Lecture21: Review

Plasma Physics 1

APPH E6101x

Columbia University

Chapters

- Ch 1 (Intro)
- Ch 2 (What is a plasma)

➡ **Ch 3 (Single particle motion)**

➡ **Ch 4 (Collisions and transport)**

➡ **Ch 5 (Plasmas as fluid, "Pinch" equilibria)**

➡ **Ch 6 (Waves)**

➡ **Ch 7 (Bohm criterion, Probes)**

➡ **Ch 8 (Beam instability, Kink Modes)**

➡ **Ch 9 (Vlasov equation)**

Notes

$$f(v) \sim \frac{n}{(2\pi kT/m)^{1/2}}\, e^{-\frac{1}{2}v^2/(kT/m)}$$

Debye Length

$$\lambda_D^2 = \frac{\epsilon_0}{n_0 e^2}(kT)$$

$$n_e \sim n_0 e^{e\phi/kT} \qquad \lambda_D^2 = \frac{v_{th}^2}{\omega_p^2}$$

Plasma Frequency $\quad \omega_p^2 = \dfrac{e^2 n}{\epsilon_0 m} = \left(v_{th}^2/\lambda_D^2\right)$

$$\boxed{\nabla^2 \Phi = -\rho/\epsilon_0} = -\frac{q}{\epsilon_0}\left(n_0\left(1 + \frac{q\Phi}{kT} \cdots\right)\right)$$

Other Frequencies

Cyclotron Frequency

Collision Frequency

Plasma Parameter $\quad N_D \sim n_0 \lambda_D^3$

$$\frac{\text{Kinetic}}{\text{Potential Energy}} \sim 6\pi N_D^{2/3}$$

Also $\quad \dfrac{\omega_{pe}}{\nu_{ei}} \sim \dfrac{N_D}{\ln(N_D)}$ so collisions are often weak

Single particle motion

$$(\bar r, \bar v)$$

$$\frac{d\bar r}{dt} = \bar v$$

$$m\frac{d\bar v}{dt} = q(\underline{E} + v \times B)$$

STRONG B important case

with E, B ..

$$\bar r = \bar\rho + \bar R$$

$$\bar\rho = \hat e_1 \rho \cos(\omega_c t) + \hat e_2 \rho \sin(\omega_c t) \quad \text{(FAST)}$$

$\bar R$ = guiding center motion (SLOW)

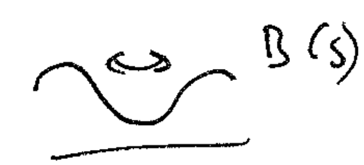
$$\mu = \tfrac{1}{2} m v_\perp^2 / B$$

$$\dot{\bar R} = \frac{\bar F \times B}{q B^2}$$

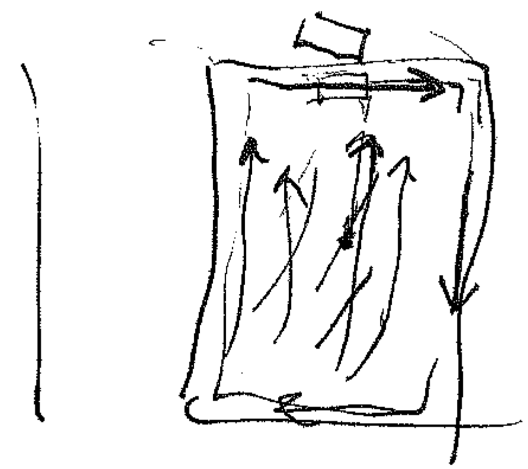
$$= \frac{\frac{1}{2} m v_\perp^2}{qB}\left(\frac{\hat b \times \nabla B}{B}\right) + \frac{m v_\parallel^2}{qB}\underbrace{\frac{\hat b \times \hat R_c}{R_c}}_{\hat b \times \bar\kappa}$$

$$+ \frac{E \times B}{B^2} + \frac{1}{\omega_c B}\frac{\partial \bar E}{\partial t}$$

μ, $J = \int ds\, v_\parallel$ ADIABATIC INVARIANTS $\omega_c, \omega_b \gg \omega$



Magnetic mirrors



Pure
Toroidal Field
Plasma

$$J = J_{m} + J_{g}$$

$$J_g = \sum n e \langle \bar{v}_g \rangle \sim \sum n e \frac{\langle w_\perp \rangle}{e B^2} \hat{\varphi} \times \nabla B$$

$$J_m \sim \nabla \times m \qquad m \sim \sum n \langle \mu \rangle \hat{B}$$

Chaos - Breakdown of
Hamiltonian - Symmetries

Waves in cold Plasma

$$A(\bar{r},t) \sim e^{-i(\omega t - \bar{k}\cdot\bar{r})}$$ ONLY PLANE WAVES

$$\frac{\partial}{\partial t} \rightarrow -i\omega$$

$$\bar{\nabla} = +i\bar{k}$$

For Linear waves

$$n = n_0 + n_1$$

$$\bar{E} = \bar{E}_0 + \bar{E}_1$$

$$\bar{B} = \bar{B}_0 + \bar{B}_1$$

$$\bar{v} = \bar{v}_0 + \bar{v}_1$$

Find DISPERSION RELATION

$$D(\omega, \bar{k}) = 0$$

And Eigenvectors POLARIZATION

$$\bar{J} = \bar{\bar{\sigma}}\cdot E$$

VACUUM $\bar{\bar{\sigma}} = 0$ $\qquad \bar{\bar{\sigma}}(\omega, k)$

$$k \times (k \times E) + i\omega\mu_0 (J_{DISP} + J_{PLAS}) = 0 \quad + i\omega\mu_0 (-i\omega\epsilon_0 \bar{\bar{I}} + \bar{\bar{\sigma}})\cdot E$$

$$\bar{k} \times (\bar{k} \times \bar{E}) + \frac{\omega^2}{c^2}\left(\bar{\bar{I}} + \frac{i\,\bar{\bar{\sigma}}}{\omega\epsilon_0}\right)\bar{E} = 0$$

EM wave DISPLACEMENT currents

PLASMA CURRENTS

$$\nabla \times B = \mu_0\left(\epsilon_0 \frac{\partial E}{\partial t}\right) + \mu_0 \bar{J}$$

$$\bar{k} \times (\bar{k} \times \bar{E}) + \frac{\omega^2/c^2}{(-i\omega\epsilon_0)}\left(-i\omega\epsilon_0\bar{\bar{I}} + \bar{\bar{\sigma}}\right)$$

$$J_{Displacement} = \epsilon_0 \frac{\partial E}{\partial t} = \underbrace{-i\omega\epsilon_0}\,\bar{E}$$

$$k \times (k \times E) + i\omega\mu_0 ($$

$$\bar{J} = \sum q n \bar{v}$$

$$\frac{d\bar{v}}{dt} = \frac{q}{m}\left[\bar{E} + v \times B\right]$$

$$k \times \bar{E} = \omega \bar{B}$$

$$\frac{d\bar{v}}{dt} = \frac{q}{m}\left[\bar{E} + \bar{v} \times (\bar{k} \times \bar{E})/\omega\right]$$

$$-i\omega \bar{v} = \frac{q}{m} \bar{E} + \bar{v} \times (\bar{k} \times \bar{E})/\omega$$

Solve for

$$\boxed{\bar{v} = \bar{\bar{\mu}} \cdot \bar{E}}$$

$$\bar{J} = \underbrace{\sum q n \bar{\bar{\mu}}}_{\bar{\bar{\sigma}}} \cdot \bar{E}$$

$$\bar{k} \times (\bar{k} \times \bar{E}) + \frac{\omega^2}{c^2}\left(\bar{\bar{I}} + i \frac{\bar{\bar{\sigma}}}{\omega \epsilon_0}\right)\bar{E} = 0$$

$$\bar{\bar{D}} \cdot \bar{E} = 0$$

$\det|D| = 0$ $\quad \bar{E}_i \sim$ Eigenmodes
Polarization

Cases

$\overline{B} = 0$ High Freq (Low Frequency ?)

$\overline{B}$ = Constant

High Frequency Parallel of

Low Frequency $\perp$ B

Wave Energy Density

Poynting's Theorem

$$\boxed{\frac{\partial w}{\partial t} + \bar{\nabla} \cdot S = - \bar{J} \cdot \bar{E}}$$

$$w = \frac{|B^2|}{2\mu_0} + \frac{\epsilon_0}{2} |E|^2 \qquad \text{For EM waves}$$

Can we look at plasma waves ....

Oscillations have wave energy density.

$$\langle w_k \rangle = \frac{\epsilon_0}{2} |E_k^2| \frac{\partial}{\partial \omega} (\omega D_R) = \frac{\epsilon_0}{2} \frac{\partial}{\partial \omega} \omega (E_k \cdot \bar{\bar{D}}_R \cdot E_k)$$

$$\bar{V}_g = - \frac{\partial D_R / \partial \bar{k}}{\partial D / \partial \omega} \qquad \omega_I = - D_I \Big/ \frac{\partial D_R}{\partial \omega}$$

Simple case

$$\overline{D} = \begin{pmatrix} 1 - k^2c^2/\omega^2 & 0 & 0 \\ 0 & 1 - \frac{k^2c^2}{\omega^2} & 0 \\ 0 & 0 & \phi \end{pmatrix}$$

$\overline{E} \perp \overline{k}$

$$\omega \frac{2}{2\omega}\left(1 - \frac{k^2c^2}{\omega^2}\right) = 2\omega\frac{k^2c^2}{\omega^3} \sim 2\frac{k^2c^2}{\omega^2} = 2$$

$$\overline{V}_g = -\frac{-2\overline{k}c^2/\omega^2}{2k^2c^2/\omega^3} \sim + k\,\omega/k^2 = \frac{\overline{k}}{k}c$$

wrong Z

$D_I = \omega_2 = 0$

$$\frac{\epsilon_0}{2}|E_k|^2 + \frac{1}{2\mu_0}|B_k|^2 + \frac{1}{2}mn|V_k^2|$$

Moments of Distribution

$$n = \iiint d^3v\, f$$

$$n\bar{V} = \iiint d^3v\, \bar{v}\, f$$

$$\tfrac{1}{2} n m V^2 = \iiint d^3v\, \tfrac{1}{2} m v^2 f$$

$$\bar{\bar{P}} = \iiint d^3v\, (\bar{v}-\bar{V})(\bar{v}-\bar{V})\, f$$

$$\bar{J} = \sum q n \bar{V}$$

$$\rho_q = \sum q n$$

$$\rho_m = \sum m n$$

Conservation of Particles

$$\frac{df}{dt} = 0 \qquad f(\bar{r}, \bar{v}, t)$$

$$\frac{\partial f}{\partial t} + v \cdot \frac{\partial f}{\partial \bar{r}} + \bar{a} \cdot \frac{\partial f}{\partial v} = 0$$

$$\text{or} \quad f(\bar{r}, \bar{v}, t) = f(\bar{r}_0, \bar{v}_0, 0)$$

$$\bar{r}(t) = \bar{r}_0 + \int_0^t dt'\, v(t')$$

$$\bar{v}(t) = \bar{v}_0 + \int_0^t dt'\, a(t')$$

$$\frac{\partial n}{\partial t} + \nabla\cdot n v = 0 \qquad \frac{Dn}{Dt} = -n\nabla\cdot\bar{v}$$

$$\frac{\partial}{\partial t}(mn\bar{v}) + \bar{\nabla}\cdot(mn\bar{v}\bar{v}) = ne(\bar{E}+\bar{v}\times B) - \bar{\nabla}\cdot\bar{\bar{P}} + \bar{R}$$

$$mn\frac{D\bar{v}}{Dt} = ne(E+\bar{v}\times B) - \nabla\cdot\bar{\bar{P}} + \bar{R}$$

$$\frac{\partial}{\partial t}W + \nabla\cdot Q = E\cdot J$$

$$W = \iiint \frac{1}{2} m v^2 f\, d^3v$$

$$\bar{Q} = \iiint \frac{1}{2} m v^2 \bar{v} f\, d^3v \qquad \text{ENERGY FLUX}$$

ADIABATIC EQUATION OF STATE

$$\frac{D}{Dt}\left(P/\rho^{\gamma}\right) = 0$$

ISOTHERMAL $\quad P/\rho = \text{CONST} \Rightarrow T = \text{CONSTANT}$

WAVES WITH FINITE PRESSURE

SOUND WAVE (ACCOUSTIC) !

DISPERSION TO PLASMA WAVES

$$\omega^2 = \omega_p^2 + 3k^2\langle v^2\rangle$$

EFFECT of COLLISIONS

$$\rho \frac{d\bar{v}}{dt} = qn\bar{E} - \nu\rho\bar{v}$$

collision frequency

$$\frac{q}{m\nu}\bar{E} = \bar{v}$$

$$J = qnv = \underbrace{\frac{q^2 n}{\nu m}}_{\sigma}\bar{E}$$

σ CONDUCTIVITY

$$0 \simeq qn(E + v \times B_0) - \nu\rho\bar{v}$$

$$\bar{J} = \bar{\bar{\sigma}} \cdot \bar{E} \qquad \text{NOT ISOTROPIC}$$

AMBIPOLAR DIFFUSION

DIFFUSION w.r.t. magnetic field.

$$n_e \simeq n_i$$

HEAVY IONS $\quad \bar{V} = \bar{V}_i$

$$\bar{J} = en(\bar{V} - \bar{V}_e) \Rightarrow \bar{V}_e = \bar{V} - \bar{J}/en$$

IONS + ELECTRONS

$$\rho \frac{d\bar{V}}{dt} = -\nabla p + \bar{J} \times B$$

ELECTRON

$$0 \approx qn(E + v_e \times B) - \nabla p_e$$
$$\approx E + (v - J/en) \times B + \nabla p_e/en$$

$$0 \approx E + v \times B - \frac{J \times B}{en} + \frac{\nabla p_e}{en}$$

NORMAL MHD $\quad en\bar{V} \gg J_{DIA}$ $\quad \frac{B \times \nabla p}{B^2}$

$$0 \approx E + v \times B$$

WITH SOME COLLISIONS

$$\bar{J}/\sigma = E + v \times B \qquad \text{RESISTIVE MHD}$$

BUT THIS REALLY ISN'T RIGHT (USUALLY)

BECOMES

$$\bar{J}/\sigma \approx \frac{J \times B}{en} + \frac{\nabla p_e}{en}$$

Now we can look for MHD Equilibrium

FAST Force Balance

$$\nabla \times B = \mu_0 J \qquad \nabla \times E = -\frac{\partial B}{\partial t} \to 0 \quad \text{EQUIL}$$

$$\nabla \cdot B = 0 \qquad 0 = E + \bar{v} \times \bar{B}$$

$$\nabla \cdot E = \nabla \cdot J = 0 \qquad (v = 0 \text{ THEN } \bar{E} = 0)$$

$$\rho \frac{d\bar{v}}{dt} = J \times B - \nabla p$$

$B \cdot \nabla p = 0$ (NO PARALLEL pressure gradients)

$$\frac{D}{Dt}\left(p / \rho_m^{\gamma} \right) \approx 0$$

Magnetic Pressure $\frac{B^2}{2\mu_0}$

Magnetic Tension $\frac{B^2}{\mu_0}$ Curved Field Lines

$$\beta \sim p / (B^2/2\mu_0)$$

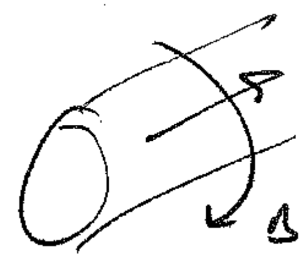
$0 = E + V \times B \Rightarrow$ FROZEN FLUX CONDITION

PLASMA AND FIELD MOVE TOGETHER

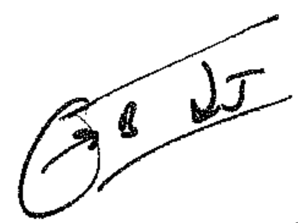
WAVES MHD with Alfvén

Magneto Accoustic (Fast + Slow)

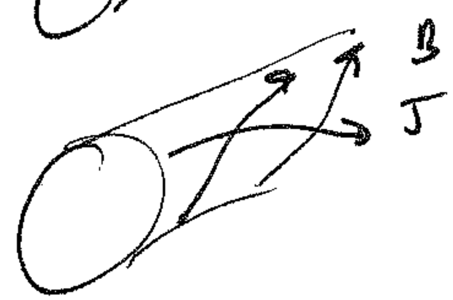
Pinch Equilibria



Z-Pinch



θ-Pinch



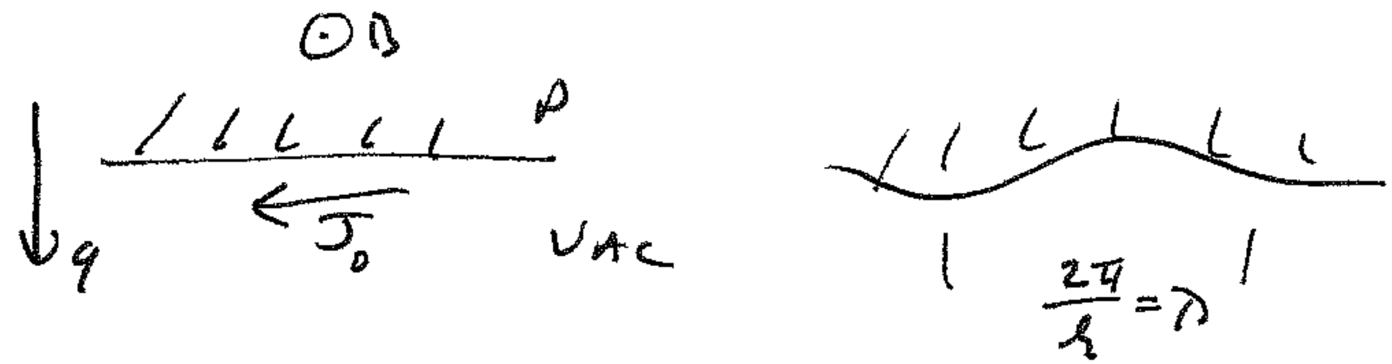
Screw Pinch

Only a family of possible

Equilibria not completely

specified

LIMITED MHD STABILITY



$$J \times B + \rho \bar{g} = 0$$

$$J_0 = -\frac{\rho}{B^2}\,\bar{B} \times \bar{g}$$

LINEARIZED

$$-j\omega \bar v \rho_0 = \rho_0 \frac{\partial \bar V_1}{\partial t} = \rho_1 \bar g + \bar J_1 \times B_0 + \bar J_0 \times B_1$$

$$\frac{\partial \rho_1}{\partial t} + j\bar k\cdot \bar V_1 \rho_0 + \bar V_1 \cdot \frac{\partial \rho_0}{\partial x}$$

$$-j\omega \rho_1 = -j\bar k\cdot\bar V_1\rho_0 + \bar V_1\cdot\frac{\partial\rho_0}{\partial x}$$

$$\rho_1/\rho_0 = \frac{k\cdot \bar V_1}{\omega} + j\frac{\bar V_1}{\omega}\cdot\frac{\partial}{\partial x}\ln\rho_0$$

$$\nabla\times B_1 = \mu_0 J_1$$

$$j\bar k\times B_1 = \mu_0 \bar J_1 \qquad E_1 + v_1\times B_0 = 0$$

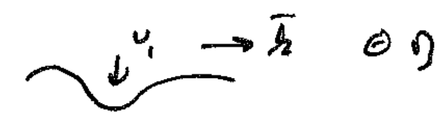
$$\bar V_1 = \frac{\bar E_1\times B_0}{B^2} \qquad k\times E_1 = \omega B_1$$

$$k\times(\bar V_1\times B_0) = -\omega B_1$$

$$\bar V_1(\bar k\cdot \bar B_0) - \bar B_0\,(k\cdot\bar V_1) = -\omega B_1$$

LET $k\cdot B_0 = 0$

$\bar k\cdot\bar V_1 = 0$



incompressible

THEN $B_1 = 0$

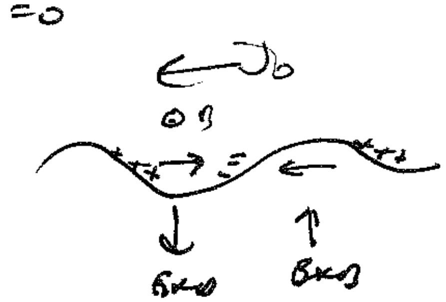
$J_1 = 0$

$$\omega = \pm i\sqrt{\frac{g}{\cancel{B}}\frac{\partial}{\partial x}\ln\rho_0}$$

$$-j\omega\rho_0\bar V_1 = \rho_1 \bar g$$

$$-j\omega\rho_0 v_1 = \frac{j\rho_0 \bar V_1\cdot\frac{\partial}{\partial x}\ln\rho_0}{\omega}\, g$$

$$\therefore\ \omega^2 = -g\,\frac{\partial}{\partial x}\ln\rho_0$$



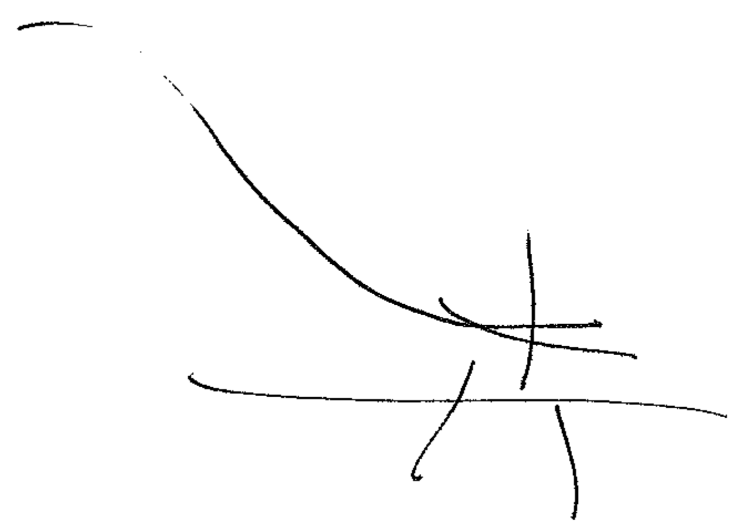
$$\nabla^2 \Phi = -\sum \frac{q}{\epsilon_0} \iiint d^3v \, f$$

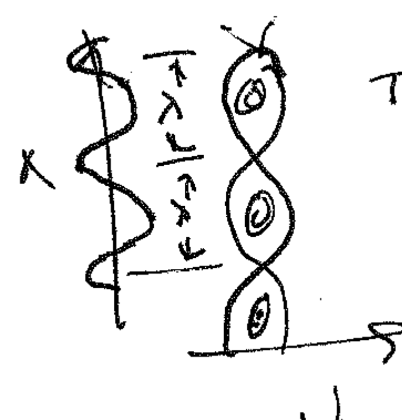
$$\frac{\partial f}{\partial t} + v \cdot \frac{\partial f}{\partial x} - \frac{q}{m} \nabla \Phi \cdot \frac{\partial f}{\partial v} = 0$$

$$f_1 = \frac{q}{m} k \Phi_k \frac{\partial f / \partial v}{\omega - k \cdot v}$$

$$k^2 \Phi = \sum \frac{q}{\epsilon_0} \iiint d^3v \, f_1$$

RESONANCE





TRAPPED ORBITS

(SOUND WAVE)
COLLISIONLESS DAMPING

Chapter Summaries

Ch 2

The Basics in a Nutshell

- Plasmas are quasineutral: $n_\mathrm{e} = \sum_k Z_k n_k$.
- Quasineutrality can be violated within a Debye length λ_D

$$\lambda_\mathrm{D} = \frac{\lambda_\mathrm{De}\lambda_\mathrm{Di}}{(\lambda_\mathrm{De}^2 + \lambda_\mathrm{Di}^2)^{1/2}} \quad , \quad \lambda_\mathrm{De,Di} = \left(\frac{\varepsilon_0 k_\mathrm{B} T_\mathrm{e,i}}{n_\mathrm{e,i}^2}\right)^{1/2} \quad .$$

- Quasineutrality can be established by the electrons within $\tau = \omega_\mathrm{pe}^{-1}$, with the plasma frequency

$$\omega_\mathrm{pe} = \left(\frac{n_\mathrm{e} e^2}{\varepsilon_0 m_\mathrm{e}}\right)^{1/2} \quad .$$

- The coupling parameter Γ determines the state of each plasma component (electrons, ions, dust)

$$\Gamma = \frac{q^2}{4\pi\varepsilon_0 a_{WS}^2 k_\mathrm{B} T} \; .$$

  Γ may be different for the components, depending on the individual temperatures and densities. A gaseous phase is found for $\Gamma \ll 1$, the liquid state for $1 < \Gamma < 180$ and the solid phase for $\Gamma > 180$.

The Basics in a Nutshell

- The complex trajectory of a charged particle in a magnetic field has been decomposed into a hierarchy of (periodic) motions

  1. gyration about the field line at the cyclotron frequency,
  2. periodic bouncing between mirror points,
  3. curvature and gradient drift, which can lead to a very slow periodic motion about the axis of the magnetic mirror.

- Each of these periodic motions is associated with an adiabatic invariant, which has a decreasing degree of conservation: the magnetic moment, the longitudinal invariant, and the flux invariant. Therefore, in the guiding center model, the real particle is replaced by a small ring current with an associated magnetic moment.
- The guiding center of this ring current performs various types of drift motion

E×B drift	$\mathbf{v}_E = (\mathbf{E} \times \mathbf{B})/B^2$
Gravitational drift	$\mathbf{v}_\mathrm{g} = (m/q)(\mathbf{g} \times \mathbf{B})/B^2$
Gradient drift	$\mathbf{v}_{\nabla B} = (m/q)(\frac{1}{2}v_\perp^2/R_c^2)(\mathbf{R}_\mathrm{c} \times \mathbf{B})/B^2$
Curvature drift	$\mathbf{v}_R = (m/q)(v_z^2/R_c^2)(\mathbf{R}_\mathrm{c} \times \mathbf{B})/B^2$
Polarization drift	$\mathbf{v}_\mathrm{p} = (m/q)(\partial \mathbf{E}/\partial t)/B^2$

- In a tokamak, the twist of the confining magnetic field is effected by the toroidal current, which is induced by a big transformer with the plasma torus as secondary winding. The rotational transform of the field lines counteracts the losses arising from plasma drifts.
- In a stellarator, the rotational transform is effected by external helical currents. Modern stellarators use modular coils which produce both the confining magnetic field and the rotational transform.

Ch 3

The Basics in a Nutshell

- The various definitions of a Maxwell distribution are

1-dimensional	$f_M^{(1)}(v_x) = n\left(\frac{m}{2\pi k_B T}\right)^{1/2} \exp\left(-\frac{m v_x^2}{2k_B T}\right)$
3-dimensional	$f_M^{(3)}(\mathbf{v}) = n\left(\frac{m}{2\pi k_B T}\right)^{3/2} \exp\left(-\frac{m(v_x^2+v_y^2+v_z^2)}{2k_B T}\right)$
distribution of speed	$f_M(v) = 4\pi v^2 n\left(\frac{m}{2\pi k_B T}\right)^{3/2} \exp\left(-\frac{m v^2}{2k_B T}\right)$
distribution of energy	$F_M(W) = n\frac{2}{\sqrt{\pi}}\frac{1}{(k_B T)^{3/2}} W^{1/2} \exp\left(-\frac{W}{k_B T}\right)$

- The various definitions of "thermal velocity" are:

mean thermal velocity	$v_{th} = \left(\frac{8k_B T}{\pi m}\right)^{1/2}$
most probable velocity	$v_T = \left(\frac{2k_B T}{m}\right)^{1/2}$

- The mean free path is $\lambda_{mfp} = (n\sigma)^{-1}$ and the collision frequency $\nu_c = v/\lambda_{mfp}$.
- The number of collision events in a hot gas per volume and second is given by the rate coefficient $\langle \sigma v \rangle$, in which the angle brackets denote averaging over the distribution function.
- The Coulomb collision frequency decreases for rising temperature as $\nu_{ei} \propto T^{-3/2}$ and is independent of plasma density.
- Transport of plasma particles is accomplished by electric fields or density gradients. The individual transport coefficients are the mobilities $\mu_{e,i} = e/(m_{e,i}\nu_{e,i})$ and the diffusion coefficients $D_{e,i} = \mu_{e,i} k_B T_{e,i}/e$.
- The global transport coefficients are the electrical conductivity $\sigma = ne(\mu_e + \mu_i)$ and the ambipolar diffusion coefficient $D_a = (D_i\mu_e + D_e\mu_i)/(\mu_e + \mu_i)$.
- In the presence of a magnetic field, the transport coefficients become tensors that link the velocity to the force. This applies to mobility, conductivity and diffusivity. The Pedersen conductivity is the diagonal element and the Hall conductivity the off-diagonal element of the conductivity tensor.

Ch 4

Ch 5

The Basics in a Nutshell

- The fluid models treat the electrons and ions as fluids and seek self-consistency of the problems by combining the fluid equations with the set of Maxwell's equations:

Faraday's induction law	$\nabla \times \mathbf{E} = -\partial \mathbf{B}/\partial t$
Ampere's law	$\nabla \times \mathbf{B} = \mu_0 (\mathbf{j} + \varepsilon_0 \partial \mathbf{E}/\partial t)$
Poisson's law	$\nabla \cdot \mathbf{E} = \rho/\varepsilon_0$
no magnetic monopoles	$\nabla \cdot \mathbf{B} = 0$

- The two-fluid model is based on separate equations for electrons and ions and describes the continuity and momentum flow of the fluids:

continuity	$\partial n/\partial t + \nabla \cdot (n\mathbf{u}) = 0$
momentum tansport	$nm\left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u}\right) = nq(\mathbf{E} + \mathbf{u} \times \mathbf{B}) - \nabla p$

- The MHD-equations describe the mass transport and the electric current in a single fluid:

momentum transport	$\rho_\mathrm{m} \frac{\partial \mathbf{v}_\mathrm{m}}{\partial t} = \mathbf{j} \times \mathbf{B} - \nabla p + \rho_\mathrm{m} \mathbf{g}$
generalized Ohm's law	$\mathbf{E} + \mathbf{v}_\mathrm{m} \times \mathbf{B} = \eta \mathbf{j} + \frac{1}{ne}(\mathbf{j} \times \mathbf{B} - \nabla p_\mathrm{e})$

- The diamagnetic drift is a net effect in an inhomogeneous distribution of guiding centers. A net electric current is established without motion of the guiding centers.
  The diamagnetic drift velocity is $\mathbf{v}_\mathrm{D} = -[\nabla p \times \mathbf{B}](qnB^2)^{-1}$.
- A magnetic field exerts an isotropic magnetic pressure $p_\mathrm{mag} = B_0^2(2\mu_0)^{-1}$ and has a field line tension $\mathscr{T} = 2p_\mathrm{mag}$.
- When the plasma is an ideal conductor, the magnetic field is frozen in the plasma. The combined motion of plasma and magnetic field leads to Alfvén waves, which propagate at the Alfvén speed $v_\mathrm{A} = B_0(\mu_0\rho_\mathrm{m})^{-1/2}$.

The Basics in a Nutshell

- In Fourier notation, Maxwell's equations become:

Induction law	$\mathbf{ik} \times \hat{\mathbf{E}} = \mathrm{i}\omega\hat{\mathbf{B}}$
Ampere's law	$\mathbf{ik} \times \hat{\mathbf{B}} = -\mathrm{i}\omega\varepsilon_0\mu_0\hat{\mathbf{E}} + \mu_0\hat{\mathbf{j}}_0$
Poisson's law	$\mathbf{ik} \cdot \hat{\mathbf{E}} = \hat{\rho}/\varepsilon_0$
no longitudinal $\hat{B}$	$\mathbf{ik} \cdot \hat{\mathbf{B}} = 0\,.$

- The wave equation: $\{\mathbf{kk} - k^2\boldsymbol{I} + \frac{\omega^2}{c^2}\boldsymbol{\varepsilon}_\omega\} \cdot \hat{\mathbf{E}} = 0$.
- The phase and group velocities are defined as $v_\varphi = \omega/k$, $v_\mathrm{g} = \mathrm{d}\omega/\,\mathrm{d}k$.
- Transverse electromagnetic waves in an unmagnetized plasma have the refractive index $\mathcal{N} = \varepsilon(\omega) = (1 - \omega_\mathrm{pe}^2/\omega^2)^{1/2}$. They exist only above a cut-off frequency, $\omega > \omega_\mathrm{pe}$.
- The transverse mode is used for plasma interferometry to determine the plasma density. The phase shift of an interferometer is proportional to the product $n_\mathrm{e}L\lambda$.
- The dispersion of an electrostatic wave in an unmagnetized plasma is determined by $\varepsilon(\omega) = 0$.
- Electrostatic waves have $\mathbf{k}||\hat{\mathbf{E}}$ and are found in two frequency regimes: Bohm-Gross modes for $\omega > \omega_\mathrm{pe}$ and ion-acoustic waves for $\omega < \omega_\mathrm{pi}$. The ion-acoustic speed is $C_\mathrm{s} = (k_\mathrm{B}T_\mathrm{e}/m_\mathrm{i})^{1/2}$.
- In magnetized plasma, the fundamental modes for propagation along the magnetic field line have circular polarization. The refractive index of the R-wave and L-wave are different. This leads to Faraday rotation of a linearly polarized wave. The R-wave (L-wave) has a resonance at the electron (ion) cyclotron frequency.
- Resonances correspond to $\mathcal{N}^2 \to \infty$, cut-offs to $\mathcal{N}^2 \to 0$.
- Waves propagating perpendicular to a magnetic field are the O-mode ($\mathbf{E}||\mathbf{B}_0$), which is unaffected by the magnetic field, and the X-mode, which has resonances at the upper hybrid frequency $\omega_\mathrm{uh} = (\omega_\mathrm{pe}^2 + \omega_\mathrm{ce}^2)^{1/2}$ and lower hybrid frequency $\omega_\mathrm{lh} \approx (\omega_\mathrm{ce}\omega_\mathrm{ci})^{1/2}$.

Ch 6

The Basics in a Nutshell

- The Child-Langmuir Law

$$j = \frac{4}{9}\left(\frac{2e}{m}\right)^{1/2}\frac{U^{3/2}}{d^2}$$

  describes the maximum, space-charge limited current in a single-species system of length d for an applied voltage U.
- Space-charge limited currents appear in plasma sheaths and in grid regions for ion extraction.
- The Bohm criterion for a sheath, $v_\mathrm{i} = v_\mathrm{B} = (k_\mathrm{B}T_\mathrm{e}/m_\mathrm{i})^{1/2}$ states that ions must enter the sheath with ion-sound speed.
- The current–voltage characteristic of a plane Langmuir probe has the parts: ion saturation regime, electron retardation regime and electron saturation regime. The floating potential is defined by $I = 0$, the plasma potential is the transition point from electron retardation to electron saturation current.
- The ion saturation curent of a plane probe is $I_{\mathrm{i,sat}} = \exp(-1/2)env_\mathrm{B}$. The electron saturation current is $I_{\mathrm{e,sat}} = -(1/4)env_{\mathrm{th,e}}$. Both currents can be used to determine the plasma density n, when the electron temperature is known.
- The electron temperature is obtained from a semi-log plot of the electron retardation current vs. the probe voltage.
- A current-carrying collisionless plasma can spontaneously form a localized internal potential drop, called a double layer.

Ch 7

Ch 8

The Basics in a Nutshell

- Plasma instabilities fall into two classes, macroscopic instabilities in real space, like the Rayleigh-Taylor instability, and microinstabilities in velocity space, like the beam-plasma instability.
- The directed flow of a group of fast electrons (beam) can excite electrostatic waves near the electron plasma frequency. This beam-plasma instability has a tremendous growth rate, which depends on $(n_\mathrm{b}/n_\mathrm{p})^{1/3}$.
- The instability of the slow wave can be understood from the concept of negative mass or negative energy waves.
- In a system of finite length (Pierce diode) the maximum electron current is limited by the onset of purely-growing, non-oscillating disturbances of the electron beam.

Ch 9

The Basics in a Nutshell

- The Vlasov equation

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla_r f + \frac{q}{m}(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \nabla_v f = 0$$

  describes the evolution of collisionless plasmas with an arbitrary distribution function in a 6-dimensional phase space spanned by position $\mathbf{x}$ and velocity $\mathbf{v}$ under the action of self-consistent electric and magnetic fields.
- Plasma waves can be treated by the linearized Vlasov model in combination with Maxwell's equations. The terminologies "cold plasma" and "hot plasma" refer to the ratio of thermal speed and phase velocity of the wave.
- Landau damping describes the exponential decay of a macroscopic wave electric field while the information is retained in the perturbed distribution function. The information can be partly recovered in echo experiments.
- The rate of Landau damping is determined by the slope of the unperturbed distribution function at the wave's phase velocity. For an unshifted Maxwellian, this is always negative. Velocity distributions having an additional shifted Maxwellian can produce a velocity interval, where the slope becomes positive leading to inverse Landau damping or instability.
- A physical picture of the mechanisms behind Landau damping involves charge bunching, ballistic response of particles and phase mixing.
- Plasma simulation with particle codes is complementary to the Vlasov approach. It describes the motion of superparticles that represent clumps of some thousand real particles. The particle-in-cell technique overcomes the limitation of N^2 scaling of the computation time for particle-particle codes.
- Plasma simulations make the nonlinear evolution of plasma processes accessible. Examples are: the trapping of electrons in beam-plasma interaction or the onset of blocking oscillations in diodes above the critical current.

Previous Exams

APPH 6101 Plasma Physics I
Final Examination: 15 December, 2005.

This is a *closed book* exam. You may use a copy of the *NRL Plasma Formulary.* All other information needed for this exam is contained within the exam booklet. If you have any questions about the exam, please clearly state these in your answer booklet and make your best answer.

You have at most 3 hours to answer all questions. All ten questions have equal "weight". (Therefore, please read all questions first, and answer the easiest questions before the more difficult ones.) When possible and appropriate, please *show your work* and write neatly.

Please write your name on your exam and exam booklet.

Final Score:

___ Question 1 (Basic definitions)

___ Question 2 (Basic definitions)

___ Question 3 (Particle motion)

___ Question 4 (Particle motion/Confinement)

___ Question 5 (Plasma waves)

___ Question 6 (Plasma waves)

___ Question 7 (Plasma transport)

___ Question 8 (Plasma stability)

___ Question 9 (Plasma equilirbium)

___ Question 10 (Vlasov theory)

___ **Total Score**

Question 1

What is the plasma frequency for an electron-positron plasma?

Question 2

Consider a proton-electron plasma with $T_e \sim 10 \times T_i$.

Part A

If a conducting sphere is inserted into the plasma, *approximately* what would be the electric potential of the sphere relative to the plasma potential?

Part B

A thin boundary between the plasma and the sphere would exist, called the "plasma sheath". Write an *approximate* expression for the thickness of this sheath.

Question 3

Consider a plasma "blob" (containing heavy ions and electrons) in a straight, uniform magnetic field under the influence of a constant gravitational acceleration, $\mathbf{g} = -\hat{z}g$.

The equation for the guiding center motion of the plasma particles is

$$\mathbf{V} = \frac{\mathbf{E} \times \mathbf{B}}{B^2} + \frac{M}{q} \frac{\mathbf{g} \times \mathbf{B}}{B^2} + \frac{M}{qB^2} \frac{d\mathbf{E}}{dt}$$

Part A

What are the names given to the three terms on the right-hand-side of the equation above?

Part B

If the plasma is sufficiently dense, such that $\omega_{pi}^2/\omega_{ci}^2 \gg 1$, then dynamics of the plasma can be described (approximately) with the equation $\nabla \cdot \mathbf{J} = 0$. Using this equation, show that the plasma blob falls down at the same rate as Newton's apple.

Question 4

Consider a plasma "blob" (containing heavy ions and electrons) located initially in a purely toroidal magnetic field. In cylindrical coordinates, the magnetic field is $\mathbf{B} = \hat{\phi} B_0 \left(r_0 / r \right)$, where r_0 is the initial radial location of the center of the blob and B_0 is the magnetic field strength at $r = r_0$.

The equation for the guiding center motion of the plasma particles is

$$\mathbf{V} = \frac{\mathbf{E} \times \mathbf{B}}{B^2} + \frac{\left(T_\perp + 2T_{||} \right)}{q} \frac{\hat{b} \times \nabla B}{B^2} + \frac{M}{qB^2} \frac{d\mathbf{E}}{dt}$$

where the low-beta assumption, $\hat{b} \cdot \nabla \hat{b} = \nabla \log B$, was used for the magnetic drifts.

Part A

In which direction does the plasma blob move? What is the direction of the electric field?

Part B

If the plasma is sufficiently dense, such that $\omega_{pi}^2 / \omega_{ci}^2 \gg 1$, then dynamics of the plasma can be described (approximately) with the equation $\nabla \cdot \mathbf{J} = 0$. Using this equation, give an expression for the motion of the plasma blob.

Part C

If the plasma motion occurs adiabatically, what happens to the plasma temperature as a result of the blob's motion?

Question 5

The whistler wave propagates along the magnetic field with a frequency range, $\omega_{ci} \ll \omega < \omega_{ce}$. An approximate dispersion relation for the whistler wave is

$$\frac{k^2 c^2}{\omega^2} = \frac{\omega_{pe}^2}{\omega(\omega_{ce} - \omega)}$$

where the sign of all frequencies are positive in the expression above.

Consider the propagation of whistler's in the magnetospheres of planets caused by atmospheric lightning. At the moment of the lightning strike, a broad band of electromagnetic frequencies are excited in a single pulse. Some of this wave energy couples to whistler waves that propagate from one pole to the other. Because the waves have different group velocities, waves at different frequencies arrive at different times.

Part A

Since the group velocity is $v_g = \partial\omega/\partial k$, given an expression for the time of arrival of whistler pulse as a function of the wave frequency.

Part B

A "cut-off" frequency is a frequency beyond (or below) which the wave can not propagate. Is there a "cut-off" for whistler waves propagating through a magnetospheric plasma? If so, what information is obtained from measurement of the "cut-off" frequency?

Question 6

Consider a uniform plasma with $T_e \gg T_i$.

Part A

Derive the dispersion relation for an ion acoustic wave by linearizing the following fluid equations:

$$
\begin{aligned}
\frac{\partial n}{\partial t} + \nabla \cdot n\mathbf{V} &= 0 \\
\frac{\partial \mathbf{V}}{\partial t} + \mathbf{V} \cdot \nabla \mathbf{V} &= -\frac{\nabla P}{nM} + \frac{q}{M}(\mathbf{E} + \mathbf{V} \times \mathbf{B}) \\
\nabla \cdot \mathbf{E} = -\nabla^2 \Phi &= \sum_{i,e} \frac{q_s n_s}{\epsilon_0} \\
P_i &\approx 0 \\
\delta \left(\frac{P_e}{n_e^\gamma} \right) &= 0
\end{aligned}
$$

Part B

The wave energy density for the ion acoustic wave is proportional to

$$W_k = \frac{\epsilon_0}{2} \left| \tilde{E} \right|^2 + \frac{1}{2} n_i M_i \left| \tilde{V}_i \right|^2 .$$

What is the ratio of the electric field energy density to the kinetic energy density for an acoustic wave?

Part C

Show the relationship between the wave energy density and the expression below for the ion acoustic wave.

$$W_k \propto \omega_R \frac{dD_R}{d\omega} \frac{\epsilon_0}{2} \left| \tilde{E} \right|^2$$

where $D_R(\omega, k)$ is the real part of the dispersion relation and ω_R is the real part of the wave frequency. [Hint: You should be able to show that $D_r \approx 1 - \omega_{pi}^2/\omega^2 + 1/\gamma k^2 \lambda_e^2$.]

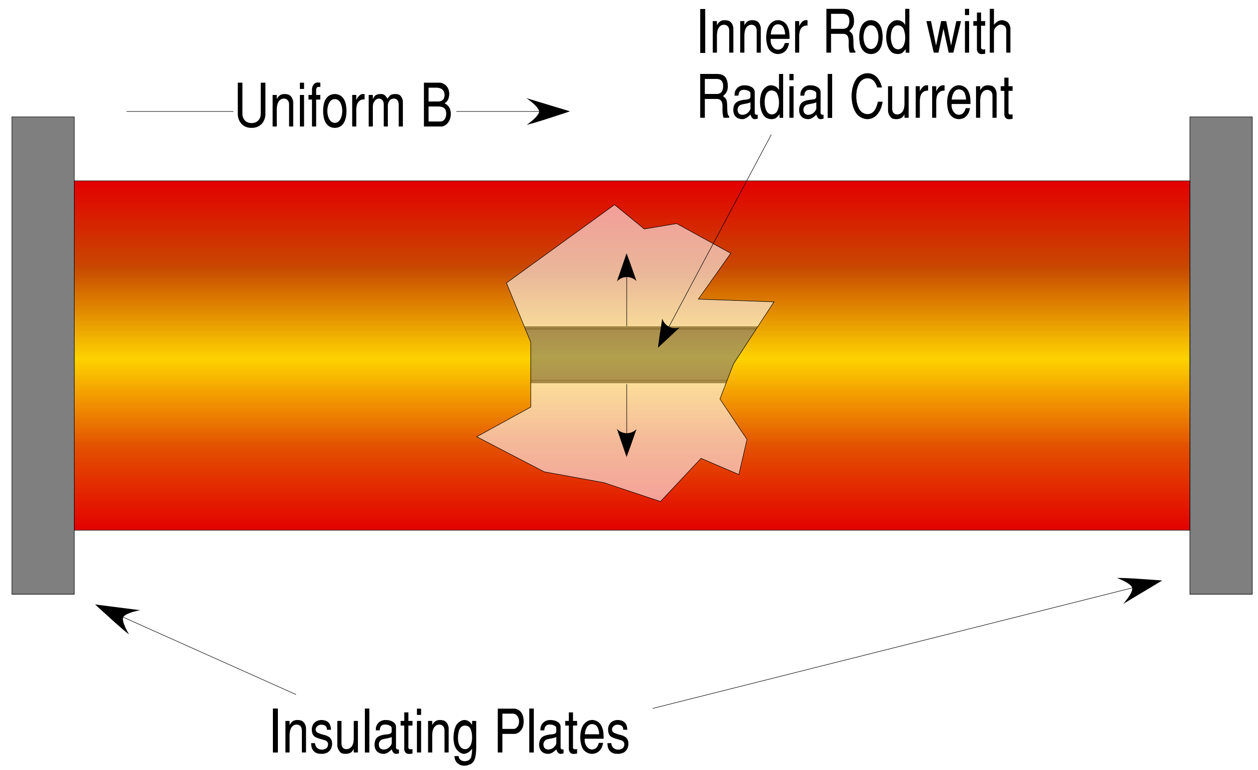


Figure 1: A cylindrical plasma column with a strong uniform magnetic field with two perfectly insulating end plates.

Question 7

Consider a uniform, magnetized plasma cylinder terminated at each end with a perfectly insulating end plate. Let the magnetic field be strong such that the plasma conductivity is given by the tensor $\mathbf{J} = \Sigma \cdot \mathbf{E}$, where

$$\Sigma = \begin{pmatrix} \sigma_\perp & \sigma_H & 0 \\ -\sigma_H & \sigma_\perp & 0 \\ 0 & 0 & \sigma_{||} \end{pmatrix}$$

where $\sigma_\perp \approx q^2 n \nu_i / M_i \omega_{ci}^2$, $\sigma_H \approx \sum_{e,i} q_s^2 n / m_s \omega_{cs}$, and $\sigma_{||} \approx q^2 n / m_e \nu_e$.

If there exists a wire (with radius a) along the axis of the plasma column, and if this wire is biased with respect to the outer surface of the plasma (at radius b), then a radial current causes the plasma to spin.

What must be the ion collision rate, $\nu_i(r)$, as a function of radius, if the plasma density and plasma rotation are constant, independent of the radius? [Hint: Work in cylindrical coordinates and use the condition $\nabla \cdot \mathbf{J} = 0$, with $\mathbf{J}$ being the radial current from the rod to the outer plasma surface.]

Question 8

Consider a plasma with a strong magnetic field, $\mathbf{B} = \hat{x} B_0$, under the influence of a vertical gravitational acceleration, $\mathbf{g} = -\hat{z} g$. Furthermore, assume that the plasma density *exponentially increases* with height such that $d \log n / dz = 1/h$, or $n(z) \propto e^{z/h}$. This plasma is unstable to the Rayleigh-Taylor instability.

Assuming that the plasma is incompressible, *i.e.* $\nabla \cdot \mathbf{V} = 0$, and the instability to propagate *perpendicular* to $\mathbf{B}$, what is the growth rate of the instability?

The ideal MHD equations are:

$$
\begin{aligned}
nM \frac{d\mathbf{V}}{dt} &= nM\mathbf{g} + \mathbf{J} \times \mathbf{B} \\
\frac{\partial n}{\partial t} + \nabla \cdot n\mathbf{V} &= 0 \\
\mathbf{E} + \mathbf{V} \times \mathbf{B} &= 0 \\
\nabla \times \mathbf{B} &= \mu_0 \mathbf{J} \\
\nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t}
\end{aligned}
$$

Question 9

A θ-pinch is a class of cylindrical plasma equilibria with azimuthal (θ-directed) currents and axial magnetic fields.

Part A

Derive the MHD equilibrium condition for a θ-pinch.

Part B

If the plasma pressure is *constant* from $0 < r < a$ and zero outside $r > a$, what must characterize the profile of the magnetic field?

Part C

Can the direction of the magnetic field *inside* the pinch be opposite of the direction of the magnetic field *outside* the pinch?

Question 10

The dispersion relation for electrostatic plasma waves in a collisionless plasma is

$$D(\omega,\ k) = 1 - \frac{\omega_{pe}^2}{k^2} \iiint d^3v \frac{\mathbf{k} \cdot df_0/d\mathbf{v}}{\mathbf{k} \cdot \mathbf{v} - \omega}$$

where $f_0(\mathbf{v})$ is the unperturbed velocity distribution function.

Imagine that the velocity-space distribution was a "cube". In other words, let $f_0(\mathbf{v}) = n/8c^3$ when $|v_x| < c$, $|v_y| < c$, and $|v_z| < c$ and $f_0 = 0$ when $\{v_x, v_y, v_z\}$ is outside of the velocity-space cube.

Now, when the plasma waves are propagating along either the $\hat{x}$, $\hat{y}$, or $\hat{z}$ directions, then the velocity-space integrals in the dispersion relation are relatively easy to evaluate. *For this special case of propagation along an axis*, what is the frequency of long-wavelength plasma waves? Also, under what conditions are the plasma waves undamped?

APPH 6101 Plasma Physics I
Final Examination: December, 2015.

This is a *closed book* exam. If you have any questions about the exam, please clearly state these in your answer booklet and make your best answer.

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Please write your name on your exam and exam booklet.

Final Score:

____ Question 1 (Particle motion)

____ Question 2 (Plasma equilibrium)

____ Question 3 (Plasma Waves)

____ Question 4 (Plasma Stability)

____ Question 5 (Vlasov Equation)

____ **Total Score**

Question 1

A charged particle is confined along a strong magnetic field and trapped between two "magnetic mirrors" separated by a distance L. See Fig. 1 above.

At the minimum of the magnetic field strength, $|B| = B_0$, and the magnitude of the particle's velocity perpendicular and parallel to the magnetic field are equal when $|B| = B_0$.

During some time, the length between the magnetic mirrors is reduced from L to $L/2$ while keeping B_0 constant.

Part A

What are the conditions for the first and second adiabatic invariants to be constant during this process?

Part B

Assuming the first and second adiabatic invariants to be constant as the length decreases, what is the ratio of the particle's kinetic energy *before*, when the magnetic mirrors are separated by L, and *after*, when they are separated by $L/2$?

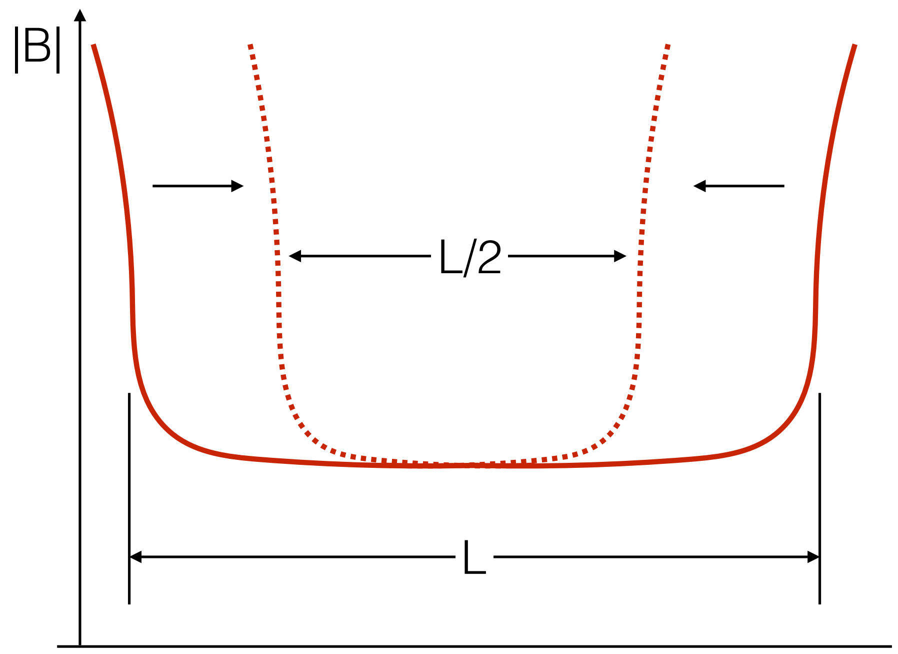

Figure 1: The magnetic field strength along a field line increases rapidly at the two "magnetic mirrors" separated by a length L. After some time, the length of the field line decreases to $L/2$.

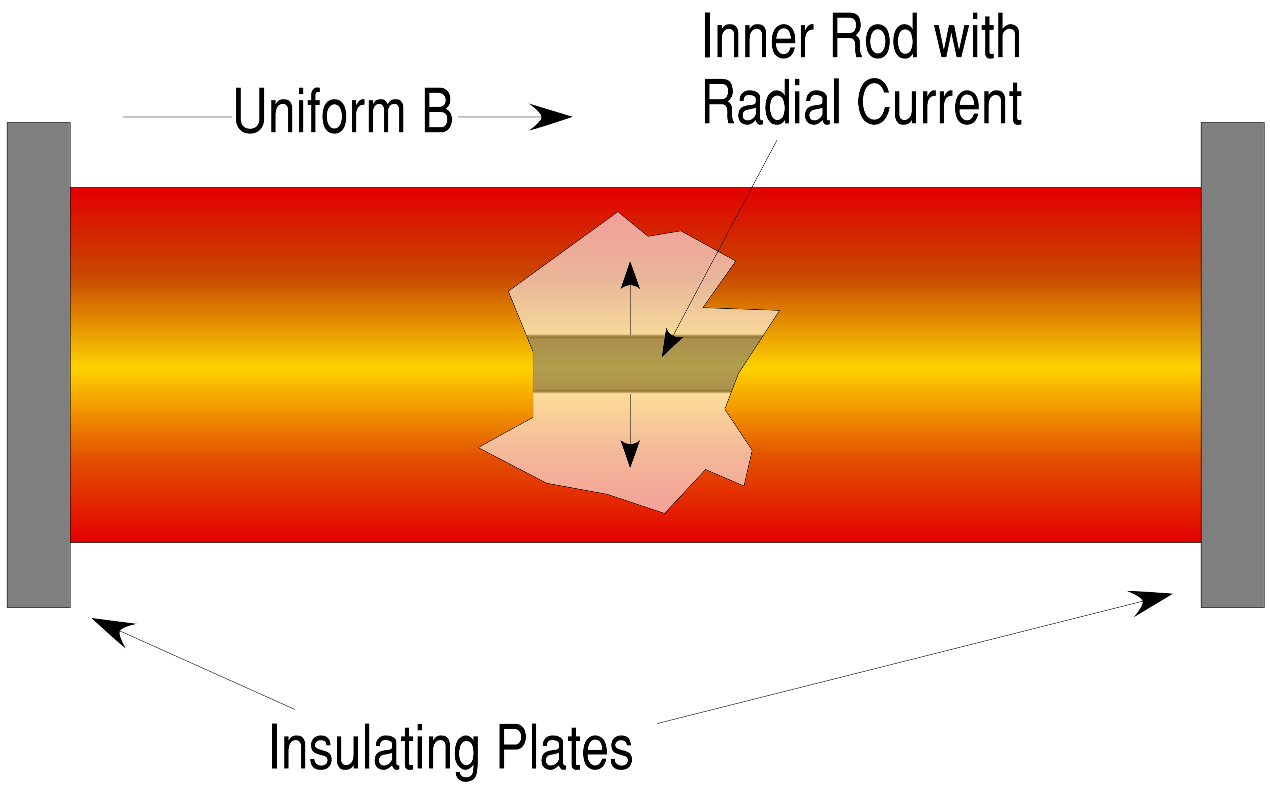

Figure 2: A cylindrical plasma column with a strong uniform magnetic field with two perfectly insulating end plates.

Question 2

Question 3

Consider the effect of electron collisions on plasma waves in a uniform cold plasma. The equation for electron motion is

$$m_e \frac{d\mathbf{v}_e}{dt} = -e\mathbf{E} - e\mathbf{v}_e \times \mathbf{B} - m_e \nu \mathbf{v}_e$$

where $\mathbf{E}$ and $\mathbf{B}$ are the electric and magnetic field of the electromagnetic wave. (There is *no equilibrium magnetic field* in this problem.)

Part A

Show that the effect of collisions can be represented by the substitution

$$m_e \to m_e \left(1 + \frac{i\nu}{\omega}\right)$$

where ω is the wave frequency.

Part B

Find the linear dispersion relation for longitudinal electron plasma oscillations including the effects of collisions. Briefly discuss the dissipation of these oscillations when $\nu \ll \omega_{pe}$.

Part C

For electromagnetic waves in a cold plasma, find approximate expressions for the real and imaginary parts of the wave number ($k = k_r + ik_i$) when $\nu \ll \omega$ and $\omega_{pe} \ll \omega$.

Question 4

Consider a cold plasma with a strong magnetic field, $\mathbf{B} = \hat{x} B_0$, under the influence of a vertical gravitational acceleration, $\mathbf{g} = -\hat{z} g$. Furthermore, assume that the plasma density *exponentially increases* with height such that $d \log n / dz = 1/h$, or $n(z) \propto e^{z/h}$. This plasma is unstable to the Rayleigh-Taylor instability.

Assuming that the plasma is incompressible, *i.e.* $\nabla \cdot \mathbf{V} = 0$, and the instability to propagate *perpendicular* to $\mathbf{B}$, what is the growth rate of the instability?

The ideal MHD equations are:

$$\begin{aligned}
nM \frac{d\mathbf{V}}{dt} &= nM\mathbf{g} + \mathbf{J} \times \mathbf{B} \\
\frac{\partial n}{\partial t} + \nabla \cdot n\mathbf{V} &= 0 \\
\mathbf{E} + \mathbf{V} \times \mathbf{B} &= 0 \\
\nabla \times \mathbf{B} &= \mu_0 \mathbf{J} \\
\nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t}
\end{aligned}$$

Question 5

When the electrostatic potential, $\Phi(\mathbf{r})$, is a constant in time, one solution to the Vlasov equation is

$$F(\mathbf{r}, \mathbf{v}, t) = n\left(\frac{m}{2\pi kT}\right)^{3/2} \exp\left[-(mv^2/2 + q\Phi(\mathbf{r}))/kT\right]$$

Part A

Show that this distribution function satisfies the Vlasov equation:

$$\frac{\partial F}{\partial t} + \mathbf{v} \cdot \nabla_r F + \frac{q}{m}\mathbf{E} \cdot \nabla_v F = 0$$

where $\mathbf{E} = -\nabla\Phi$.

Part B

For a plasma with positrons and electrons described by the same distribution above, integrate over velocity space and show that the electrostatic potential must satisfy

$$\nabla^2\Phi = -\frac{en}{\epsilon_0}\left[\exp(-e\Phi/kT) - \exp(e\Phi/kT)\right]$$

Part C

Assuming that $e\Phi/kT \ll 1$, show that the potential around a charge Q located at $r = 0$ is given by

$$\Phi(r) = \frac{1}{4\pi\epsilon_0}\frac{Q}{r}e^{-r/\lambda_D}$$

What is λ_D?