Lecture19: Kinetic Theory

Plasma Physics 1

APPH E6101x

Columbia University

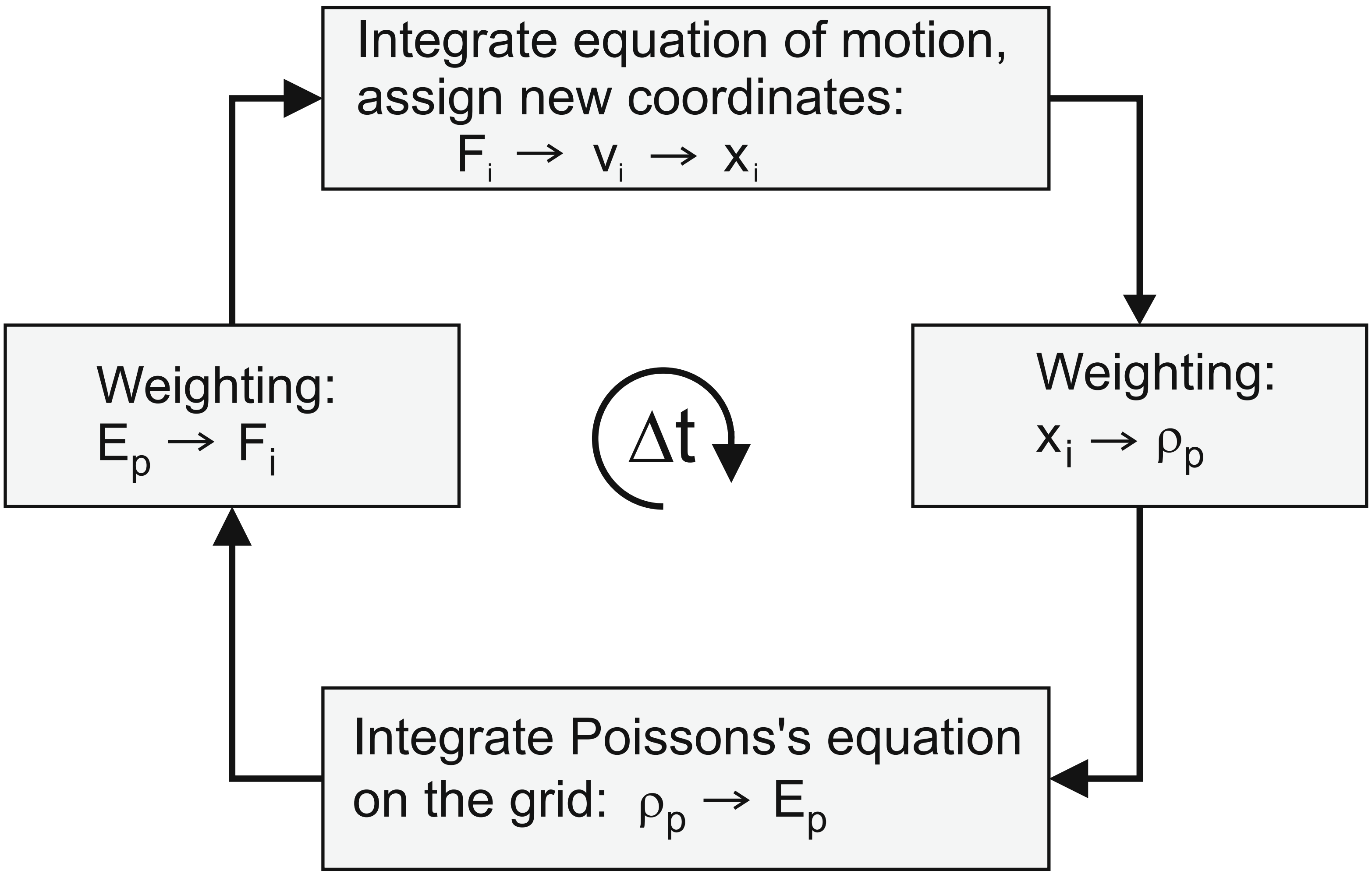
Past Lectures

- PIC simulation of kinetic instabilities

  Chapter 9, Section 9.4

https://plasmasim.physics.ucla.edu/codes

Calculate and Repeat



Dynamics (Leap-Frog)

$$x_{n+1} = x_n + v_{n+1}$$
$$v_{n+1} = v_n - E_n$$

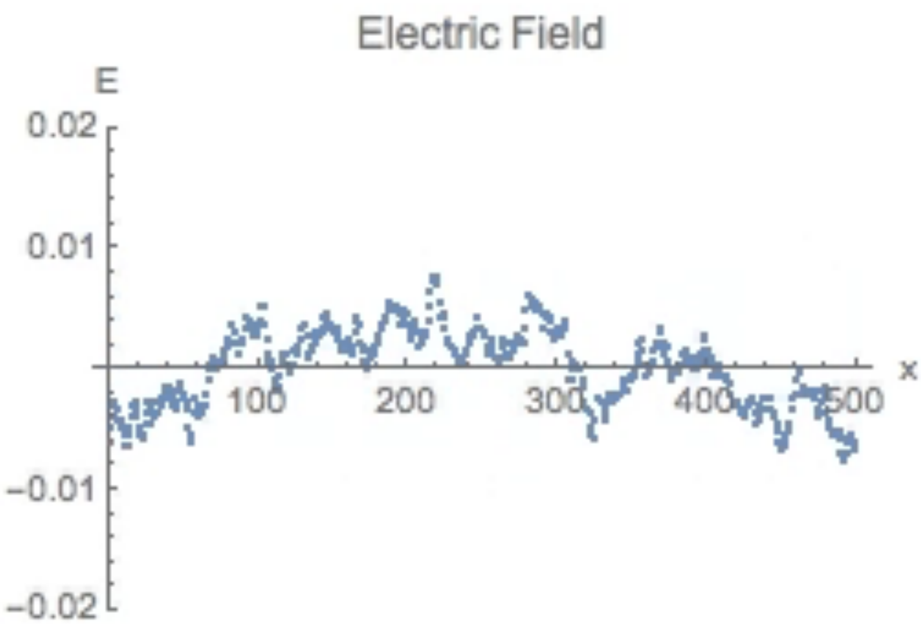
$$x_{n+1} = x_n + v_{n+1}$$
$$v_{n+1} = v_n + E_n/M_i$$

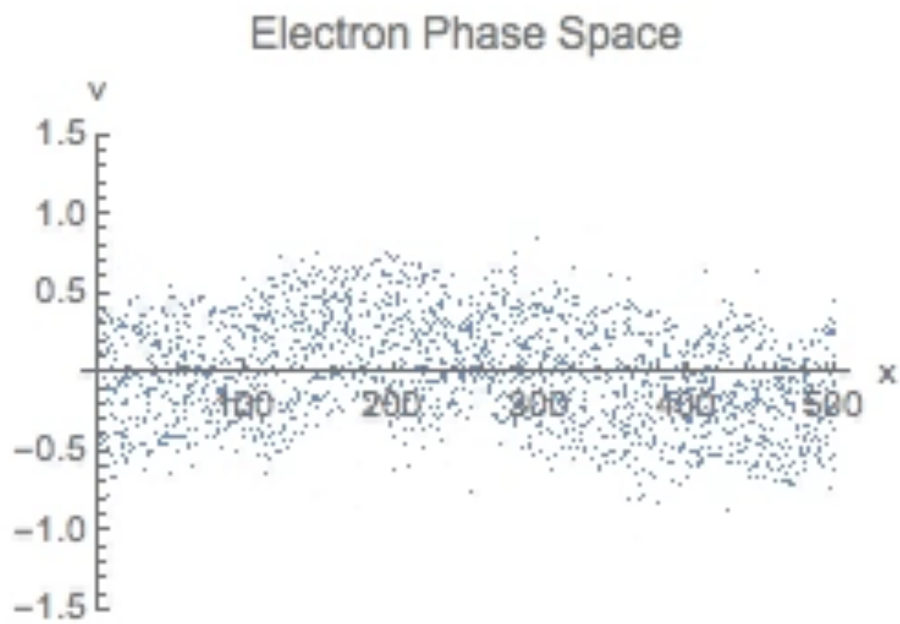
Poisson's Eq

$$\frac{d^2\Phi}{dx^2} = -\frac{q}{\epsilon_0} \left(\rho_i(x) - \rho_e(x)\right)$$

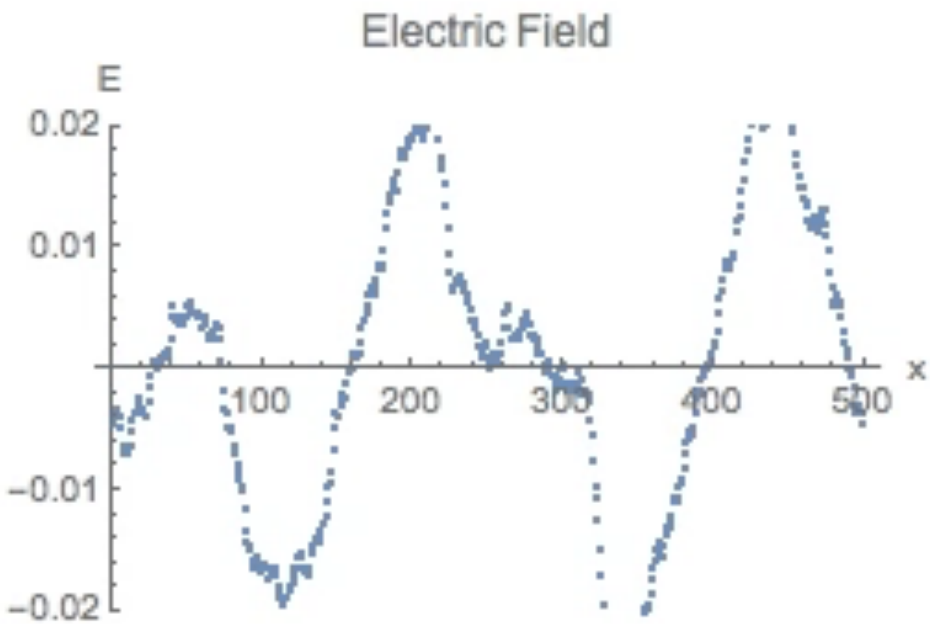
$$\Phi_k = \frac{1}{k^2}\frac{q}{\epsilon_0} FFT[\rho_i(x) - \rho_e(x)]$$

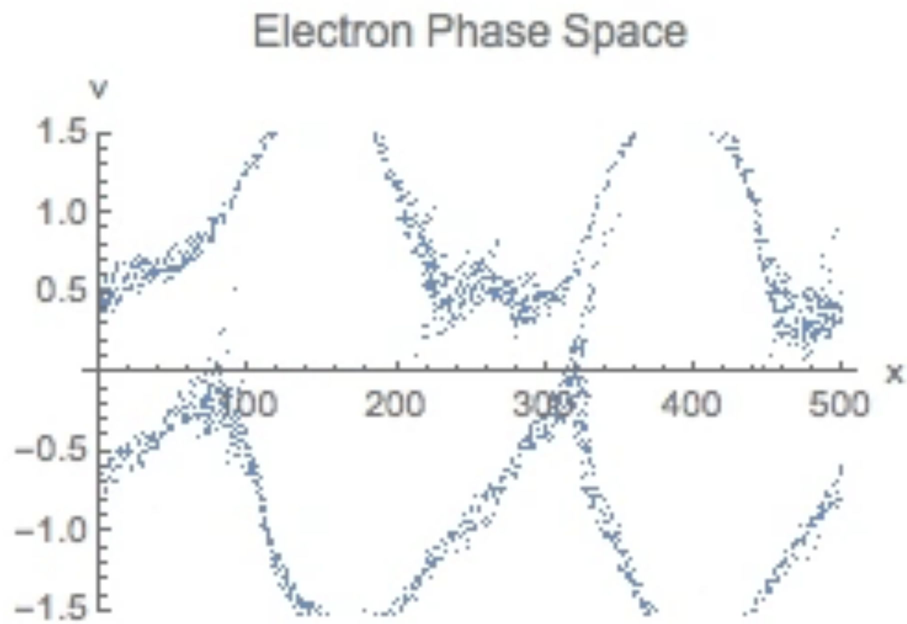
Simple Example





Two-Stream Instability





This Week

- Ch. 9: Kinetic Theory
- Vlasov's Equation
- Landau Damping

Nonlinear Plasma Dynamics

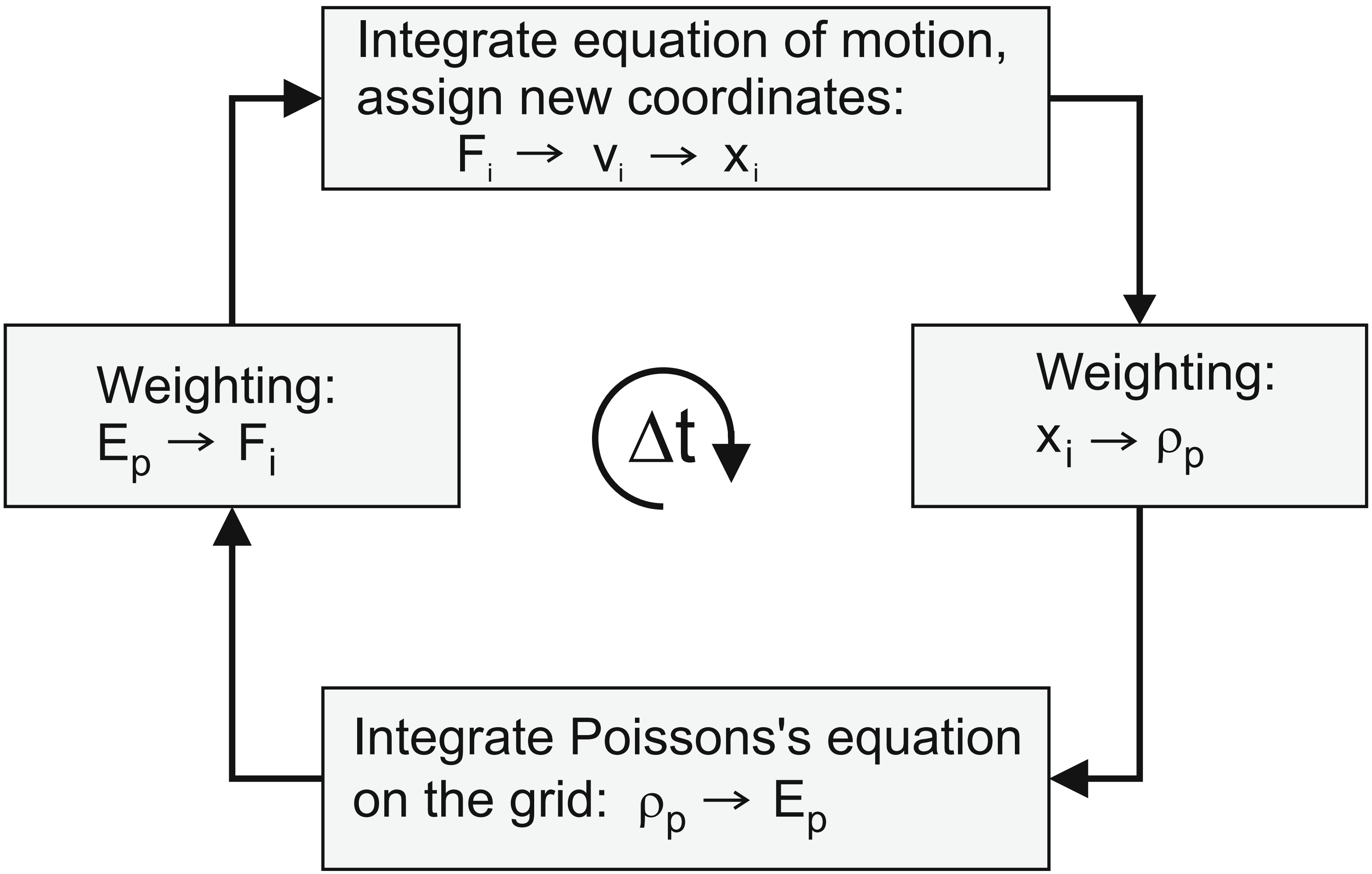
$$f^{(\alpha)}(\mathbf{r},\mathbf{v},t) = \sum_k \delta(\mathbf{r}-\mathbf{r}_k(t))\delta(\mathbf{v}-\mathbf{v}_k(t))$$

$$\nabla^2\Phi(\mathbf{r},t) = -\sum_{k,\alpha} q_\alpha \delta(\mathbf{r}-\mathbf{r}_k(t))$$

$$\frac{d\mathbf{v}_k}{dt} = -\frac{q_\alpha}{m_\alpha}\nabla\Phi(\mathbf{x}_k,t)$$

$$\frac{d\mathbf{x}_k}{dt} = \mathbf{v}_k$$

Nonlinear Plasma Dynamics



Vlasov EM Dynamics

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla_r f + \mathbf{a} \cdot \nabla_v f = 0$$

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla_r f + \frac{q}{m}(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \nabla_v f = 0$$

$$\rho_\alpha = \iiint d^3v \, f_\alpha(\mathbf{x}, \mathbf{v}, t)$$

$$\mathbf{J}_\alpha = \iiint d^3v \, \mathbf{v} \, f_\alpha(\mathbf{x}, \mathbf{v}, t)$$

Vlasov Poisson Dynamics

$$\frac{\partial f_\alpha}{\partial t} + v\frac{\partial f_\alpha}{\partial x} - \frac{q_\alpha}{m_\alpha}\frac{\partial \Phi}{\partial x}\frac{\partial f_\alpha}{\partial v} = 0$$

$$\frac{\partial^2 \Phi}{\partial x^2} = -\sum_\alpha q_\alpha \int dv\, f_\alpha(x, v, t)$$

Properties of Vlasov Equation

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla_r f + \mathbf{a} \cdot \nabla_v f = 0$$

1. The Vlasov equation conserves the total number of particles N of a species, which can be proven, for the one-dimensional case, as follows:

$$\frac{\partial N}{\partial t} = \frac{\partial}{\partial t} \iint f \, \mathrm{d}x \mathrm{d}v = -\iint v \frac{\partial f}{\partial x} \, \mathrm{d}x \mathrm{d}v - \iint a \frac{\partial f}{\partial v} \, \mathrm{d}x \mathrm{d}v$$

Properties of Vlasov Equation

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla_r f + \mathbf{a} \cdot \nabla_v f = 0$$

2. Any function, $g[\frac{1}{2}mv^2 + q\Phi(x)]$, which can be written in terms of the total energy of the particle, is a solution of the Vlasov equation (cf. Problem 9.1).

Properties of Vlasov Equation

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla_r f + \mathbf{a} \cdot \nabla_v f = 0$$

3. The Vlasov equation has the property that the phase-space density f is constant along the trajectory of a test particle that moves in the electromagnetic fields $\mathbf{E}$ and $\mathbf{B}$. Let $[\mathbf{x}(t), \mathbf{v}(t)]$ be the trajectory that follows from the equation of motion $m\dot{\mathbf{v}} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$ and $\dot{\mathbf{x}} = \mathbf{v}$, then

$$\begin{aligned} \frac{\mathrm{d} f(\mathbf{x}(t), \mathbf{v}(t), t)}{\mathrm{d}t} &= \frac{\partial f}{\partial t} + \frac{\partial f}{\partial \mathbf{x}} \cdot \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} + \frac{\partial f}{\partial \mathbf{v}} \cdot \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} \\ &= \frac{\partial f}{\partial t} + \frac{\partial f}{\partial \mathbf{x}} \cdot \mathbf{v} + \frac{\partial f}{\partial \mathbf{v}} \cdot \frac{q}{m}(\mathbf{E} + \mathbf{v} \times \mathbf{B}) = 0 \, . \end{aligned} \tag{9.15}$$

Properties of Vlasov Equation

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla_r f + \mathbf{a} \cdot \nabla_v f = 0$$

4. The Vlasov equation is invariant under time reversal, $(t \rightarrow -t)$, $(\mathbf{v} \rightarrow -\mathbf{v})$. This means that there is no change in entropy for a Vlasov system.

Relationship between Vlasov Eq and Fluid Eqs

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla_r f + \mathbf{a} \cdot \nabla_v f = 0$$

$$0 = \frac{\partial}{\partial t} \int f \, \mathrm{d}v + \frac{\partial}{\partial x} \int v \, f \, \mathrm{d}v + a \big[f \big]_{-\infty}^{\infty} = \frac{\partial n}{\partial t} + \frac{\partial}{\partial x}(nu)$$

Relationship between Vlasov Eq and Fluid Eqs

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla_r f + \mathbf{a} \cdot \nabla_v f = 0$$

$$\times\, m$$

$$0 = \frac{\partial}{\partial t} \int m v f \, \mathrm{d}v + \frac{\partial}{\partial x} \int v^2 f \, \mathrm{d}v + a \int v \frac{\partial f}{\partial v} \, \mathrm{d}v \quad \times\, m$$

$$= \frac{\partial}{\partial t} \int m v f \, \mathrm{d}v + \frac{\partial}{\partial x} \left[\int m(v-u)^2 f \, \mathrm{d}v + n m u^2 \right]$$

$$+ a \left(\left[v f \right]_{-\infty}^{\infty} - \int f \, \frac{\mathrm{d}v}{\mathrm{d}v} \, \mathrm{d}v \right) \quad \times\, m$$

$$= \frac{\partial}{\partial t}(nmu) + \frac{\partial p}{\partial x} + u \frac{\partial}{\partial t}(nmu) + (nmu) \frac{\partial u}{\partial x} - nma$$

$$= nm \left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} \right) + \frac{\partial p}{\partial x} - nma \, ,$$

$$p = \int m(v-u)^2 f \, \mathrm{d}v$$

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Exact Nonlinear Plasma Oscillations

IRA B. BERNSTEIN, JOHN M. GREENE, AND MARTIN D. KRUSKAL
Project Matterhorn, Princeton University, Princeton, New Jersey
(Received July 1, 1957)

The problem of a one-dimensional stationary nonlinear electrostatic wave in a plasma free from interparticle collisions is solved exactly by elementary means. It is demonstrated that, by adding appropriate numbers of particles trapped in the potential-energy troughs, essentially arbitrary traveling wave solutions can be constructed.

When one passes to the limit of small-amplitude waves it turns out that the distribution function does not possess an expansion whose first term is linear in the amplitude, as is conventionally assumed. This disparity is associated with the trapped particles. It is possible, however, to salvage the usual linearized theory by admitting singular distribution functions. These, of course, do not exhibit Landau damping, which is associated with the restriction to well-behaved distribution functions.

The possible existence of such waves in an actual plasma will depend on factors ignored in this paper, as in most previous works, namely interparticle collisions, and the stability of the solutions against various types of perturbations.

Steady Solution to Vlasov-Poisson Eqs

$$v\frac{\partial f(x,v)}{\partial x} + \frac{e}{m_\mathrm{e}}\frac{\partial \Phi}{\partial x}\frac{\partial f(x,v)}{\partial v} = 0$$

$$\frac{\partial^2 \Phi}{\partial x^2} = \frac{e}{\varepsilon_0}\int_{-\infty}^{\infty} f(x,v)\,\mathrm{d}v\,.$$

Steady Solution to Vlasov-Poisson Eqs

$$v \frac{\partial f(x,v)}{\partial x} + \frac{e}{m_{\mathrm{e}}} \frac{\partial \Phi}{\partial x} \frac{\partial f(x,v)}{\partial v} = 0$$

The phase space trajectories of test particles form the *characteristic curves* of the Vlasov equation and result from integrating the equation of motion for

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = v \quad \text{and} \quad \frac{\mathrm{d}v}{\mathrm{d}\tau} = \frac{e}{m} \frac{\mathrm{d}\Phi}{\mathrm{d}x} \,. \tag{9.20}$$

Here we have introduced the *transit time* τ, which must be distinguished from the absolute time. The considered problem of a stationary flow is independent of absolute time.

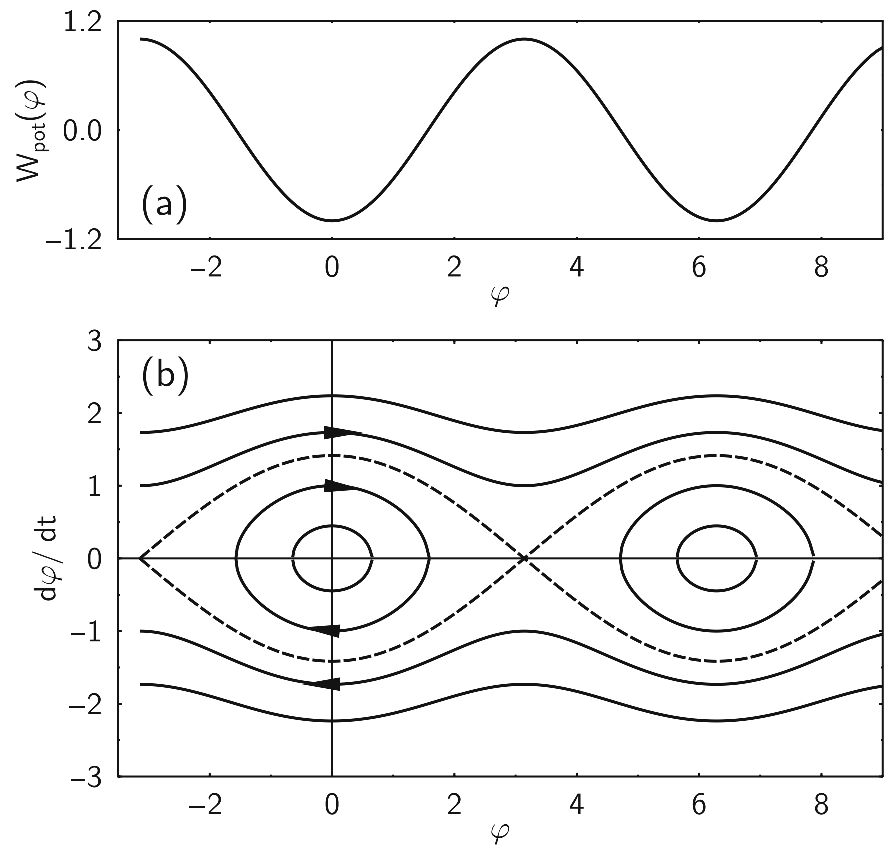
$$\frac{\partial^2 \Phi}{\partial x^2} = \frac{e}{\varepsilon_0} \int_{-\infty}^{\infty} f(x,v)\, \mathrm{d}v \,.$$

"Resonant Particles"

We will use this phase space picture to study the motion of nearly resonant electrons in a wave field. The resonance condition $v \approx v_\varphi$ ensures that the electron "sees" a nearly constant potential well of the wave. Therefore, in a first approximation, its motion is described by energy conservation in the moving frame of reference:

$$W_{\mathrm{tot}} = \frac{1}{2} m_{\mathrm{e}} (v - v_\varphi)^2 + e\hat{\Phi} \cos(kx) = \mathrm{const}\,. \tag{9.86}$$

Therefore, we can expect free electron streaming w.r.t. the wave when $W_{\mathrm{tot}} > 2e\hat{\Phi}$. This defines the trapping potential $\Phi_{\mathrm{t}} = m(v - v_\varphi)^2/(4e)$. Electrons with an energy less than this critical value are trapped by the wave and perform bouncing oscillatiuons in the wave potential.



Small-Amplitude Plasma Wave

$$f_{\mathrm{e}}(x, v, t) = f_{\mathrm{e}0}(v) + f_{\mathrm{e}1}(x, v, t)$$

$$f_{\mathrm{e}0}(v) = n_{\mathrm{e}0}\left(\frac{m_{\mathrm{e}}}{2\pi k_{\mathrm{B}} T_{\mathrm{e}}}\right)^{1/2} \exp\left\{-\frac{m_{\mathrm{e}} v^2}{2k_{\mathrm{B}} T_{\mathrm{e}}}\right\}$$

$$f_{\mathrm{e}1} = \hat{f}_{\mathrm{e}1} \exp[\mathrm{i}(kx - \omega t)]\,.$$

Linearizing the Vlasov equation, and using the wave representation (9.36), we obtain

$$\frac{\partial f_{\mathrm{e}1}}{\partial t} + v\frac{\partial f_{\mathrm{e}1}}{\partial x} - \frac{e}{m_{\mathrm{e}}} E_1 \frac{\partial f_{\mathrm{e}0}}{\partial v} = 0 \tag{9.38}$$

$$-\mathrm{i}\omega \hat{f}_{\mathrm{e}1} + \mathrm{i}kv\hat{f}_{\mathrm{e}1} - \frac{e}{m_{\mathrm{e}}} \hat{E}_1 \frac{\partial f_{\mathrm{e}0}}{\partial v} = 0\,, \tag{9.39}$$

Small-Amplitude Plasma Wave

$$f_{\mathrm{e}}(x, v, t) = f_{\mathrm{e}0}(v) + f_{\mathrm{e}1}(x, v, t)$$

$$f_{\mathrm{e}0}(v) = n_{\mathrm{e}0}\left(\frac{m_{\mathrm{e}}}{2\pi k_{\mathrm{B}} T_{\mathrm{e}}}\right)^{1/2} \exp\left\{-\frac{m_{\mathrm{e}} v^2}{2 k_{\mathrm{B}} T_{\mathrm{e}}}\right\}$$

$$f_{\mathrm{e}1} = \hat{f}_{\mathrm{e}1} \exp[\mathrm{i}(kx - \omega t)] .$$

$$\boxed{\hat{f}_{\mathrm{e}1} = \mathrm{i}\frac{e}{m_{\mathrm{e}}} \frac{\partial f_{\mathrm{e}0}/\partial v}{\omega - kv} \hat{E}_1 .} \tag{9.40}$$

The vanishing of the denominator $(\omega - kv)$ causes a singularity in the perturbed distribution function, which we will have to address carefully. The electrons with $v \approx \omega/k$ will be called *resonant particles*. In Sect. 8.1.2 we had already seen the particular role of resonant particles for beam-plasma interaction.

Non-resonant (most) Particles

The perturbed electron distribution function represents a space charge

$$\rho = e\left(n_\mathrm{i} - \int_{-\infty}^{\infty} f_\mathrm{e}\,\mathrm{d}v\right) = -e\int_{-\infty}^{+\infty} f_\mathrm{e1}\,\mathrm{d}v\,, \tag{9.41}$$

$$ik\hat{E}_1 = \frac{\rho}{\varepsilon_0} = \frac{1}{\mathrm{i}k}\hat{E}_1\frac{\omega_\mathrm{pe}^2}{n_\mathrm{e0}}\int_{-\infty}^{+\infty}\frac{\partial f_\mathrm{e0}/\partial v}{\omega/k - v}\,\mathrm{d}v\,. \tag{9.42}$$

When the mean thermal speed of the electrons is sufficiently small compared to the phase velocity of the wave (see Fig. 9.7), the contribution from resonant particles in (9.43) is attenuated by the exponentially small factor in the numerator. Then, the main contributions to the integral in (9.43) originate from the interval $[-v_\mathrm{Te}, v_\mathrm{Te}]$, where we can expand the function $(\omega/k - v)^{-1}$ into a Taylor series

$$\frac{1}{\omega/k - v} = \frac{k}{\omega} + \frac{k^2}{\omega^2}v + \frac{k^3}{\omega^3}v^2 + \frac{k^4}{\omega^4}v^3 + \cdots\,. \tag{9.45}$$

Non-resonant (most) Particles

The perturbed electron distribution function represents a space charge

$$\rho = e\left(n_{\mathrm{i}} - \int_{-\infty}^{\infty} f_{\mathrm{e}}\,\mathrm{d}v\right) = -e\int_{-\infty}^{+\infty} f_{\mathrm{e1}}\,\mathrm{d}v\,, \tag{9.41}$$

$$\boxed{\frac{\partial f_{\mathrm{e0}}}{\partial v} = -n_{\mathrm{e0}}\,\frac{2v}{\sqrt{\pi}\,v_{\mathrm{Te}}^3}\,\exp\left(-\frac{v^2}{v_{\mathrm{Te}}^2}\right)}$$

$$ik\hat{E}_1 = \frac{\rho}{\varepsilon_0} = \frac{1}{\mathrm{i}k}\hat{E}_1\frac{\omega_{\mathrm{pe}}^2}{n_{\mathrm{e0}}}\int_{-\infty}^{+\infty}\frac{\partial f_{\mathrm{e0}}/\partial v}{\omega/k - v}\,\mathrm{d}v\,. \tag{9.42}$$

The integral (9.43) can be solved analytically using the relations

$$\int_{-\infty}^{+\infty} x^{2n}\mathrm{e}^{-ax^2} = \frac{1\times 3\times\cdots\times(2n-1)}{(2a)^n}\left(\frac{\pi}{a}\right)^{1/2} \tag{9.46}$$

$$\int_{-\infty}^{+\infty} x^{2n+1}\mathrm{e}^{-ax^2} = 0\,. \tag{9.47}$$

Non-resonant (most) Particles

Using terms up to fourth order in the phase velocity, we obtain

$$\varepsilon(\omega, k) = 1 - \frac{\omega_{\mathrm{pe}}^2}{\omega^2} - \frac{3}{2}\frac{\omega_{\mathrm{pe}}^2}{\omega^4} k^2 v_{\mathrm{Te}}^2 = 0\,. \tag{9.48}$$

$$\frac{\omega_{\mathrm{pe}}^2}{k^2}\; P\int_{-\infty}^{+\infty} \frac{\partial f_{\mathrm{e}0}/\partial v}{\omega/k - v}\,\mathrm{d}v \approx \frac{\omega_{\mathrm{pe}}^2}{\omega^2} + \frac{3}{2}\frac{\omega_{\mathrm{pe}}^2}{\omega^4} k^2 v_{\mathrm{Te}}^2 + \cdots \tag{9.50}$$

The perturbed electron distribution function represents a space charge

$$\rho = e\left(n_{\mathrm{i}} - \int_{-\infty}^{\infty} f_{\mathrm{e}}\,\mathrm{d}v\right) = -e\int_{-\infty}^{+\infty} f_{\mathrm{e}1}\,\mathrm{d}v\,, \tag{9.41}$$

$$\hat{f}_{\mathrm{e}1} = \mathrm{i}\frac{e}{m_{\mathrm{e}}}\frac{\partial f_{\mathrm{e}0}/\partial v}{\omega - kv}\,\hat{E}_1$$

Next Lecture

- Ch. 9: Kinetic Theory
- Landau Damping

Linear Dispersion Relation in an infinite homogenous plasma

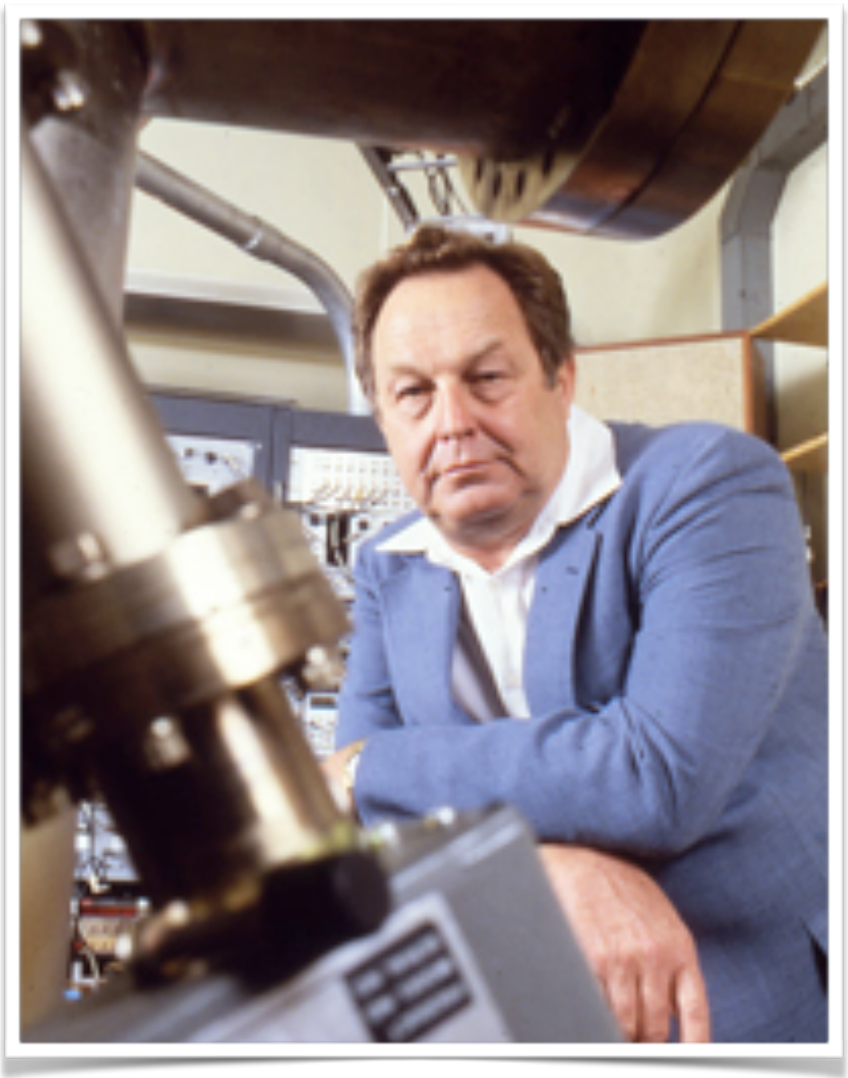
$$D(\mathbf{k}, \omega) = 1 - \sum_{\alpha} \frac{{\omega_{p\alpha}}^2}{k^2} \int_L \frac{\partial F_{\alpha 0}/\partial u}{u - \omega/|k|}\, du = 0$$

then gives for the dispersion equation for weakly damped electrostatic waves in a field-free plasma

$$1 - \sum_{\alpha} \frac{{\omega_{p\alpha}}^2}{k^2} \left(1 + i\omega_i \frac{\partial}{\partial \omega_r}\right) \left[\oint \frac{\partial F_{\alpha 0}(u)/\partial u}{u - \omega_r/|k|}\, du + \pi i \left[\frac{\partial F_{\alpha 0}(u)}{\partial u}\right]_{u=\omega_r/|k|} \right] = 0 \qquad (8.5.9)$$

John Malmberg and Chuck Wharton

The first experimental measurement of Landau Damping



John Malmberg

(obit, Nov 1992)

Prof. Malmberg joined UCSD from General Atomics in 1969 as a professor of physics. Much of his work revolved around theoretical and experimental investigations of fully ionized gases or plasmas. The field could offer insights into how stars work and how to ignite and control thermonuclear reactions to produce fusion energy--the power that drives the sun.

A plasma is the fourth state of matter, with solids, liquids and gases making up the other three. Most of the matter in the Universe is in the plasma state; for example, the matter of stars is composed of plasmas.

In recent years, Prof. Malmberg had been experimenting with pure electron plasmas that were trapped in a magnetic bottle. By contrast with electrically neutral plasmas that contain an equal number of positive and negative electrons, pure electron plasmas are rare in nature.

Before joining UCSD, Prof. Malmberg was director of the Plasma Turbulence group at General Atomics, where he carried out some of the first and most important experiments to test the basic principals of plasma physics. **Perhaps his most important experiment involved the confirmation of the phenomenon called "Landau damping," where electrons surf on a plasma wave, stealing energy from the wave and causing it to damp (decrease in amplitude).**

For his pioneering work in testing the basic principals of plasma, and for his more recent work with electron plasmas, Prof. Malmberg was named the recipient of the American Physical Society's James Clerk Maxwell Prize in Plasma Physics in 1985.

Chuck Wharton

(emeritus, Cornell)

Professor Wharton was a staff member at the University of California Lawrence Radiation Laboratory at Livermore and Berkeley, California from 1950-1962. While in this position, he spent a year (1959-60) as engineer-scientist at the Max-Planck Institute for Physics in Munich, Germany, and also as a lecturer at the International Summer Course in Plasma Physics at Riso, Denmark. For the next five years he was a staff member of the Experimental Physics Group at General Atomics in San Diego, California. He joined the EE faculty as a full professor in 1967.

In 1973 he received the Humboldt Prize awarded by the Alexander von Humboldt Foundation. He was elected a fellow of the American Physical Society in 1973. In 1976 he was elected a fellow of the IEEE "in recognition of contributions to the understanding of plasmas and to the development of plasma diagnostic techniques." In 1979 he was given the award, Socio Onorario, by the International School of Plasma Physics (Milan, Italy).

Charles (Chuck) taught undergraduate courses in electromagnetic theory, plasma physics, and electrical sciences laboratory. His research was primarily in the area of plasma-physics diagnostics, in which he is a recognized world authority, and in plasma interactions and heating with waves and beams with applications to controlled thermonuclear fusion.

COLLISIONLESS DAMPING OF ELECTROSTATIC PLASMA WAVES*

J. H. Malmberg and C. B. Wharton
John Jay Hopkins Laboratory for Pure and Applied Science,
General Atomic Division of General Dynamics Corporation, San Diego, California
(Received 6 July 1964)

DISPERSION OF ELECTRON PLASMA WAVES*

J. H. Malmberg and C. B. Wharton
John Jay Hopkins Laboratory for Pure and Applied Science,
General Atomic Division of General Dynamics Corporation, San Diego, California
(Received 31 May 1966)

Raw Data

Two probes, each consisting of a 0.2-mm diameter radial tungsten wire, are placed in the plasma. One probe is connected by coaxial cable to a chopped signal generator. The other probe is connected to a receiver which includes a sharp high-frequency filter, a string of broad-band amplifiers, an rf detector, a video amplifier, and a coherent detector operated at the transmitter chopping frequency. Provision is made to add a reference signal from the transmitter to the receiver rf signal, i.e., we may use the system as an interferometer. The transmitter is set at a series of fixed frequencies, and at each, the receiving probe is moved longitudinally. The position of the receiving probe, which is transduced, is applied to the x axis of an x-y recorder, and the interferometer output or the logarithm of the received power is applied to the y axis.

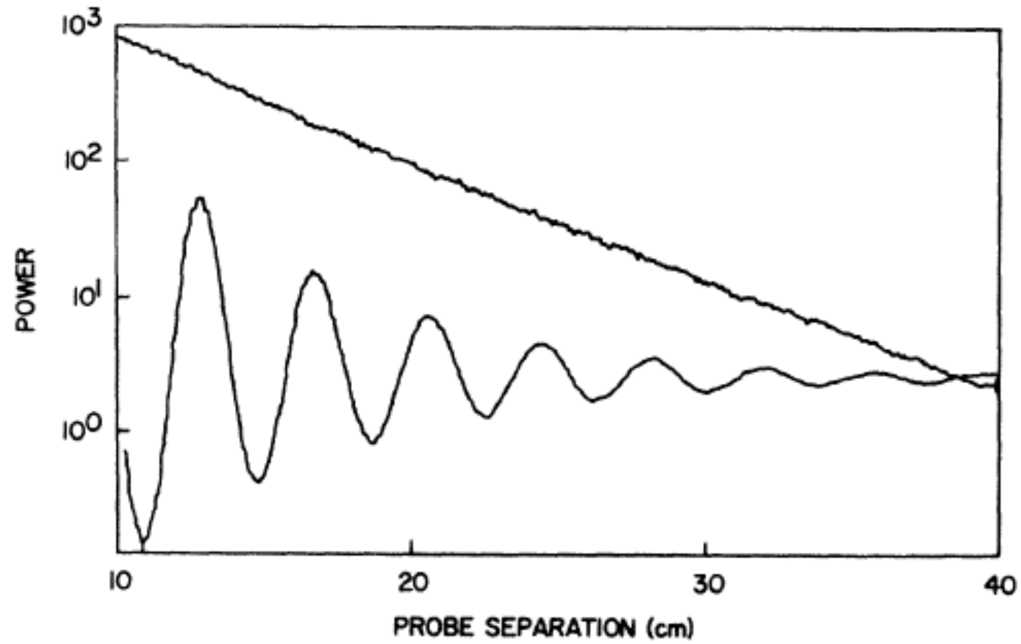


FIG. 1. Raw data. Upper curve is the logarithm of received power. Lower curve is interferometer output. Abscissa is probe separation.

Landau Damping: The Measurement

Important key observation...

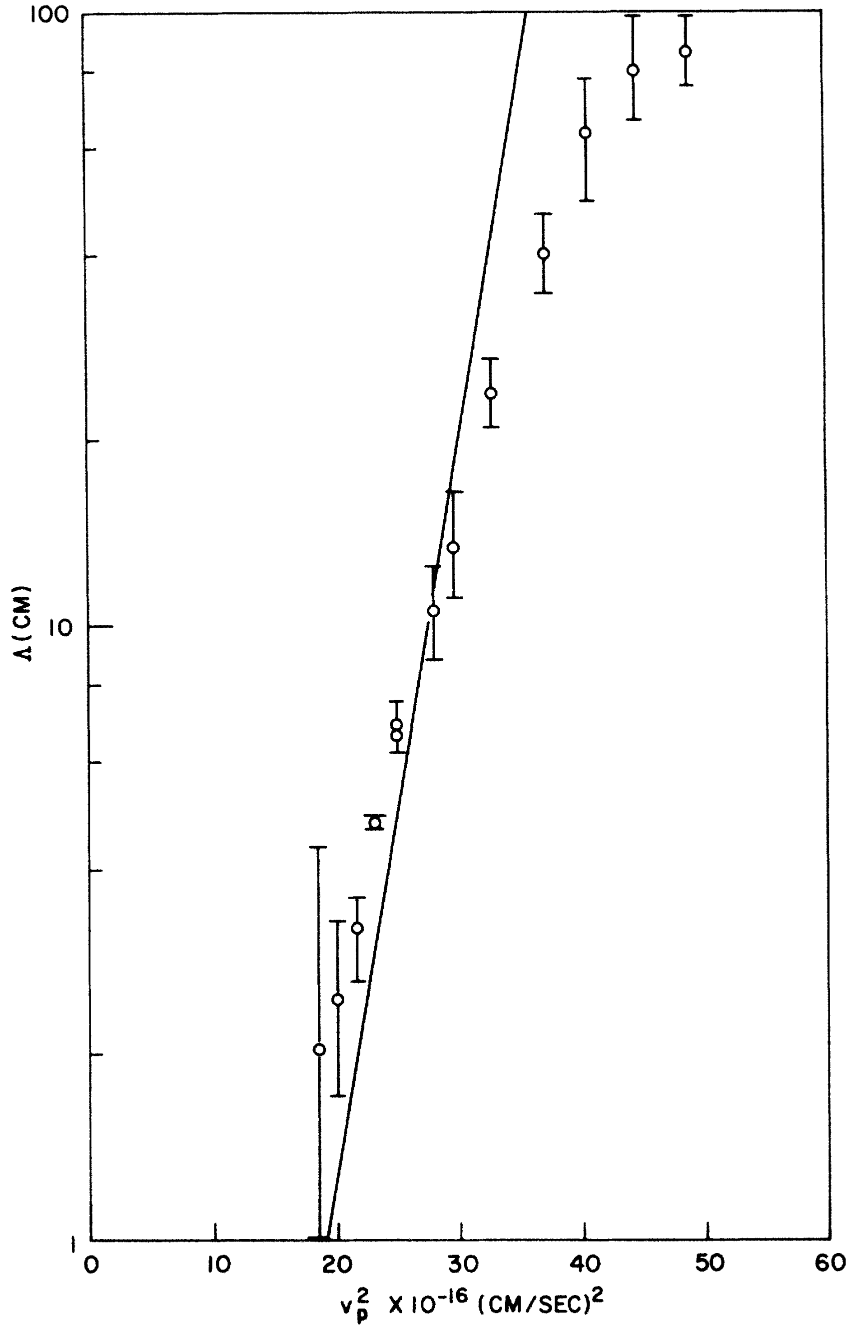


FIG. 3. Logarithm of damping length vs phase velocity squared. The solid curve is theory of Landau for a Maxwellian distribution with a temperature of 10.5 eV.