp\big|_0\right)}{r^2}$$

ETC...

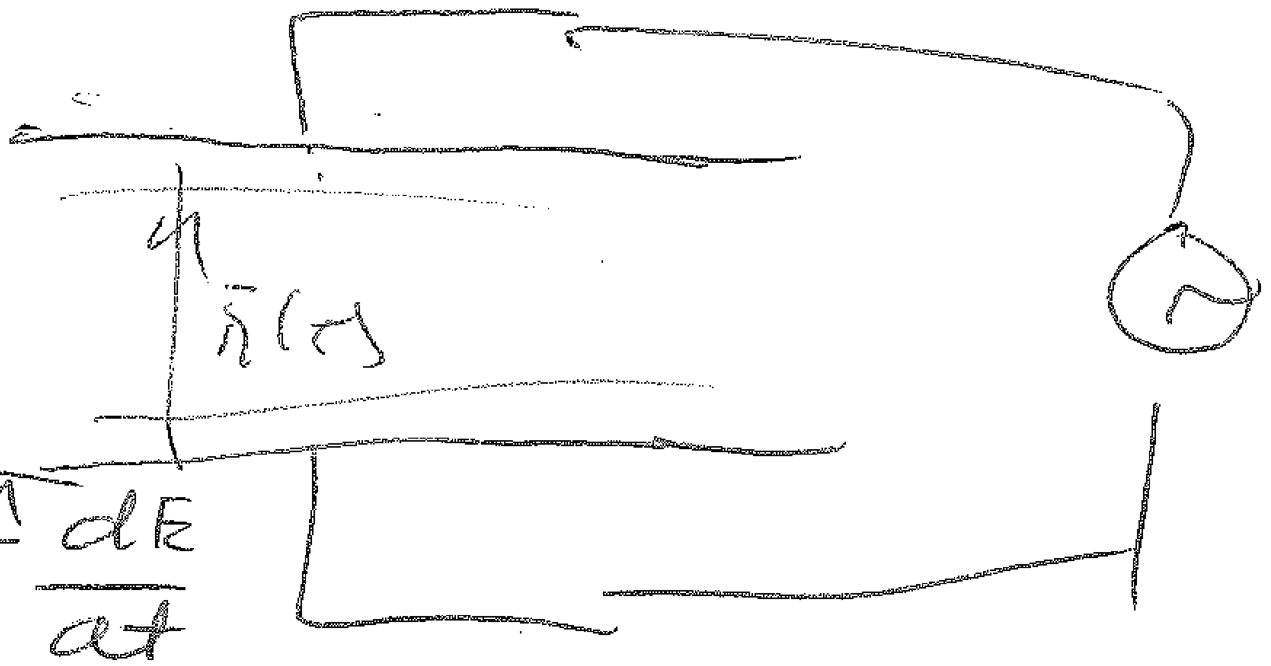
$$\bar{V}_\perp(t) \simeq \frac{\bar{a}(t) \times \bar{r}}{r^2} + \frac{1}{r^2}\,\bar{r} \times \frac{d}{dt}\left(\frac{\bar{a}(t) \times \bar{r}}{r^2} + \cdots\right)$$

$$\simeq \underset{\uparrow\ E \times B}{\frac{\bar{a}(t) \times \bar{r}}{r^2}} + \underbrace{\frac{1}{r^2}\frac{d}{dt}\bar{a}_\perp(t)}_{\text{POLARIZATION DRIFT}} + \cdots$$

WITH $\bar{a} = \frac{e}{m}\bar{E}$ $\xrightarrow{\text{THEN}}$

Polarization Drift: Plasma Capacitor

$$J_{POL} = \sum_s q_s n_s V_{POL}$$



$$V_{POL} = \frac{M^2}{\beta^2 D^2}\,\frac{q}{m}\frac{dE}{dt}\; q\,n \sim \frac{nM}{B^2}\frac{dE}{dt}$$

$$\nabla \cdot J = -\frac{\partial \rho}{\partial t}$$

$$\nabla \cdot \left(\frac{nM}{B^2}\right) \bar{E} = -\rho$$

$\uparrow$

$$\epsilon E = D \qquad \epsilon = \frac{nm}{B^2} \sim \left(\frac{e^2 n}{m \epsilon_0}\right) \epsilon_0 \frac{m^2}{e^2 B^2}$$

$$\sim \epsilon_0 \left(\frac{\omega_{pi}^2}{\omega_{ci}^2}\right) >> 1$$

$B = 1\,\mathrm{kG}$ $\qquad \omega_{pi} = 2\pi\, \underbrace{10^4 \cdot 10^3}_{10\,\mathrm{MHz}}$

$n \sim 10^{10}$ $\qquad \omega_{ce} \sim_{2\pi} 10^3\, 10^5 \sim 100\,\mathrm{MHz}$

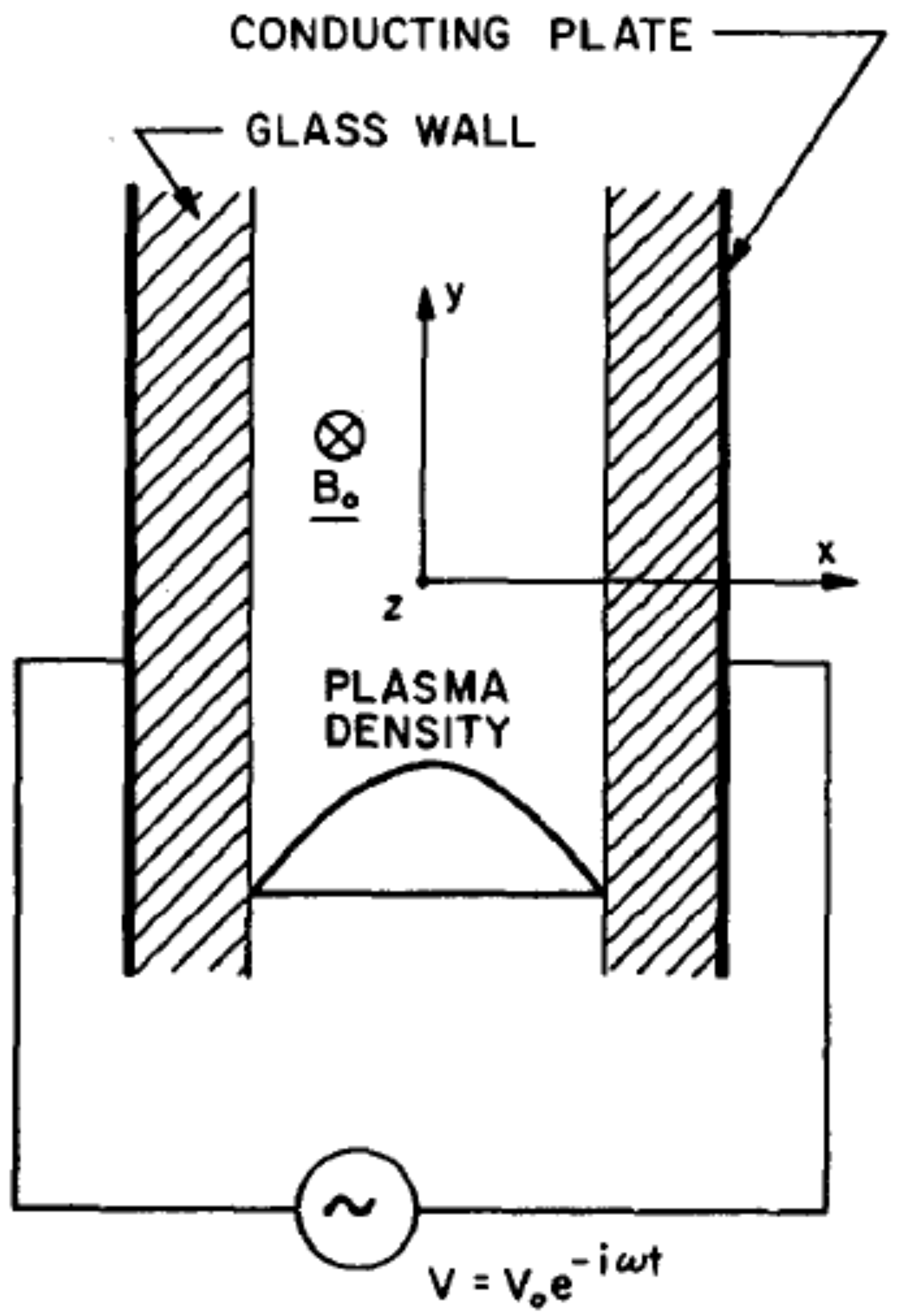
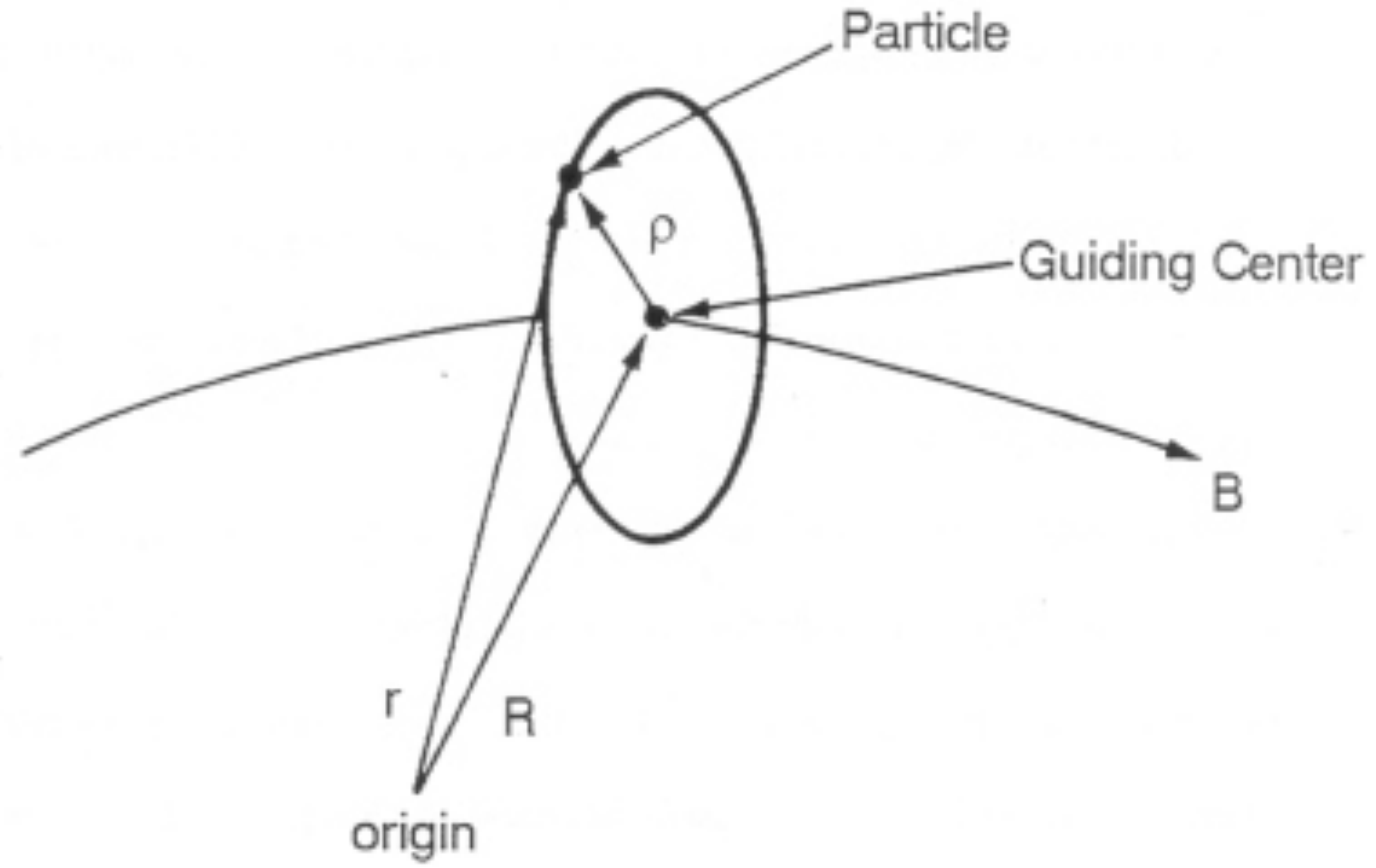


FIG. 1. Geometry of the plasma capacitor. The density variation is shown on the figure.

See: A. L. Peratt, H. H. Kuehl; Plasma Capacitor in a Magnetic Field. *Phys. Fluids* (1972); 15 (6): 1117–1127; https://doi.org/10.1063/1.1694037

Motion with a *slowly* varying B…

NOW, <u>BOTH GYROMOTION</u> AND <u>DRIFT</u> WILL CHANGE (SINCE $\vec{r}$ CHANGES) BUT TO ANALYZE THIS REQUIRES CAREFUL ALGEBRA… WE'LL DO THAT IN THE <u>NEXT</u> STEP.



$B(x)$ changing slowly…

NOW, BOTH GYROMOTION AND DRIFT WILL CHANGE (SINCE $\bar{\Omega}$ CHANGES) BUT TO ANALYZE THIS REQUIRES CAREFUL ALGEBRA... WE'LL DO THAT IN THE NEXT STEP.

HERE, WE START WITH...

$$\frac{d\dot{\bar{r}}}{dt} = \dot{\bar{r}} \times \Big(\underbrace{\bar{\Omega} + (\bar{\rho}\cdot\bar{\nabla})\bar{\Omega} + \cdots}_{\text{TAYLOR EXPAND ABOUT GYRO CENTER}} \Big)$$

$$\bar{\Omega}(\bar{r}) = \bar{\Omega}(\bar{R} + \bar{\rho})$$

$$\approx \bar{\Omega}(\bar{R}) + (\bar{\rho}\cdot\bar{\nabla})\bar{\Omega} + \cdots$$

$$\frac{d\dot{\bar{r}}}{dt} \approx \dot{\bar{r}} \times \bar{\Omega} + \dot{\bar{r}} \times (\bar{\rho}\cdot\bar{\nabla})\bar{\Omega} + \cdots$$

(1) (2) (3)

Drifts (separately)

NOTE THAT THERE ARE OSCILLATING (GYRATING) AND NON-OSCILLATING TERMS.

SO LET'S LOOK ONLY AT THE DRIFTS. SO WE LOOK AT THE GYRO-AVERAGED DRIFT MOTION

1ST TWO ARE EASY..

$$\left\langle \frac{d\dot{r}}{dt} \right\rangle_\theta = \frac{d\bar{V}}{dt} \; ; \quad \left\langle \dot{r} \times \bar{r} \right\rangle_\theta = \bar{V} \times \bar{r}$$

Drifts (separately)

LAST IS THE <u>PRODUCT</u> OF TWO OSCILLATING TERMS...

$$\frac{1}{2\pi}\int_0^{2\pi} d\theta \; \dot{\bar{r}} \times (\bar{\rho}\cdot\bar{\nabla})\bar{r} = \frac{1}{2\pi}\int_0^{2\pi} d\theta \; \bar{w} \times (\bar{\rho}\cdot\bar{\nabla})\bar{r}$$

WHICH GIVES SOMETHING LIKE A <u>CONSTANT</u> PLUS SOMETHING THAT GOES LIKE $2\omega_c$! WE WANT THE CONSTANT PART...

NOW, I WANT TO AVERAGE THE

$\bar{w}\,\bar{\rho}$ TERMS

SO I going TO DO THE FOLLOWING

$$\bar{w} \times (\bar{\rho}\cdot\bar{\nabla})\bar{r} = \bar{w}\,\underbrace{(\bar{\rho}\cdot\bar{\nabla})}\times \bar{r}$$

THIS IS JUST AN OPERATOR WHICH ACTS ON $\bar{r}$ (IT'S THE SAME THING)

Drifts (separately)

THEN USING

$$\bar{w} = w\left[\hat{e}_1 \cos\theta - \hat{e}_2 \sin\theta\right]$$

$$\bar{\rho} = \frac{w}{\Omega}\left[\hat{e}_1 \sin\theta + \hat{e}_2 \cos\theta\right]$$

$$\langle \bar{w}\bar{\rho} \rangle_\theta = \frac{w^2}{\Omega} \frac{1}{2\pi}\int_0^{2\pi} d\theta \left[\hat{e}_1 \cos\theta - \hat{e}_2 \sin\theta\right]\left[\hat{e}_1 \sin\theta + \hat{e}_2 \cos\theta\right]$$

$$\to \frac{w^2}{\Omega} \frac{1}{2\pi}\int_0^{2\pi} d\theta \left[\hat{e}_1\hat{e}_2 \cos^2\theta - \hat{e}_2\hat{e}_1 \sin^2\theta\right]$$

OR

$$\langle \bar{w}\,\bar{\rho} \rangle_\theta = \frac{w^2}{2\Omega}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\langle \dot{\bar\rho} \times (\bar\rho\cdot\bar\nabla)\bar\Omega \rangle_\theta = \frac{\omega^2}{2\Omega}\left[\hat e_1\hat e_2 - \hat e_2\hat e_1\right]\cdot\bar\nabla\times\bar\Omega$$

$$= -\frac{\omega^2}{2\Omega}(\hat b\times\bar\nabla)\times\bar\Omega$$

OK

since $\hat b\times\bar\nabla = (\hat e_1\times\hat e_2)\times\bar\nabla$

$$= \hat e_2\,\hat e_1\cdot\bar\nabla - \hat e_1\,\hat e_2\cdot\bar\nabla$$

$$\frac{d\bar V}{dt} = \bar V\times\bar\Omega - \frac{\omega^2}{2\Omega}(\hat b\times\bar\nabla)\times\bar\Omega$$

$$= \bar V\times\bar\Omega - \frac{\omega^2}{2\Omega}\Big[\bar\nabla(\bar\Omega\cdot\hat b) - \hat b\,\underbrace{\bar\nabla\cdot\bar\Omega}_{\hookrightarrow\, 0}\Big]$$

$\nabla\cdot\bar B = 0$

NOW DEFINE $\mu \equiv \dfrac{\frac{1}{2}m\omega^2}{B}$

$$\boxed{\frac{d\bar V}{dt} = \bar V\times\bar\Omega - \frac{\mu}{m}\bar\nabla|B|}$$

!!

Gradient-B Drift

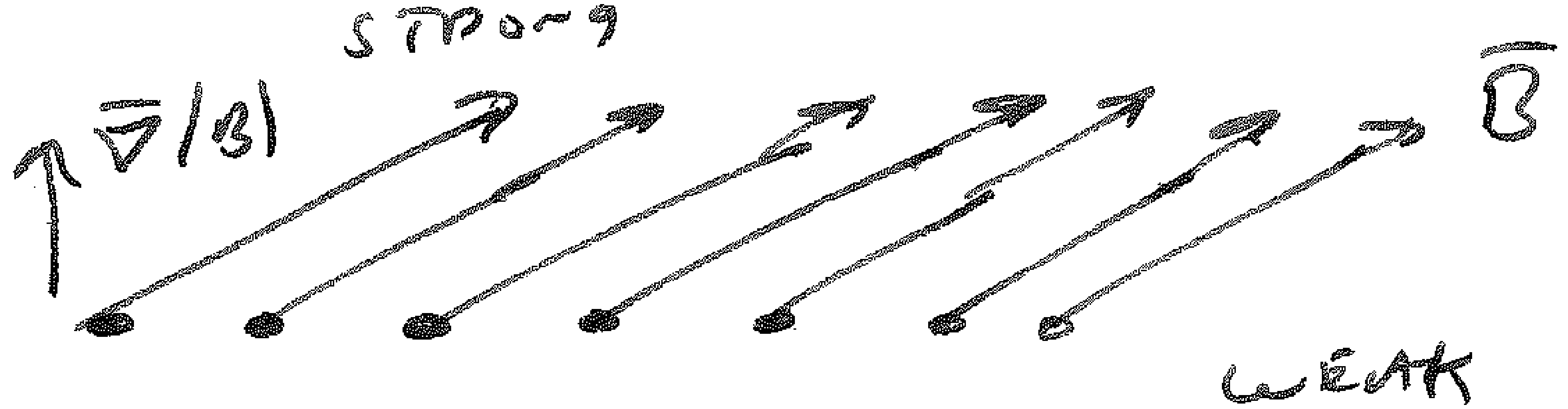
IF $\nabla|B|$, μ AND $\bar{r}(\bar{R})$ ARE CONSTANT, THEN

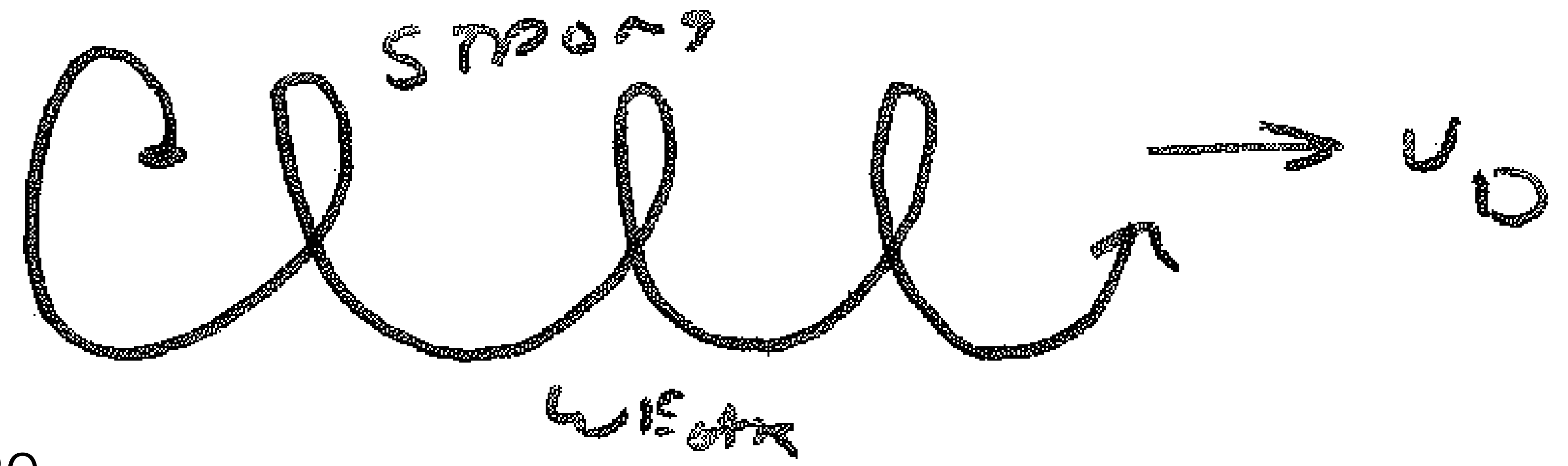
AND $\cancel{\hat{b}\cdot\nabla|B| = 0}$

$$\bar{V} r^2 = + \frac{\mu}{m} \bar{r} \times \nabla|B|$$

$$\bar{V} = + \frac{\mu}{m} \frac{\bar{r} \times \nabla|B|}{r^2}$$

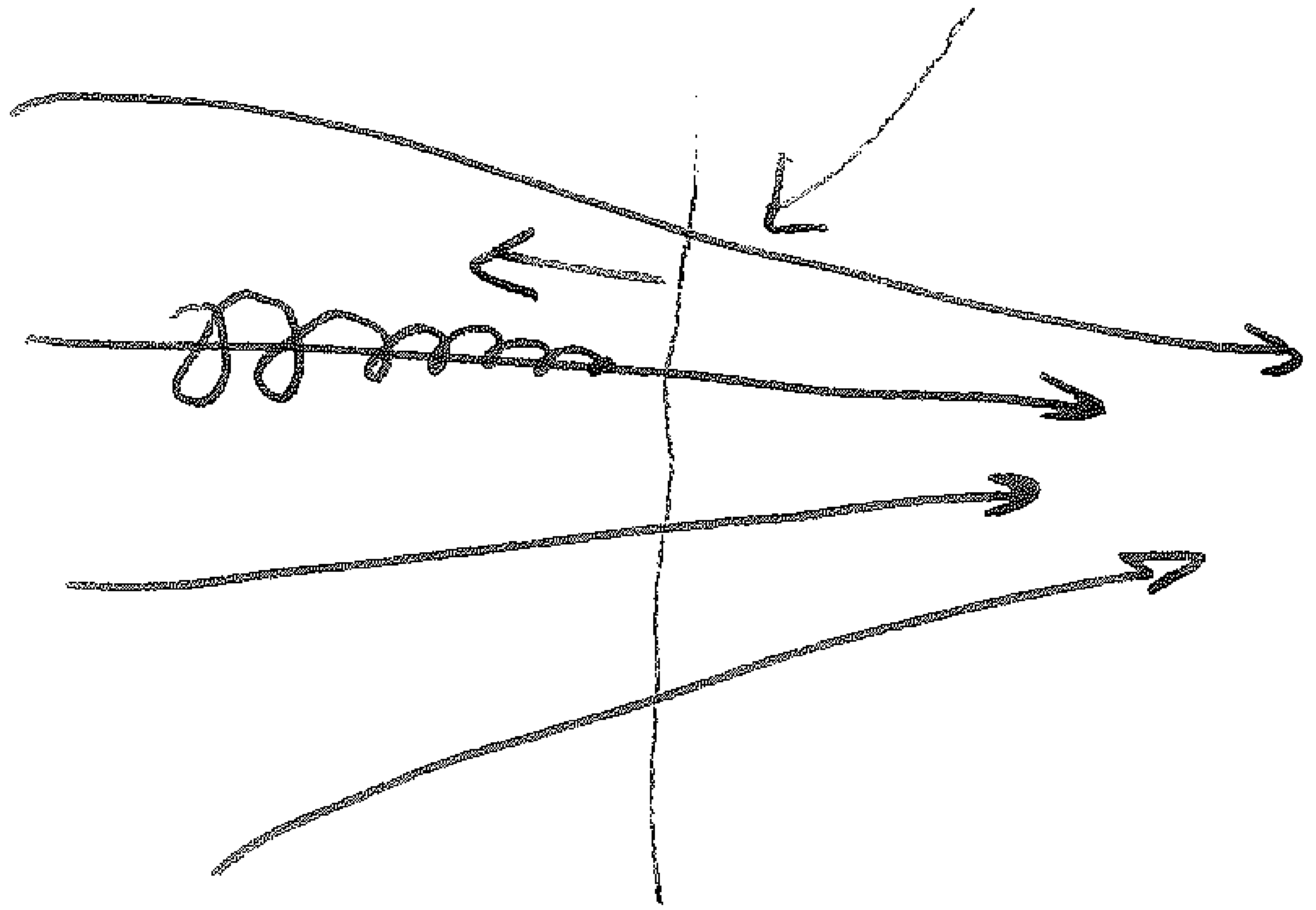
!!





(Parallel to B) Mirroring…

$$\hat{b}\cdot\frac{d\bar{V}}{dt} = -\underbrace{\frac{\mu}{m}\,\hat{b}\cdot\nabla|B|}_{\mu\cdot\ \mathrm{GRAD}\ B\ \mathrm{FORCE}} \quad !!$$



$$\frac{d\hat{b}}{dt} = V_\parallel\,\bar{\kappa} = V_\parallel\left(\hat{b}\cdot\bar{\nabla}\right)\hat{b}$$

$$\frac{d}{dt}\hat{b}\cdot\bar{V} - \bar{V}\cdot\frac{d\hat{b}}{dt} = \frac{d}{dt}V_\parallel - V_\parallel\left(\bar{V}_\perp\cdot\bar{\kappa}\right)$$

← VERY SMALL SINCE $V_\perp \sim \mathcal{O}(\kappa)$

$$\vec{r} \times \frac{d\vec{v}}{dt} = r^2 \vec{V}_\perp - \frac{\mu}{m} \vec{r} \times \vec{\nabla}|B|$$

$$\vec{V}_\perp = \frac{\mu}{m} \frac{1}{r} \hat{b} \times \vec{\nabla}|B| + \frac{\hat{b} \times \frac{d\vec{v}}{dt}}{r}$$

NOTICE TOTAL $\vec{V}$

NOW LET'S SOLVE THIS BY ITERATION...

(Perpendicular to B)
Curvature Drift

$$\vec{V}_\perp \simeq \underbrace{\frac{\mu}{m} \frac{1}{r} \hat{b} \times \vec{\nabla}|B|}_{\text{THIS IS A DRIFT}} + \underbrace{v_\parallel \hat{b}}_{\text{THIS IS}}$$

SLOW THIS IS A DRIFT $V\left(\frac{\rho}{L_B}\right)$

THIS IS A REAL FAST SPEED

$$\vec{V} \simeq v_\parallel \hat{b}$$

!!

$$V_\perp \simeq \frac{\mu}{m} \frac{1}{r} \hat{b} \times \vec{\nabla}|B| + \frac{\hat{b} \times \frac{d}{dt}(v_\parallel \hat{b})}{r}$$

Curvature Drift

$$\bar{V} \approx \underbrace{\frac{\mu}{m}\frac{1}{\Omega}\hat{b}\times\bar{\nabla}|B|}_{\text{THIS IS A DRIFT}} + \underbrace{v_\parallel \hat{b}}_{\text{THIS IS}}$$

SLOW

THIS IS A DRIFT

$v\left(\frac{\rho}{L_B}\right)$

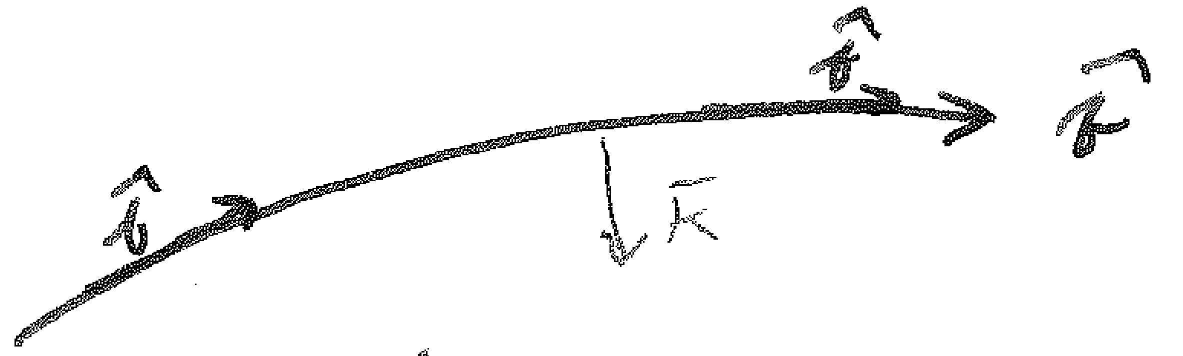
THIS IS A REAL FAST SPEED

$$\bar{V} \approx v_\parallel \hat{b}$$

$$V_\perp \approx \frac{\mu}{m}\frac{1}{\Omega}\hat{b}\times\bar{\nabla}|B| + \frac{\hat{b}\times\frac{d}{dt}(v_\parallel\hat{b})}{\Omega}$$

$$\frac{d}{dt}(v_\parallel \hat{b}) = \hat{b}\frac{dv_\parallel}{dt} + v_\parallel \frac{d}{dt}\hat{b}$$

$$= \underbrace{\hat{b}\frac{dv_\parallel}{dt}}_{\text{THIS is} \parallel \text{to } B} + v_\parallel^2 \underbrace{\hat{b}\cdot\bar{\nabla}\hat{b}}_{\bar{\kappa}}$$



$\frac{d}{dt} \approx v_\parallel \hat{b}\cdot\nabla$

$\bar{\kappa}$ = CURVATURE !!

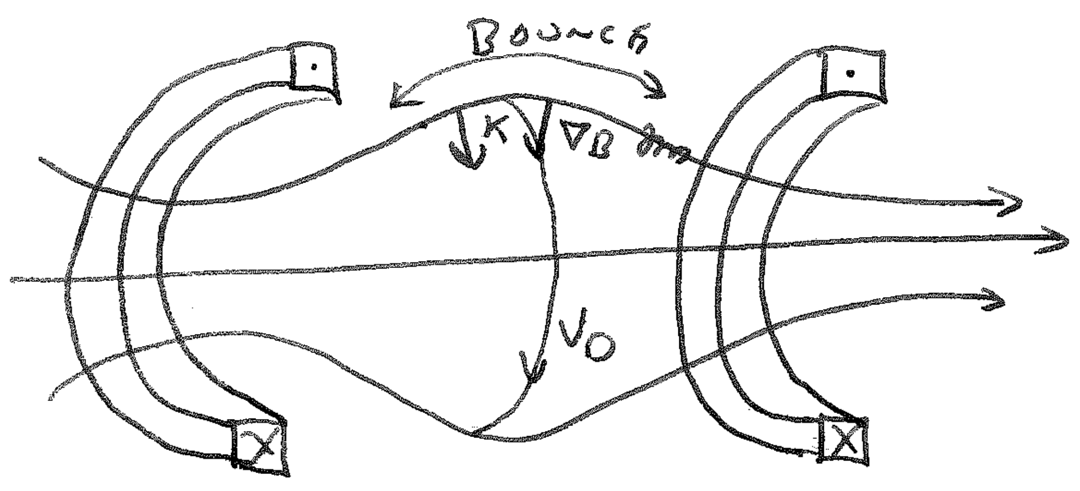
$\bar{\kappa} = \frac{1}{R_c}$

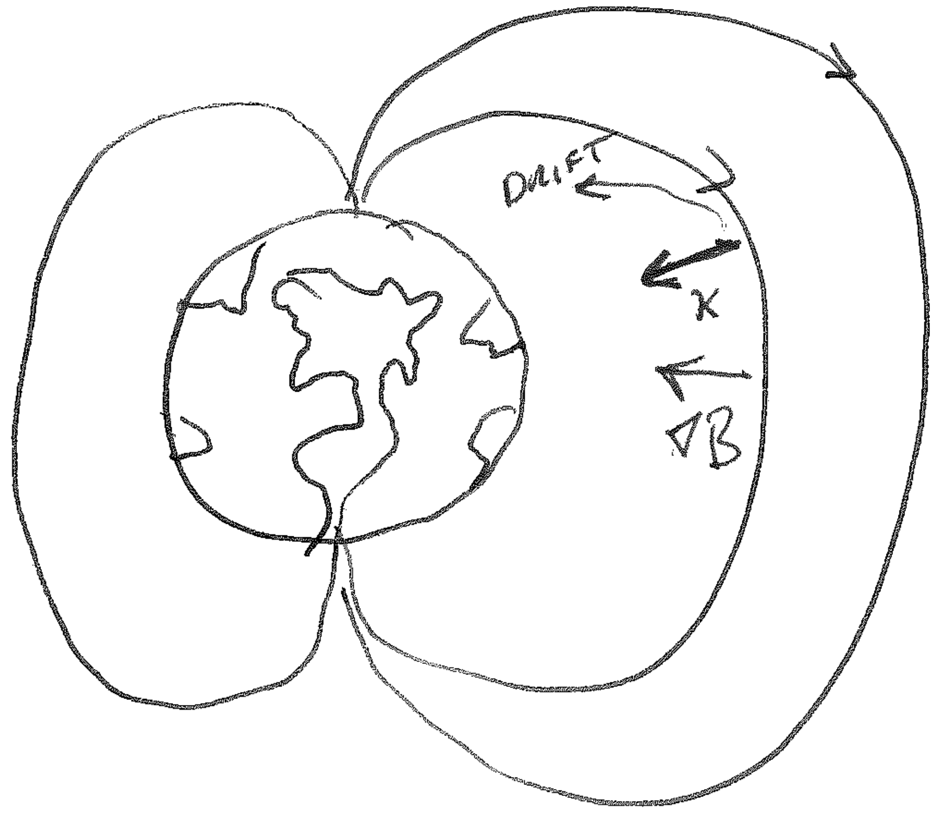
Summary

$$\bar{V} = v_\parallel \hat{b} + \frac{\hat{b}}{\Omega} \times \left(\frac{\mu}{m} \nabla |B| + v_\parallel^2 \bar{\kappa} \right)$$

$$\frac{d v_\parallel}{dt} = - \frac{\mu}{m} \hat{b} \cdot \nabla |B|$$

$\rho =$
$\omega =$ SAME AS BEFORE (but Ω changes...)





Another Simple Proof for Constant Adiabatic Invariant

$$m \frac{dv_\parallel}{dt} = -\mu \frac{dB}{ds}$$

$$\frac{d}{dt} \frac{1}{2} m v_\parallel^2 = -\mu v_\parallel \frac{dB}{ds}$$

BUT $$\frac{d}{dt}\left(\frac{1}{2} m v_\parallel^2 + \frac{1}{2} m v_\perp^2\right) = 0$$

SO $$\frac{d}{dt}\left(\frac{1}{2} m v_\perp^2\right) - \mu v_\parallel \frac{dB}{ds} = 0$$

BUT $$v_\parallel \frac{dB}{ds} = \frac{dB}{dt}$$

SO $$\frac{d}{dt}\left(\frac{1}{2} m v_\perp^2\right) - \frac{1}{2} m v_\perp^2 \frac{1}{B} \frac{dB}{dt} = 0$$

or $$B \frac{d}{dt}\left(\frac{\frac{1}{2} m v_\perp^2}{B}\right) = 0$$

$$\boxed{\frac{d}{dt} \mu = 0}$$

Next Monday

- Homework #2
- Guiding Center Hamiltonian (Littlejohn & Boozer)
- Tokamaks and bananas