Lecture 2: Plasma Physics I

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Plasma Parameters

• $n(r,t)$ – density - plasma frequency, $\omega_p$

• $T(r,t)$ – temperature, $v_{th} = (kT/m)^{1/2}$

• $\lambda_D$ – Debye length, $v_{th}/\omega_p$

• $N_D$ - plasma parameter, $(4 \lambda_D^3/3) n >> 1$
FIGURE S.1 Plasmas that occur naturally or can be created in the laboratory are shown as a function of density (in particles per cubic centimeter) and temperature (in kelvin). The boundaries are approximate and indicate typical ranges of plasma parameters.

Distinct plasma regimes are indicated:

- For thermal energies greater than that of the rest mass of the electron ($k_B T > m_e c^2$), relativistic effects are important.
- At high densities, where the Fermi energy is greater than the thermal energy ($E_F > k_B T$), quantum effects are dominant.
- In strongly coupled plasmas (i.e., $n\lambda_D^{-3} < 1$, where $\lambda_D$ is the Debye screening length), the effects of the Coulomb interaction dominate thermal effects; and
- When $E_F > e^2 n^{1/3}$, quantum effects dominate those due to the Coulomb interaction, resulting in nearly ideal quantum plasmas.
- At temperatures less than about 105 K, recombination of electrons and ions can be significant, and the plasmas are often only partially ionized.

http://www.nap.edu/catalog/4936/plasma-science-from-fundamental-research-to-technological-applications
The Equations of Plasma Physics

\[ \nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0} \]  
(5.1)

\[ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \]  
(5.2)

\[ \nabla \cdot \mathbf{B} = 0 \]  
(5.3)

\[ \nabla \times \mathbf{B} = \mu_0 \left( \mathbf{j} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right) \]  
(5.4)

\[ f^{(\alpha)}(\mathbf{r}, \mathbf{v}, t) = \sum_k \delta(\mathbf{r} - \mathbf{r}_k(t))\delta(\mathbf{v} - \mathbf{v}_k(t)), \]  
(9.4)

\[ N^{(\alpha)} = \iiint f^{(\alpha)}(\mathbf{r}, \mathbf{v}, t) \, d^3r \, d^3v, \]  
(9.5)

\[ \rho_m(\mathbf{r}, t) = \sum_{\alpha} m^{(\alpha)} n^{(\alpha)}(\mathbf{r}, t) \]  
(9.7)

\[ \rho(\mathbf{r}, t) = \sum_{\alpha} q^{(\alpha)} n^{(\alpha)}(\mathbf{r}, t). \]  
(9.8)

\[ \frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla_r f + \frac{q}{m} (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \nabla_v f = 0. \]  
(9.13)

\[ m \dot{\mathbf{v}} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}), \]  
(3.1)
Debye Length:
The small scale of electric fluctuations

\[ f(W = \text{energy}) \propto e^{-W/NkT} \]

\[ W = \frac{1}{2} m \dot{v}^2 + V(\vec{r}) \]

\[ m_e(z) \propto e^{-e \Phi(z)/\kappa T} \]

\[ m_i(z) \propto e^{-e \Phi(z)/\kappa T} \]

\[ m_e(z) = m_i(z) \approx e^{-e \Phi(z)/\kappa T} \left( \frac{1}{kT_e} + \frac{1}{kT_i} \right) \]

\[ \nabla \cdot E = \rho/\varepsilon_0 \]

\[ E = -\nabla \Phi \]

\[ \nabla^2 \Phi = -\rho/\varepsilon_0 \]

**NOTE:** In mass units, \( \rho = \text{mass} \)
Debye Length:
Potential near a change within a plasma

\[ \nabla^2 \phi = \frac{1}{\varepsilon_0} (\varepsilon_0 \frac{\partial^2 \phi}{\partial n^2}) = \frac{2 \phi}{\varepsilon_0} + \frac{e}{\lambda_D} \nabla \cdot \nabla \phi \]

\[ \oint \nabla \cdot \nabla \phi = -\frac{Q}{\varepsilon_0} \]

\[ \oint \partial \hat{n} \cdot \nabla \phi \Rightarrow \nabla \cdot \phi = \frac{Q}{4\pi \varepsilon_0} \frac{1}{r} \]

But with plasma shielding, we can find a solution

\[ \phi \sim \frac{Q}{4\pi \varepsilon_0} \frac{1}{r} f(r) \]

\[ \frac{\partial \phi}{\partial n} = -\frac{Q}{4\pi \varepsilon_0} \frac{1}{r} f'(r) \]

\[ \frac{\partial^2 \phi}{\partial n^2} = \frac{\partial^2 f}{\partial n^2} - \frac{Q}{4\pi \varepsilon_0} \frac{1}{r^2} f' + \frac{Q}{4\pi \varepsilon_0} \frac{1}{r^2} f'' \]

\[ f'' = \frac{1}{r^2} f' \]

\[ \phi = \frac{Q}{4\pi \varepsilon_0} \frac{1}{r} - \frac{Q}{4\pi \varepsilon_0} \frac{1}{r_D} \]
Plasma Parameter

\[ N_D = \frac{4}{3\pi} \frac{3}{\Gamma_i} n > 1 \quad \text{For a "useful" plasma} \]

\[ N_D = \left( \frac{1}{3\Gamma_i} \right)^{3/2} \quad \Gamma_i \sim \frac{8^{2/3} \epsilon \langle \langle \rangle \rangle}{\hbar^2 T_i} \left( \frac{4\epsilon_0}{3} c^3 \sim \frac{1}{m_i} \right) \]

When \( \Gamma_i \) is large, then “plasma” is strongly coupled, usually only for cases where there exists a large or small \( T_i \) component with plasma.

Like dusty plasma

\[ \Gamma_i = \frac{1}{3} N_{Di}^{-2/3} \quad \text{Coupling Parameter} \]

\[ \Gamma_i = \frac{e^2}{4\pi \epsilon_0 a_{WS} k_B T_i} \]

\[ n_i \frac{4\pi}{3} a_{WS}^3 = 1 \]

\( a_{WS} = \) Wigner-Seitz radius
Plasma Frequency: "Fast" Electron Motion of Plasma

\[ \nabla \cdot \mathbf{E} = \frac{P}{e_0}, \]

\[ P = P_i + P_e \approx -e \vec{v}_e \quad \text{(ions do not move)} \]

\[ \frac{2f}{2t} + \nabla \cdot \nabla f + \frac{e}{m} \vec{E} \cdot \nabla f = 0 \]

\[ \iiint d^3v \Rightarrow \frac{2m}{2t} + \nabla \cdot \nabla m = 0 \quad \text{(no pressure)} \]

\[ \nabla m = \iiint d^3v \nabla f \]

\[ \Rightarrow \text{LINEARIZE} \quad m = m_0 + \vec{m} \quad (\vec{m}/m \ll 1) \]

\[ \frac{2\vec{m}}{2e} + m_0 \nabla \cdot \nabla \vec{v} = 0 \]

\[ \frac{2\vec{m}}{2e} + m_0 \nabla \cdot \frac{\vec{v}}{2e} = 0 \]

\[ \frac{2\vec{m}}{2e} + \frac{e^2 m_0}{m_0 e_0} \nabla \cdot \vec{v} = 0 \]
Collisions
Collisions

Review Fundamental Length Scales of a Plasma

- \( \lambda_0 \) : \( n^{-1/3} \)
- \( \lambda_d \) : \( n^{-1/3} \)
- \( \lambda_{\text{mfp}} \)
- \( b \) : \( n^{-1/3} \)
- \( \lambda \) : \( \lambda_{\text{mfp}} \)
- \( \Lambda \) : \( \Lambda_{\text{mfp}} \)

- Average path between collisions

\( \Lambda \approx n \lambda_0^3 \) = # of particles in Debye sphere

- \( \lambda_0 \approx 10^{-2} \text{ cm} \)
- \( \lambda_{\text{mfp}} \approx 100 \text{ m} \)

- Distance of closest approach for 90° scatter

- Average interparticle spacing

- Debye shielding length

- Mean-free path between collisions

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OSCILLATIONS IN IONIZED GASES

BY LEWI TONKS AND IRVING LANGMUIR

Abstract

A simple theory of electronic and ionic oscillations in an ionized gas has been developed. The electronic oscillations are so rapid (ca. $10^9$ cycles) that the heavier positive ions are unaffected. They have a natural frequency $v_e = (ne^2/\pi m)^{1/2}$ and, except for secondary factors, do not transmit energy. The ionic oscillations are so slow that the electron density has its equilibrium value at all times. They vary in type according to their wave-length. The oscillations of shorter wave-length are similar to the electron vibrations, approaching the natural frequency $v_p = v_e(m_e/m_p)^{1/2}$ as upper limit. The oscillations of longer wave-length are similar to sound waves, the velocity approaching the value $v = (kT_e/m_p)^{1/2}$. The transition occurs roughly (i.e. to 5% of limiting values) within a 10-fold wave-length range centering around $2(2)^{1/2}\pi \lambda_D$, $\lambda_D$ being the "Debye distance." While the theory offers no explanation of the cause of the observed oscillations, the frequency range of the most rapid oscillations, namely from 300 to 1000 megacycles agrees with that predicted for the oscillations of the ultimate electrons. Another observed frequency of 50 to 60 megacycles may correspond to oscillations of the beam electrons. Frequencies from 1.5 megacycles down can be attributed to positive ion oscillations. The correlation between theory and observed oscillations is to be considered tentative until simpler experimental conditions can be attained.
II. EXPERIMENTS

A. Discharge tubes. On the experimental side we have worked with two tubes, both containing filamentary cathodes used as electron sources, collectors so placed as to receive a portion of the direct beam of primary electrons from a filament, and an anode off to one side to maintain the discharge. The two tungsten filaments in the first tube used were supported near the middle of the 18 cm spherical bulb by long glass-covered leads. Their exposed portions were about 1.1 cm long, parallel and about 0.5 cm apart. At a distance of 4.2 cm from them was the collector, a circular disk backed by mica and 1.1 cm in diameter.

The second tube was similar except that it contained three vertical tungsten filaments, g, c, and d. Fig. 1, g above, and c and d about 2.5 cm below it. These two were 0.4 cm apart and all three lay in the same plane. Their diameter was 0.025 cm and their vertical and active portions were 1.1 cm long. Opposite them and about 4 cm away were supported the 1.1 cm disk collectors h and b. The primary electrons are somewhat deflected by the magnetic field of the heating current, and the collectors are inclined as shown in order to give perpendicular incidence of the primaries on them.

An appendix containing a little mercury extended from the bottom of the bulb and was immersed in a water bath, the temperature of which controlled the mercury pressure.
B. Detection of oscillations. The oscillations were detected with a zincite-tellurium detector X and galvanometer arranged, for most of the work, in a circuit as shown in Fig. 2. The detector was supported by a spring suspension to shield it from mechanical shocks which were found to destroy its sensitivity. The high frequency potential was applied across the two points X and Y, Y being grounded to one side of Fig. 2. Crystal detector circuit a filament and also to the metal-screen cage surrounding the apparatus. The inductance L was often only a 10 to 15 cm length of copper wire. The two condensers, which were of 0.0025 pf each shunted the galvanometer and 60-cycle crystal-calibrating circuits for the high-frequency oscillations, but at the same time allowed known 60-cycle voltages to be conveniently impressed on the crystal for calibrating purposes at frequent intervals.
The construction of a Lecher-wire system for measuring wavelength by determining the distance between successive positions of current or voltage maximum is shown in Fig. 10. If a sensitive current-reading instrument is inserted in the shorting bar, series-resonance will be indicated by maximum current, and parallel-resonance by minimum current. When the amount of power supplied by the measured circuit is small, series-resonance (i.e., the shorting bar at half-wave positions) is indicated by maximum reaction upon the circuit, and parallel resonance (i.e., the shorting bar at odd quarter-wave positions) by minimum reaction upon the circuit. To measure frequency, the line is coupled into the circuit and the shorting bar moved to a position of resonance, with fairly loose coupling. The shorting bar is then moved to the next position of similar resonance, and the distance between the two is equal to one half-wavelength, from which frequency is known.

Under good conditions, frequency measurements may be made by the Lecher-wire method to an accuracy of the order of 0.1%. At the higher frequencies the accuracy decreases due to the difficulty of measuring small distances to such a high degree of accuracy. In addition to their greater accuracy, Lecher-wire wavemeters have the further advantages over tuned-circuit wavemeters that they
C. Frequency range of oscillations. Oscillation frequencies as low as $10^6$ and as high as $10^9$ cycles per second have been observed. Over a large portion of this range it was not found possible to obtain any resonance phenomena, but oscillations were detected throughout so that we lean toward the view that these electric vibrations are irregular, thereby constituting an "electrical noise." Under certain conditions definite frequencies were observed both on the collectors and on external electrodes glued to the tube wall or placed against it.

The frequency range can roughly be divided into three parts, namely from 1 to 100 megacycles, from 100 to 300 megacycles, and from 300 to 1000 megacycles.

**Fig. 8.** Volt-ampere characteristics of a collector in the primary electron beam.

**Fig. 10.** Oscillation frequency and ionization density.
I. THEORY

Two types of oscillation seem to be theoretically possible, first oscillations of electrons which are too rapid for the ions to follow, and second, oscillations of the ions which are so slow that the electrons continually satisfy the Boltzmann Law.

A. Plasma-electron oscillations. When the electrons oscillate, the positive ions behave like a rigid jelly with uniform density of positive charge $ne$. Imbedded in this jelly and free to move there is an initially uniform electron distribution of charge density, $-ne$.

B. Plasma-ion oscillations. We are now ready to discuss the slower ionic oscillations. In this case we shall assume the same type of displacement for the ions as we did for the electrons before.
Langmuir's Electrons

HEAVY Hg IONS

\[
\frac{M_i}{m_i} \approx 200 \times 1.836 \Rightarrow 1
\]

GAUSS'S LAW

\[
\nabla \cdot \mathbf{E} = \rho / \varepsilon_0 = -\varepsilon \delta m(x) / \varepsilon_0
\]

\[
\frac{2F_x}{2x} = \frac{\varepsilon m_0}{\varepsilon_0} \frac{2\varepsilon}{2x}
\]

\[
\left\{ E_x = \frac{\varepsilon m_0}{\varepsilon_0} \mathcal{E}(x) \right\}
\]

**But force on electrons...**

\[
\frac{2F_x}{2x} = -\varepsilon E_x = -\frac{\varepsilon^2 m_0}{\varepsilon_0} \mathcal{E}(x)
\]

\[
\mathbf{F}_p = \frac{\varepsilon^2 m_0}{\varepsilon_0 e_0}
\]

ELECTROSTATIC OSCILLATIONS ARE (NEARLY) INDEPENDENT OF WAVELENGTH

\[
E(x) \sim \text{ELECTRON "COLLECTIVE" DISPLACEMENT}
\]

\[
\frac{\partial m(x)}{\partial t} + \nabla \cdot (m \mathbf{v}) = 0
\]

\[
\frac{\partial m}{\partial t} + \nabla \cdot (m \mathbf{v} \mathbf{t}) = 0
\]

\[
\mathbf{v} \cdot \mathbf{t} = 0
\]
Electromagnetic Oscillations

\[ \nabla \times B = \epsilon_0 \mu_0 \frac{2E}{2c} + \mu_0 \vec{J} \]

\[ \nabla \times \left( \nabla \times E \right) = -\epsilon_0 \mu_0 \frac{2E}{2c^2} - \frac{\mu_0}{2c} \vec{J} \]

\[ \vec{J} = -\epsilon_0 \mu_0 \vec{v} \]

\[ \frac{c^2}{2} \frac{2\vec{J}}{2c} = -\frac{1}{c^2} \epsilon_0 \mu_0 \frac{2\vec{J}}{2c} \]

\[ \lambda = \frac{1}{2\pi c} \left( \sqrt{\omega^2 - \omega_0^2} \right) \]

Two cases:

\[ \nabla \times E = 0 \quad \text{ELECTROSTATIC (LARGITUINAL)} \]

\[ \nabla \times E \neq 0 \quad \text{ELECTROMAGNETIC (TRANVERSE)} \]

Electromagnetic:

\[ \nabla \times \left( \nabla \times E \right) = \nabla (\nabla \cdot E) - \nabla^2 E \]

Then

\[ c^2 \nabla^2 E = \frac{2^2 \vec{E}}{2c^2} + \left( \frac{\epsilon_0 \mu_0}{\epsilon_0 \mu_0} \right) \vec{E} \]

\[ \lambda = \frac{1}{2\pi c} \left( \sqrt{\omega^2 - \omega_0^2} \right) \]
\[ \delta m_i = -m_0 \frac{2\hat{e}}{\lambda_0} \quad \text{(ion perturbed density)} \]

\[ \delta m_0 \approx m_0 (1 - e^{-\phi / \lambda_0}) = \frac{e m_0}{\lambda_0} \frac{\hat{E}}{e_0} \]

\[ \nabla \cdot \mathbf{E} = -\nabla^2 \Phi = \frac{\rho}{\varepsilon_0} = \frac{e m_0}{\varepsilon_0} \frac{2\hat{e}}{2\lambda} + \frac{e m_0}{\lambda^2} \Phi \]

\[ \text{Dut} \quad m_p = \frac{\partial^2 \hat{e}}{2 \varepsilon_0} = e E_x = -e \frac{2\hat{E}}{2\lambda} \]

\[ \text{So} \quad \frac{1}{m_p} \frac{2\hat{e}}{\lambda^2} = -e \frac{2\hat{E}}{2\lambda} \]

\[ \frac{2^2 \lambda^2}{2^2 \lambda^2} = -e \frac{2\hat{E}}{2\lambda} - \frac{e m_0}{\lambda^2} \frac{\hat{\Phi}}{e_0} \]

Assume plane waves: \[ (\hat{e}, \hat{\Phi}) \sim e \]

Then \[ \frac{\partial^2}{\partial t^2} \hat{e} = -\omega^2 \hat{e}, \quad \frac{\partial^2}{\partial x^2} \hat{\Phi} = -\frac{k^2}{\lambda^2} \hat{\Phi} + i \kappa \hat{\kappa} \]

\[ -i \omega^2 \hat{e} = -\frac{\omega^2}{\lambda^2} \hat{\kappa} \]

\[ \hat{\kappa} = \frac{\omega^2}{\lambda^2} \hat{\kappa} \]

\[ \omega = \frac{k \omega_c \lambda_0}{\sqrt{1 + k^2 \lambda^2}} \]
Problems

2.1 Prove that the electron Debye length can be written as

\[ \lambda_{\text{De}} = 69 \text{ m} \left[ \frac{T (\text{K})}{n_e (\text{m}^{-3})} \right]^{1/2} \]

2.2 Calculate the electron and ion Debye length
   (a) for the ionospheric plasma \( T_e = T_i = 3000 \text{ K}, n = 10^{12} \text{ m}^{-3} \).
   (b) for a neon gas discharge \( T_e = 3 \text{ eV}, T_i = 300 \text{ K}, n = 10^{16} \text{ m}^{-3} \).

2.3 Consider an infinitely large homogeneous plasma with \( n_e = n_i = 10^{16} \text{ m}^{-3} \). From this plasma, all electrons are removed from a slab of thickness \( d = 0.01 \text{ m} \) extending from \( x = -d \) to \( x = 0 \) and redeposited in the neighboring slab from \( x = 0 \) to \( x = d \). (a) Calculate the electric potential in this double slab using Poisson’s equation. What are the boundary conditions at \( x = \pm d \)? (b) Draw a sketch of space charge, electric field and potential for this situation. What is the potential difference between \( x = -d \) and \( x = d \)?

2.4 Show that the equation for the shielding contribution (2.24) results from (2.21) and (2.23).

2.5 Derive the relationship between the coupling parameter for ion-ion interaction \( \Gamma \) Eqs. (2.15) and \( N_D \) (2.33) under the assumption that \( T_e = T_i \).

2.6 Show that the second Lagrange multiplier in Eq. (2.6) is \( \lambda = (k_B T)^{-1} \).
   Hint: Start from

\[ \frac{1}{T} = \left\langle \frac{\partial S}{\partial \lambda} \frac{\partial \lambda}{\partial U} \right\rangle \]

and use \( \sum n_i = 1 \).
Next Week: Chapter 3 
Mechanics of Charged Particles

- Charged particle motion in inhomogeneous, static and slowly-varying electric and magnetic fields

\[ m \ddot{v} = q(E + v \times B) , \]  
(3.1)