Lecture18:
Plasma Physics 1

APPH E6101x
Columbia University

Last lecture: Reduced MHD and Kink Modes
This lecture: Numerics, Perturbed Energy, Tearing Modes

Ch. 9 Fitzpatrick: Magnetic Reconnection

Magnetic Reynolds Number
Lundquist Number

Reduced MHD: Incompressible

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla\times(\mathbf{V}\times\mathbf{B}) + \frac{\eta}{\mu_0}\nabla^2\mathbf{B}.$$

$$\nabla\cdot\mathbf{V} = 0, \qquad \mathbf{V} = \nabla\phi\times\mathbf{e}_z,$$

$$\rho\left[\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V}\cdot\nabla)\mathbf{V}\right] + \nabla p - \mathbf{j}\times\mathbf{B} = 0, \qquad \mathbf{B} = \nabla\psi\times\mathbf{e}_z,$$

$$S = \frac{\mu_0\, V\, L}{\eta} \simeq \frac{|\nabla\times(\mathbf{V}\times\mathbf{B})|}{|(\eta/\mu_0)\,\nabla^2\mathbf{B}|}, \quad \rightarrow \infty \text{ (“Ideal”)}$$

$$\mathbf{E} + \mathbf{V}\times\mathbf{B} = \eta\,\mathbf{j}. \quad \approx 0 \text{ (“Ideal”)}$$

$$S = \frac{\tau_R}{\tau_H}. \qquad \tau_H = \frac{k^{-1}}{(B_0^2/\mu_0\,\rho)^{1/2}} \qquad \tau_R = \frac{\mu_0\, a^2}{\eta}$$

Summary of Reduced MHD

$$\bar{B}_\perp(r,\theta,z,t) = \hat{z}\times\nabla\psi \qquad \bar{V}_\perp(r,\theta,z,t) = \hat{z}\times\nabla\chi$$

$$\nabla\cdot\bar{B} = 0 \qquad \nabla\cdot\bar{V} = 0$$

$$\frac{d}{dt}\psi = (\bar{B}\cdot\bar{\nabla})\chi$$

"POLOIDAL FLUX EVOLVES DYNAMICALLY DUE TO FIELD-ALIGNED CHANGES IN THE STREAM FUNCTION"

$$\rho\frac{d}{dt}\Omega_z = (\bar{B}\cdot\bar{\nabla})J_z$$

"AXIAL VORTICITY CHANGES ACCORDING TO FIELD-ALIGNED VARIATION OF AXIAL CURRENT"

$\mu_0 J_z = \nabla^2_\perp\psi$

Linearized Reduced ("*Ideal*") MHD

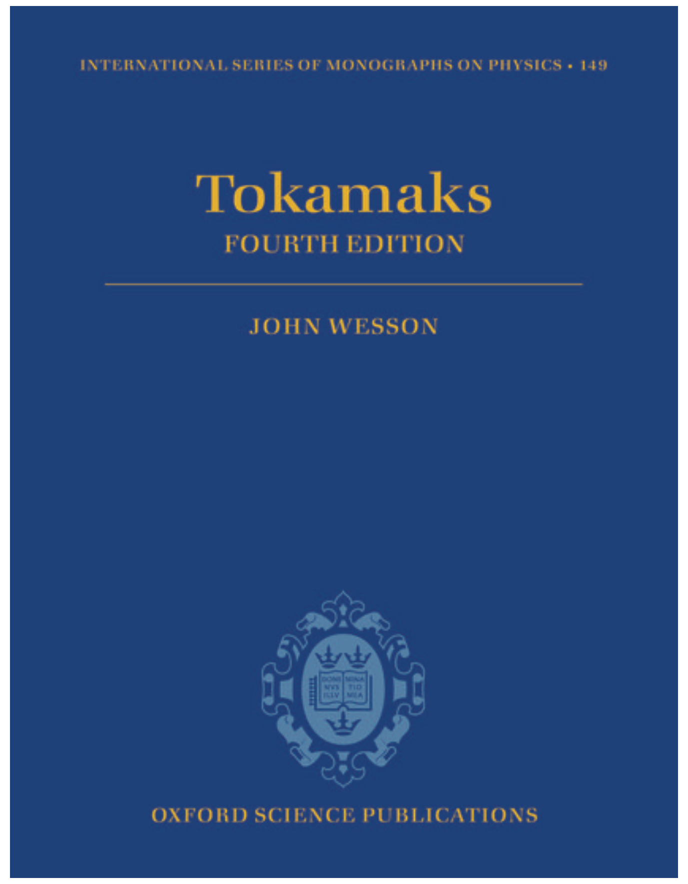
$$
\begin{aligned}
-\rho\omega\nabla_\perp^2\chi &= -\frac{m}{r}\boxed{\frac{\partial J_{z,0}}{\partial r}}\tilde{\psi} + \frac{B_p}{\mu_0 r}(m-nq)\nabla_\perp^2\tilde{\psi} \\
-\omega\tilde{\psi} &= \frac{B_p}{r}(m-nq)\chi\,.
\end{aligned}
$$

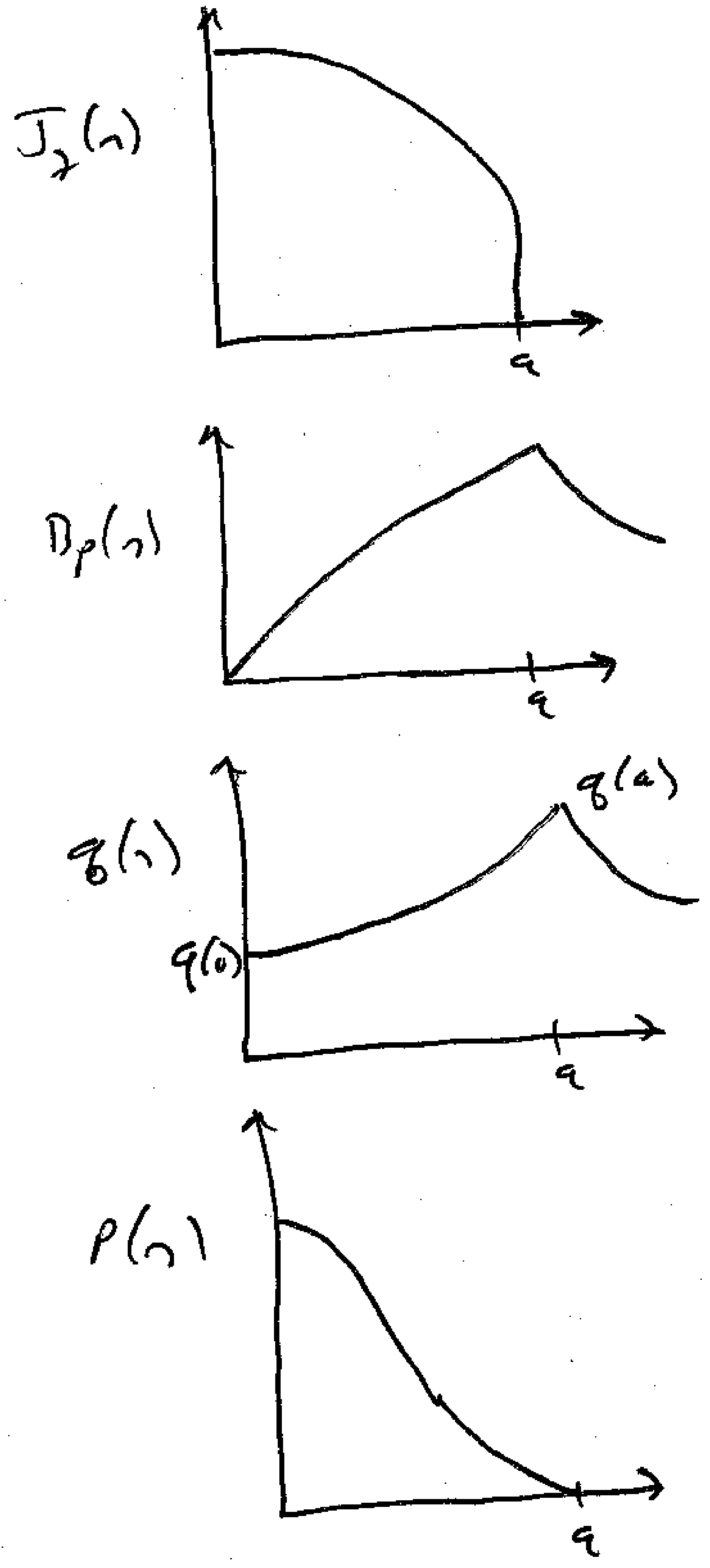
Shafranov "Constant *J*" Case ($dJ_0/dr = 0$)
Wesson "Model $J_0(r)$" Case ($dJ_0/dr \neq 0$)

Wesson's Cylindrical Equilibrium ($dJ_0/dr \neq 0$)

J.A. Wesson 1978 *Nucl. Fusion* 18 87; http://doi.org/10.1088/0029-5515/18/1/010

John Weston
1932 - 2020





$J_z(r) = J_0 (1 - r^2/a^2)^\nu$

J_0 = CENTRAL CURRENT DENSITY

$q(0) = \frac{2B_z}{\mu_0 R J_0}$ $\quad \mu_0 I_p = \frac{2\pi^2 a^2 B_z}{\mu_0 q(a) R(1+\nu)}$

$\nu = \frac{q(a)}{q(0)} - 1$

$\ell_i \neq \frac{1}{2}$

$\langle\beta\rangle \sim \epsilon^2/q_a^2 \quad \beta_p \sim 1 \quad \langle B_r\rangle \sim 20\,\epsilon/q_a$

EQUILIBRIUM SET BY $q(0), q(a)$
TWO PARAMETERS

$\frac{r}{q}\frac{dq}{dr} = s$ = MAGNETIC SHEAR $\neq 0$

PRESSURE PROFILE CAN BE MORE PEAKED

$B\cdot\nabla \to 0$ AT INTERNAL RESONANT LAYER

$(m - n/q(r)) \cong 0$

Wesson's Kink Modes

LINEARIZED EQUATIONS FOR PERTURBED STREAM FUNCTION (χ) AND PERTURBED POLOIDAL FLUX (ψ)

$$-\rho\,\omega\,\nabla_\perp^2\tilde{\chi} = -\frac{m}{r}\boxed{\frac{\partial J_z}{\partial r}\,\tilde{\psi}} + \frac{B_\theta}{\mu_0 r}(m - nq)\nabla_\perp^2\tilde{\psi} \qquad \text{(MHD)}$$

$$-\omega\tilde{\psi} = \frac{B_\theta}{r}(m - nq)\tilde{\chi} \qquad \text{(INDUCTION)}$$

LET'S TAKE $\rho \simeq$ UNIFORM, WITH A SHARP JUMP AT THE PLASMA'S EDGE:

$$\omega\,\rho\,\frac{\partial\chi}{\partial r}\bigg|_{a^-} = \frac{B_\theta(a)}{\mu_0 a}(m - nq_a)\,\tilde{\psi}_a\,\Delta'(a)$$

$\Delta'(a)$ ← PERTURBED SURFACE CURRENT AT PLASMA'S EDGE

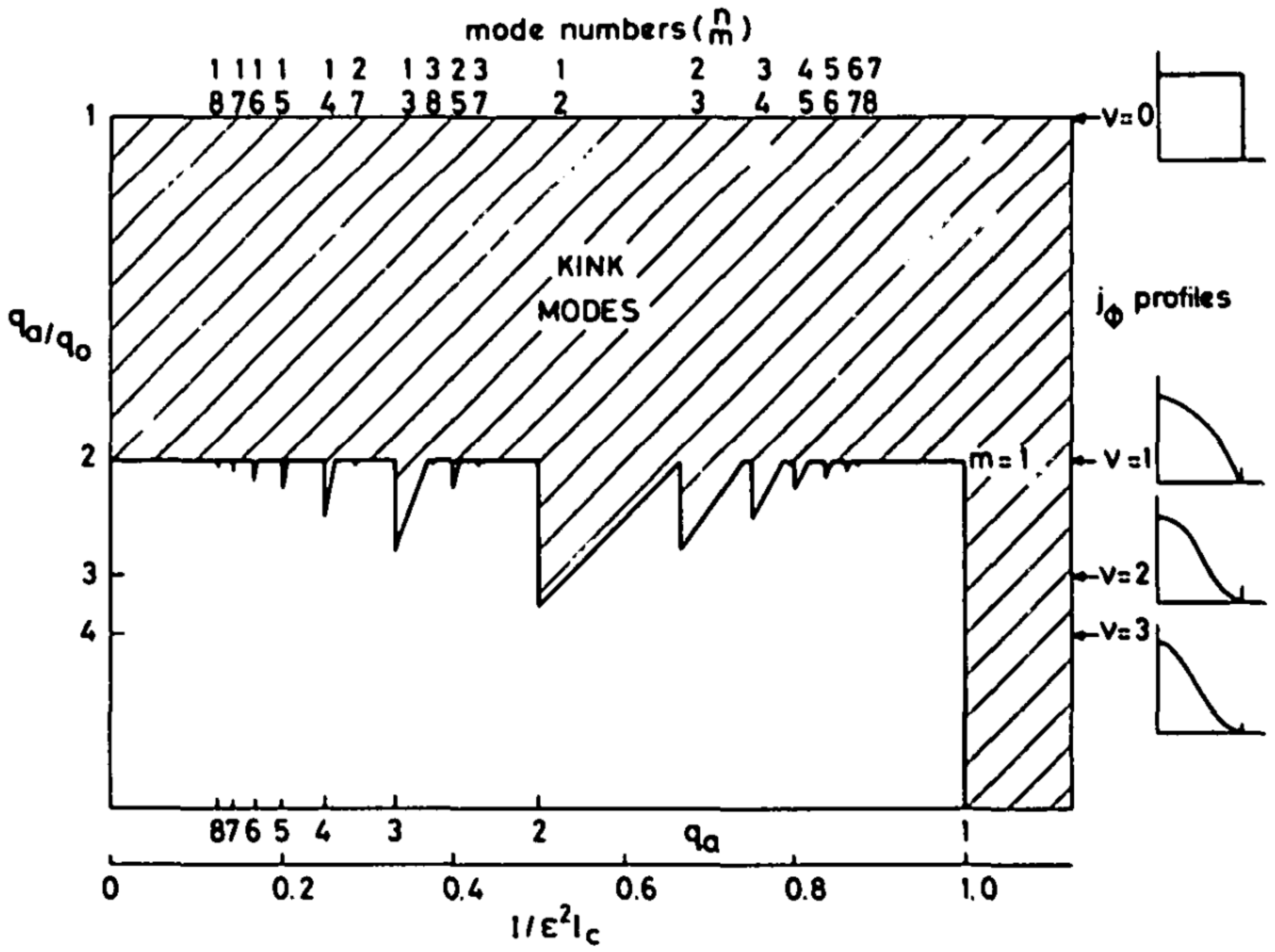
WITH INDUCTION EQUATION:

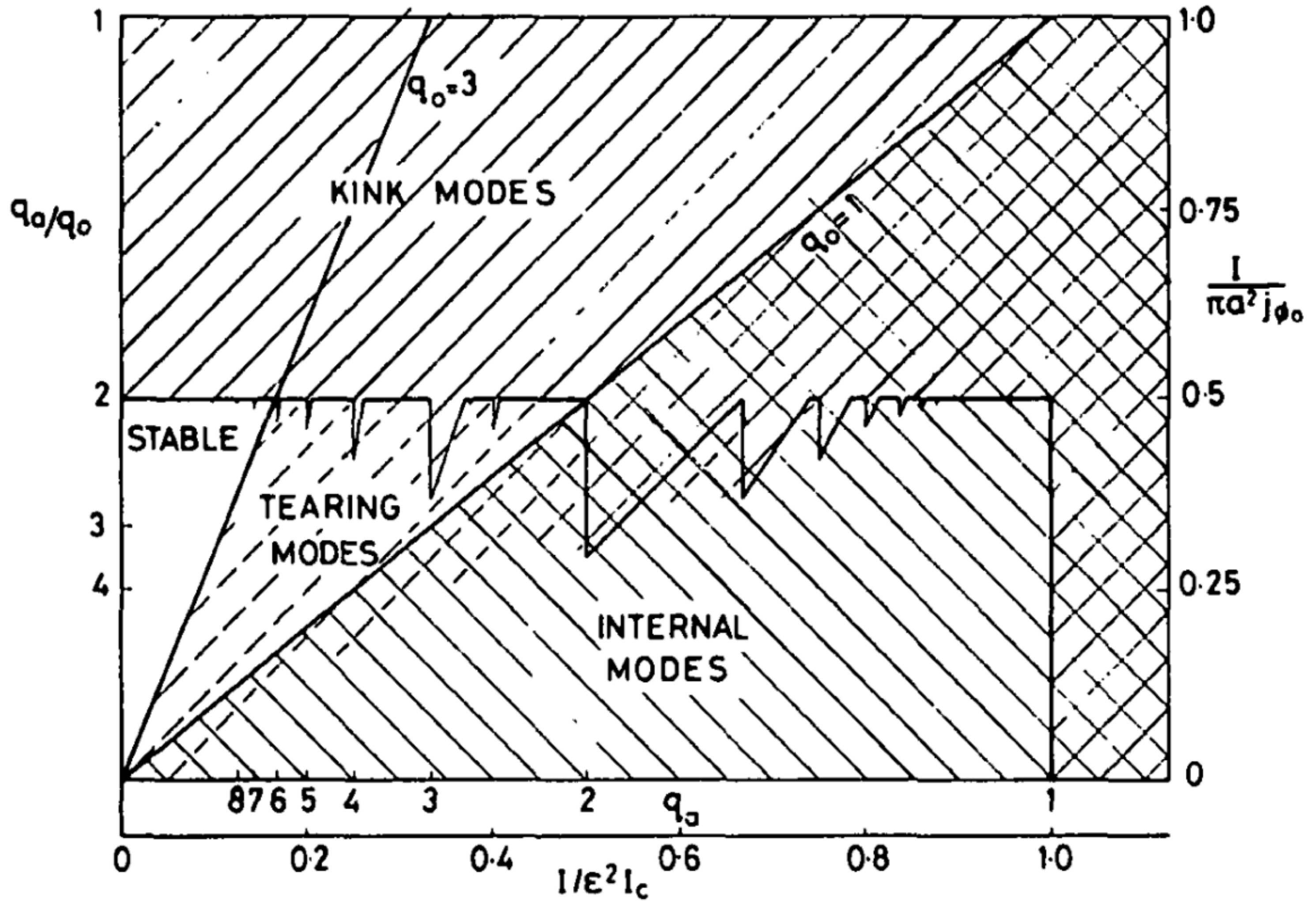
$$\omega^2 = -\omega_A^2(m - nq_a)^2\,\Delta'(a)\,\frac{\chi_a}{(\partial\chi/\partial r)_{a^-}}$$

INSTABILITY REQUIRES $\Delta'(a) > 0$

BUT, HOW TO FIGURE OUT $\tilde{\psi}(r)$?

Wesson's Kink Modes





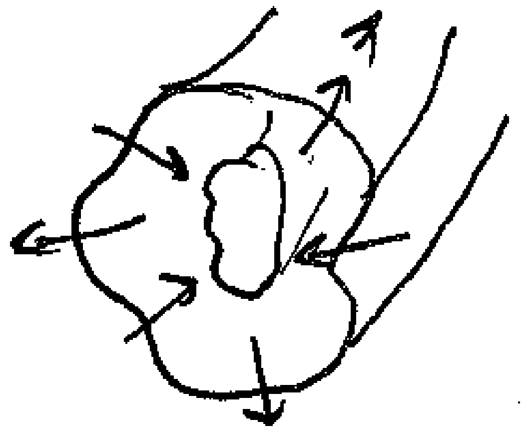
Since $|\omega| < \omega_A$, the kink mode causes the "internal" plasma to respond "quickly", so quickly that we can ignore the time it takes to form a distorted, 3D, quasi-equilibrium.

Inside, the plasma is a "forces" equilibrium

$$0 \simeq -\frac{m}{r}\boxed{\frac{\partial J_z}{\partial r}}\tilde{\Psi} + \frac{B_\theta}{\mu_0 r}(m - nq)\nabla_\perp^2 \tilde{\Psi}$$

Outside, the response is the "vacuum" response.

With $J_z(r)$, we have to solve for $\tilde{\Psi}$ using a computer. (This is very easy for the cylindrical "tokamak")



← The surface current "pushes"/"pulls" plasma, and the "distorted" plasma is measured by $\tilde{\Psi}(r, \theta, z)$

Kink Mode Plasma.nb

INSIDE, THE PLASMA IS A "FORCED" EQUILIBRIUM

$$0 \approx -\frac{m}{r}\boxed{\frac{\partial J_z}{\partial r}}\tilde{\Psi} + \frac{B_\theta}{\mu_0 r}(n - mq)\nabla_\perp^2 \tilde{\psi}$$

OUTSIDE, THE RESPONSE IS THE "VACUUM" RESPONSE.

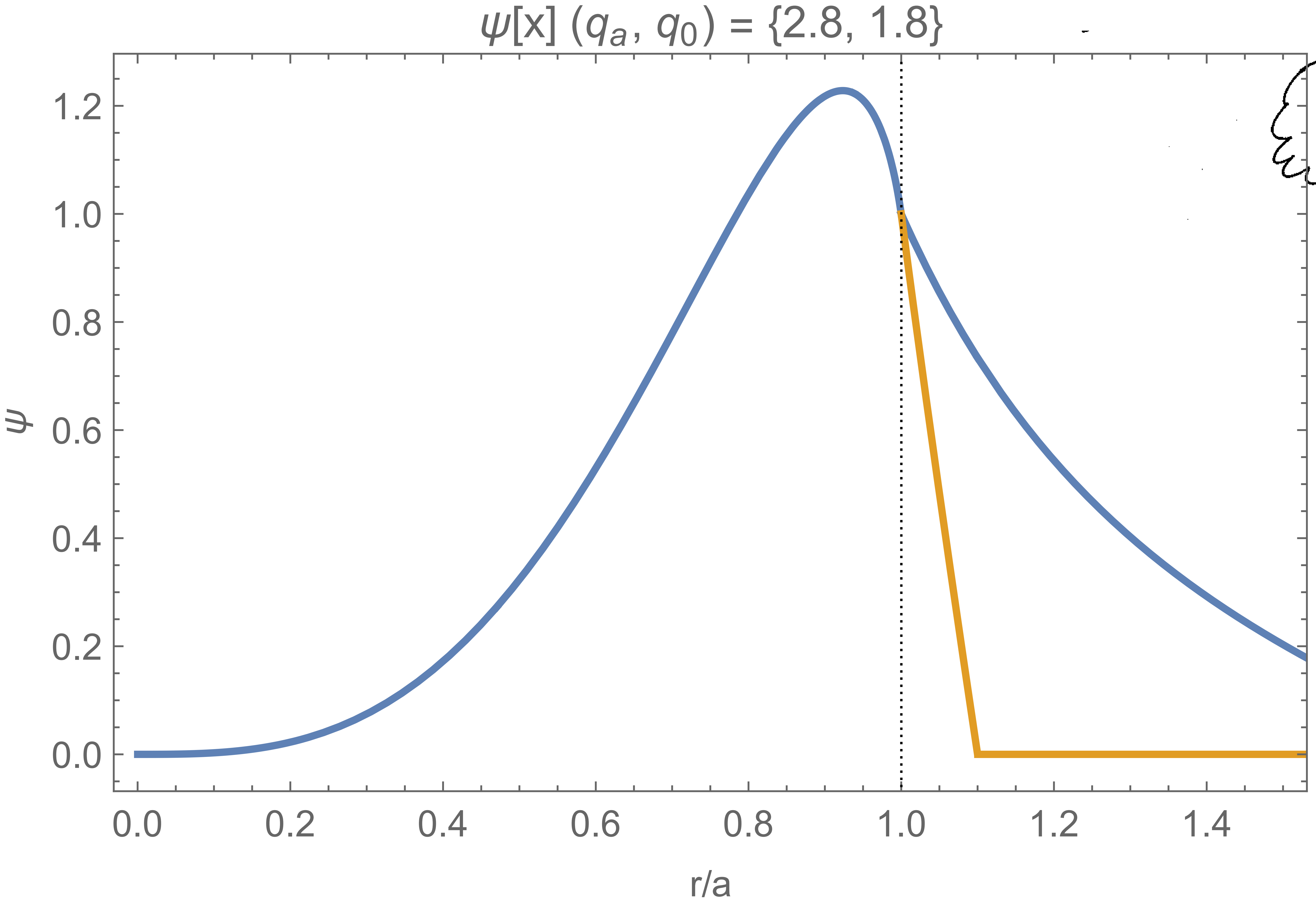
Eqs for MHD Force Balance

With $x = r/a$ and the Wesson current profile, the equation for MHD force balance is

```
In[18]:= eqsIdeal = {D[ψ[x], x] == ψp[x],
           x D[x ψp[x], x] - m^2 ψ[x] +
             4 νj (m / n / q0) x^2 (1 - x^2)^(νj - 1) ψ[x] / (m / n / q[x] - 1) == 0}
```

Out[18]= $$\left\{\psi'[x] == \psi p[x],\ -m^2\,\psi[x] + \frac{4\,m\,x^2\,\left(1-x^2\right)^{-1+\nu j}\,\nu j\,\psi[x]}{n\,q0\,\left(-1+\frac{m\,\left(1-\left(1-x^2\right)^{1+\nu j}\right)}{n\,qa\,x^2}\right)} + x\,\left(\psi p[x] + x\,\psi p'[x]\right) == 0\right\}$$

Kink Mode Plasma.nb



$$\omega^2 = -\omega_A^2 (m - nq_a)^2 \, \Delta'(a) \, \frac{\chi_a}{(\partial\chi/\partial r)_{a^-}}$$

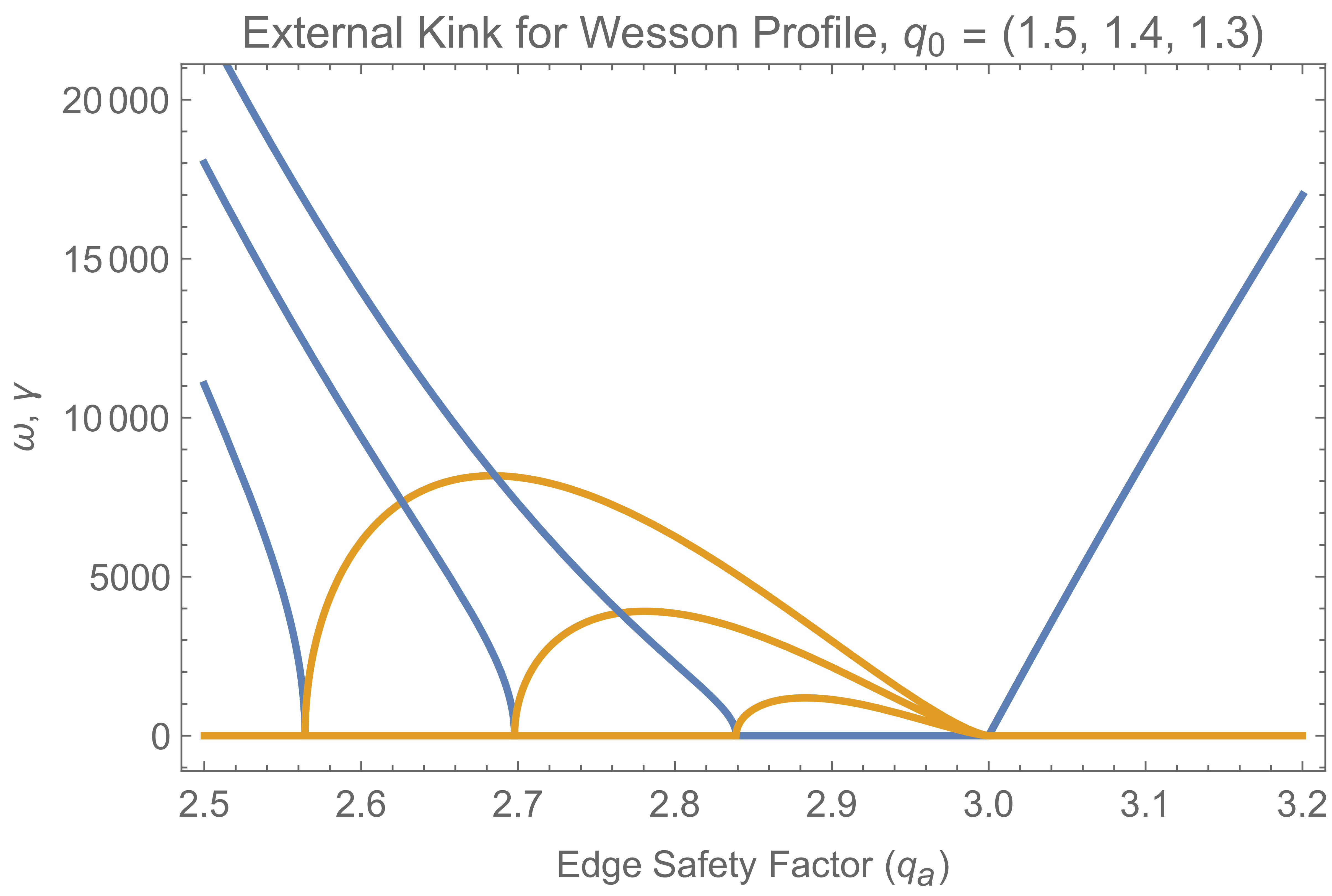
INSTABILITY REQUIRES $\Delta'(a) > 0$

$\omega_A/2\pi$ ~ 43 kHz (HBT-EP)

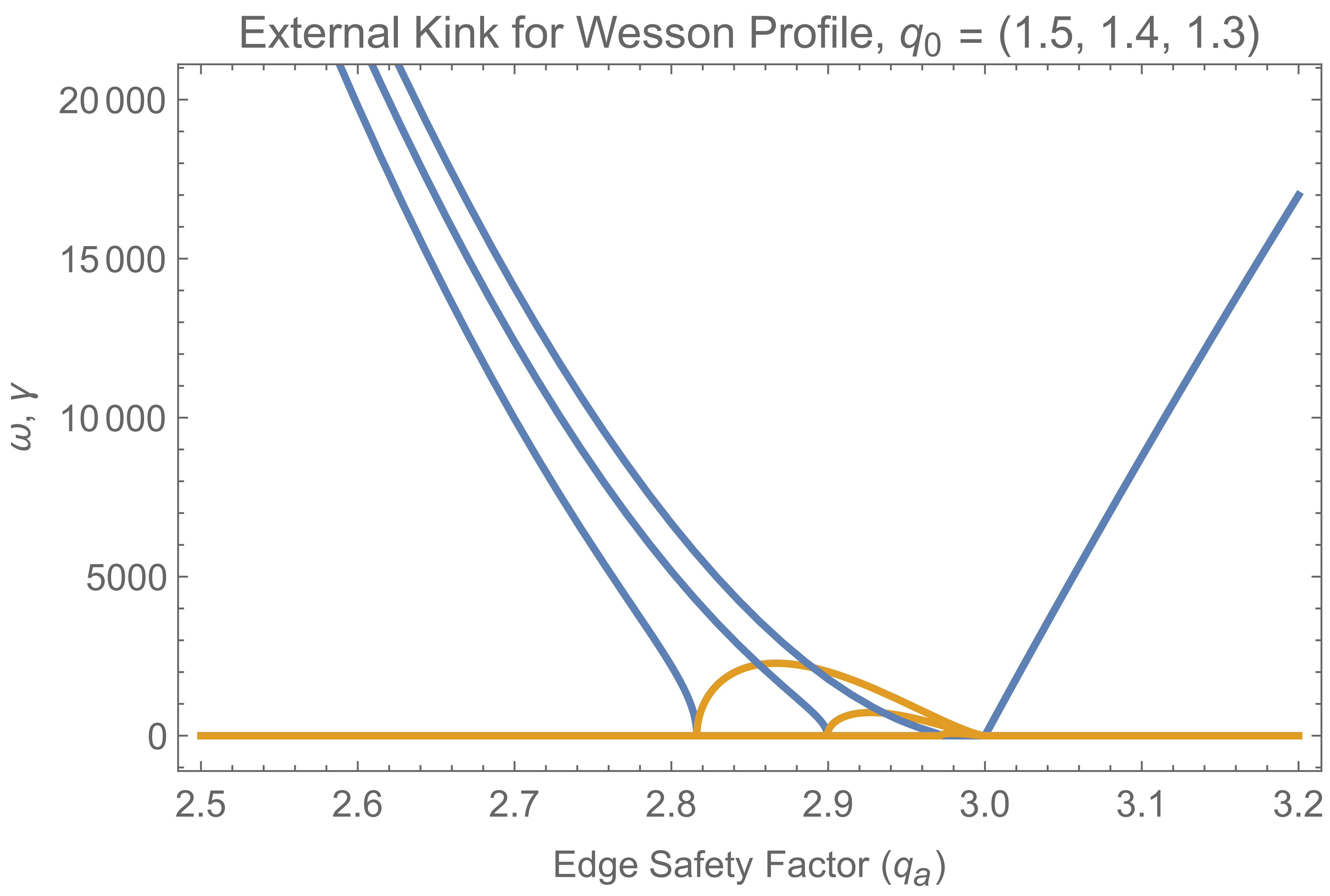
γ (b/a=1.8) = 12 msec^{-1}

ω (b/a=1.1)/2π = 270 Hz

Kink Mode Plasma.nb



Wall “out”: b/a = 1.8



Wall “in”: b/a = 1.2

Energetics of Kink Modes

$$\mathbf{v}_1 = \frac{d\boldsymbol{\xi}}{dt} \qquad \rho_0 \frac{\partial \mathbf{v}_1}{\partial t} = \boxed{\mathbf{j}_0 \times \mathbf{B}_1 + \mathbf{j}_1 \times \mathbf{B}_0} - \nabla p_1 = \mathbf{F}(\boldsymbol{\xi})$$

$$\rho_0 \frac{\partial^2 \boldsymbol{\xi}}{\partial t^2} = \mathbf{F}(\boldsymbol{\xi})$$

Total Integrated Energy

$$\frac{\omega^2}{2} \int \rho_0 |\xi|^2 \, dV = -\frac{1}{2} \int \boldsymbol{\xi}^* \cdot \mathbf{F}(\boldsymbol{\xi}) \, dV = \delta W(\boldsymbol{\xi}^*, \boldsymbol{\xi})$$

$$\boxed{\omega^2(\boldsymbol{\xi}, \boldsymbol{\xi}^*) = \frac{\delta W(\boldsymbol{\xi}, \boldsymbol{\xi}^*)}{K(\boldsymbol{\xi}, \boldsymbol{\xi}^*)} = \frac{\text{Perturbed “Potential” Energy}}{\text{Perturbed Kinetic Energy}}}$$

Sec. 7.3: *Introduction to plasma physics* by Donald Gurnett and Amitava Bhattacharjee (2nd Ed. 2017; Cambridge)
Sec 3.1: *Magnetohydrodynamic Stability of Tokamaks* by Hartmut Zohm (Wiley 2014)

Energetics of Kink Modes

$$\rho_0 \frac{\partial \mathbf{v}_1}{\partial t} = \boxed{\mathbf{j}_0 \times \mathbf{B}_1 + \mathbf{j}_1 \times \mathbf{B}_0} - \nabla p_1 = \mathbf{F}(\boldsymbol{\xi})$$

$$\frac{\omega^2}{2} \int \rho_0 |\xi|^2 \, dV = -\frac{1}{2} \int \boldsymbol{\xi}^* \cdot \mathbf{F}(\boldsymbol{\xi}) \, dV = \delta W(\boldsymbol{\xi}^*, \boldsymbol{\xi})$$

Perturbed Magnetic "Twist"
(negative or positive)

$$\xi^* \cdot (\bar{J}_0 \times \bar{B}_1)$$

$$\xi^* \cdot (\bar{J}_0 \times (\hat{z} \times \nabla \psi_1))$$

$$\xi^* \cdot (\hat{z}(\bar{J}_0 \cdot \nabla \psi_1) - \bar{\nabla}\psi_1 J_{0z})$$

$$- \xi^* \cdot (\bar{\nabla}\psi_1 J_{0z})$$

$$= -i \frac{J_{0z}}{\omega^*} \hat{z} \cdot (\bar{\nabla}\chi^* \times \nabla \psi_1)$$

Perturbed Magnetic Energy
(always positive)

$$\xi^* \cdot (\bar{J}_1 \times \bar{B}_0)$$

$$\xi^* \cdot ((\nabla \times B_1) \times \bar{B}_0)$$

$$- \xi^* \cdot (\bar{B}_0 \times (\nabla \times \bar{B}_1))$$

$$\nabla \cdot [(\xi^* \times B_0) \times B_1] + \underbrace{\bar{B}_1 \cdot \nabla \times (\xi^* \times B_0)}_{|\bar{B}_1|^2}$$

Sec. 7.3: *Introduction to plasma physics* by Donald Gurnett and Amitava Bhattacharjee (2nd Ed. 2017; Cambridge)
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Energetics of Kink Modes

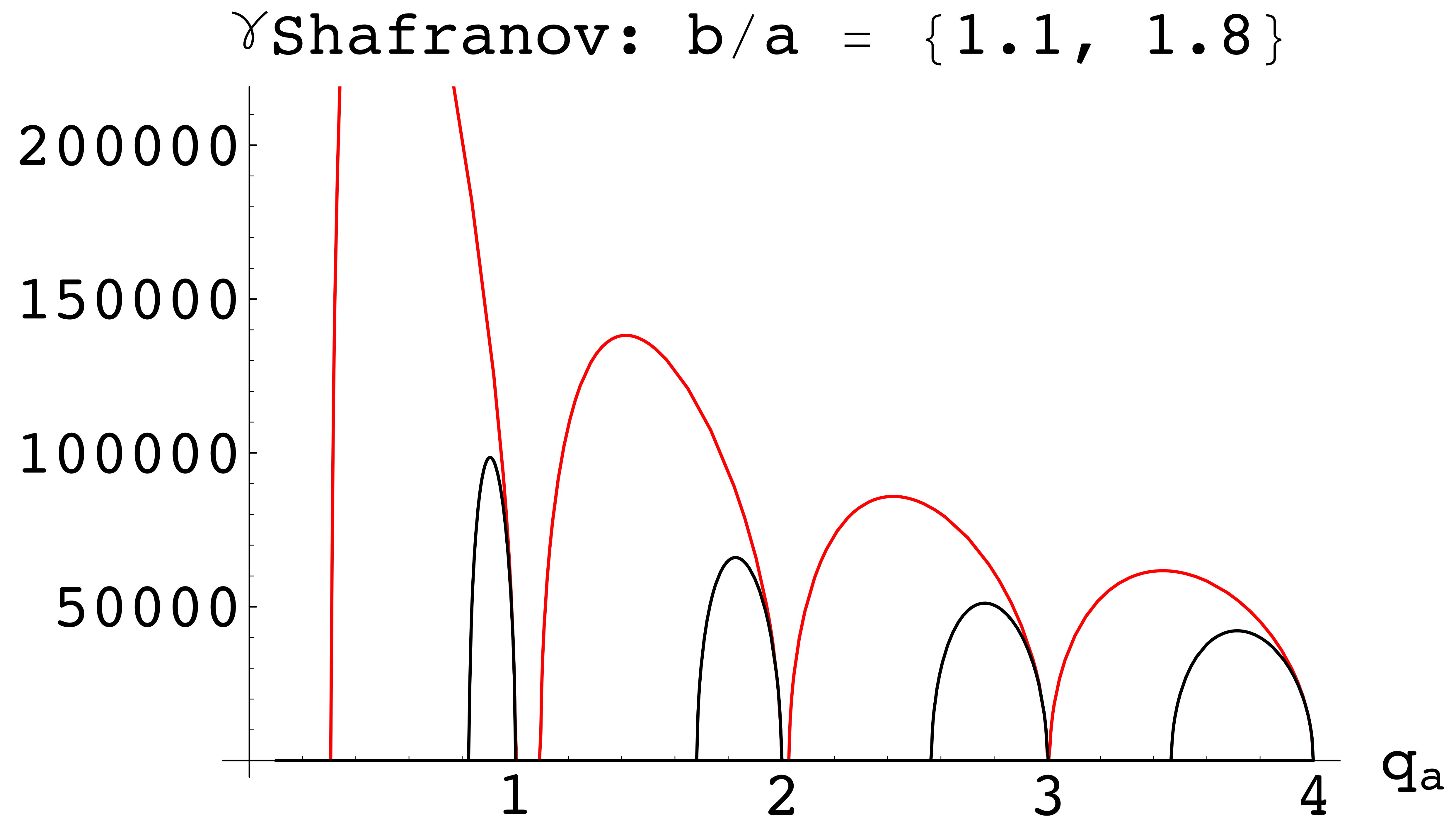
$$
\begin{aligned}
\delta W_p &= \frac{1}{2}\int_{r<a} dV \left[\frac{|\tilde{\mathbf{B}}|^2}{\mu_0} + \frac{iJ_{z,0}}{\omega^*}\,\tilde{\mathbf{v}}^*\cdot(\hat{\mathbf{z}}\times\tilde{\mathbf{B}})\right] \\
&= \frac{1}{2}\int_{r<a} dV \left[\frac{|\nabla_\perp\tilde{\psi}|^2}{\mu_0} - \frac{iJ_{z,0}}{\omega^*}\,\hat{\mathbf{z}}\cdot(\nabla_\perp\chi^*\times\nabla_\perp\tilde{\psi})\right] \\
\delta W_{v,b} &= \frac{1}{2}\int_{a<r<b} dV\,\frac{|\tilde{\mathbf{B}}|^2}{\mu_0} = \frac{1}{2}\int_{a<r<b} dV\,\frac{|\nabla_\perp\tilde{\psi}|^2}{\mu_0} \\
\delta K &= \frac{1}{2}\int_{r<a} dV\,\rho\,\frac{|\tilde{\mathbf{v}}|^2}{|\omega|^2} = \frac{1}{2}\int_{r<a} dV\,\rho\,\frac{|\nabla_\perp\chi|^2}{|\omega|^2}\,.
\end{aligned}
$$

$$
\begin{aligned}
\delta W_p &= \pi m\left[\frac{|\tilde{\psi}_a|^2}{\mu_0} + \frac{J_{z,0}}{\omega^*}(\chi_a^*\tilde{\psi}_a)\right] = \pi m\frac{|\tilde{\psi}_a|^2}{\mu_0}\left[1 - \frac{2}{m-nq_a}\right] \\
\delta W_{v,b} &= \pi m\frac{|\tilde{\psi}_a|^2}{\mu_0}\Lambda \quad \left(\text{with } \delta W_{v,\infty} = \pi m\frac{|\tilde{\psi}_a|^2}{\mu_0}\right) \\
\delta K &= \pi m\frac{|\tilde{\psi}_a|^2}{\mu_0}\,\frac{1}{\omega_A^2(m-nq_a)^2}\,,
\end{aligned}
$$

(What happens as $q_a \rightarrow m/n$?)

$$
\boxed{\omega^2(\xi,\xi^*) = \frac{\delta W(\xi,\xi^*)}{K(\xi,\xi^*)} = \frac{\text{Perturbed “Potential” Energy}}{\text{Perturbed Kinetic Energy}}}
$$

Energetics of Kink Modes



(What happens as $q_a \rightarrow m/n$?)

$$\omega^2(\xi, \xi^*) = \frac{\delta W(\xi, \xi^*)}{K(\xi, \xi^*)} = \frac{\text{Perturbed “Potential” Energy}}{\text{Perturbed Kinetic Energy}}$$

Resistive MHD

"IDEAL" MHD DESCRIBES THE "FAST" TIME DYNAMICS OF A MAGNETIZED PLASMA, $\omega \sim \omega_A$.

BUT, ON LONGER TIME SCALES, RESISTIVITY (AND OTHER NON-IDEAL EFFECTS) ARE IMPORTANT.

MHD

$$\rho \frac{d\bar{v}}{dt} = -\nabla p + J \times B$$

$$\frac{\partial \rho}{\partial t} = -\bar{\nabla} \cdot \rho \bar{v}$$

$$\bar{E} = -\bar{v} \times \bar{B} + \eta \bar{J}$$

$$\nabla \times B = \mu_0 \bar{J}$$

$$\frac{\partial \bar{B}}{\partial t} = -\nabla \times \bar{E}$$

η (STAINLESS STEEL) $\sim 0.7\, \mu\Omega \cdot m$

η (HBT-EP) $\sim 4-10 \times \eta_{SS}$

$$\eta = \frac{m_e}{e^2 n \tau_e} \propto \frac{1}{T_e^{3/2}}$$

WHAT IS η?

FOR A CONSTANT $\bar{E}$, COLLISIONS LEAD TO A CONSTANT DRIFT VELOCITY.

$$0 \simeq q\bar{E} + m\bar{v}/\tau_{col} \Rightarrow \bar{v} = \frac{q\tau_{col}}{m}\bar{E}$$

$$\bar{J} = en(\bar{v}_i - \bar{v}_e) \doteq -en\bar{v}_e = \frac{e^2 n \tau_{col}}{m_e}\bar{E}$$

There is no "equilibrium" in resistive MHD

MHD:

$$\mu_0 \nabla P = (\nabla \times B) \times \bar{B}$$

FORCE BALANCE IS EASY TO FIND
i.e. GRAD-SHAFRANOV

FINITE RESISTIVITY IMPLIES NO EQUILIBRIUM

PLASMA DISCHARGES (LIKE A TOKAMAK) MUST BE ACTIVELY SUSTAINED *

* NOT SO FOR SOME STELLARATORS AND THE LEVITATED DIPOLE

INDUCTION

$$\frac{\partial B}{\partial t} \approx 0 \simeq \nabla \times (v \times \bar{B}) - \nabla \times \left(\frac{\eta}{\mu_0} \nabla \times B\right)$$

STATIC EQUILIBRIUM $\bar{v} \approx 0$

SO

$$0 \simeq \nabla \times \left(\frac{\eta}{\mu_0} \nabla \times B\right)$$

NO EQUILIBRIUM IN GENERAL

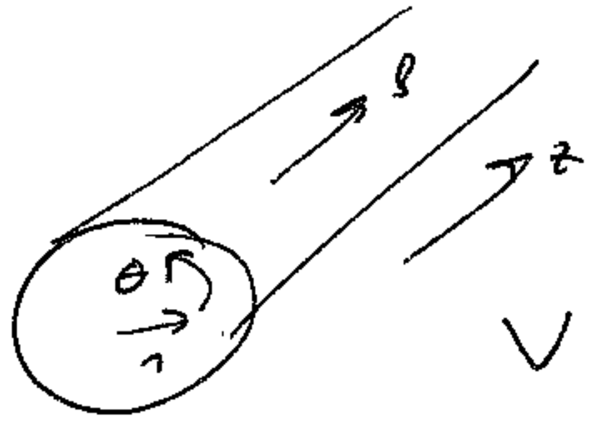
e.g. $\nabla \times J = 0 \Rightarrow$ NO TOROIDAL EQUILIBRIUM
NO POLOIDAL, DIAMAGNETIC CURRENT

RESISTIVE EQUILIBRIUM NEEDS SOURCE/SINKS OF CURRENT
i.e. ELECTRODES

"Simple" Resistive Equilibrium ⇒ Transport

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot \rho \bar{V}$$

$$\frac{\partial B}{\partial t} = \nabla \times (v \times B) - \nabla \times \left(\frac{\eta}{\mu_0} \nabla \times B\right)$$



$$V \sim \frac{E \times B}{B^2} + \frac{\eta \bar{B} \times \bar{J}}{B^2}$$

MOSTLY POLOIDAL

MOSTLY RADIAL

$$\frac{\partial \rho}{\partial t} = -\nabla \left(\frac{\rho \eta}{B^2} B \times J\right) \qquad J \sim \frac{B \times \nabla P}{B^2}$$

$$= \nabla \cdot \left(\frac{\rho \eta}{B^2} \nabla_\perp P\right) \qquad P = \frac{\rho k T}{m_i}$$

$$\cong \nabla \cdot \underbrace{\frac{\rho k T \eta}{B^2 m_i}}_{} \nabla_\perp \rho$$

$$D_\rho \sim \frac{\rho \eta}{B^2} V_{THi}^2 = \frac{\eta}{\mu_0} \left(\frac{V_{THi}}{V_A}\right)^2 \cong (\eta / \mu_0)\, \beta$$

$$\frac{\partial B}{\partial t} \cong -\nabla \times \left(\frac{\eta}{\mu_0} \nabla \times B\right)$$

$$= -\frac{\eta}{\mu_0}\left[\nabla(\nabla \cdot B) - \nabla^2 B\right] - \nabla\left(\frac{\eta}{\mu_0}\right) \times J$$

→ SMALL IF η WEAKLY OR CYLINDRICAL GEOMETRY

$$= \left(\frac{\eta}{\mu_0}\right) \nabla^2 B$$

→ MAGNETIC DIFFUSION COEFFICIENT

$$D_m = \frac{\eta}{\mu_0}$$

$$D_\rho << D_m$$

$$D_\rho \sim D_m \beta$$

“Simple” Resistive Equilibrium ⇒ Transport

$D_m \sim \frac{\eta}{\mu_0}$ → CURRENT RELAXES ON THE TIME SCALE OF $\tau_m \approx a^2/2D_m$

DEVICE	SIZE, ⟨a⟩	T_e, (keV)	n (m^{-3})	η ∧	τ_m (sec)
HBT-EP	0.14	0.05	1×10^{19}	3×10^{-6}	8 msec
T-3	0.13	1.0	2×10^{19}	3×10^{-8}	600 msec
DIII-D	0.85	6	8×10^{19}	2×10^{-9}	400 sec
JET	1.6	15	1×10^{20}	6×10^{-10}	1.5 Hours
ITER	2.6	10	1×10^{20}	1×10^{-9}	2.2 Hours

η (copper) = $1.7 \times 10^{-8}\ \Omega\, m$

Tearing Modes: Internal Resonances ($B \cdot \nabla = 0$)

REVIEW

VORTICITY ↓ AXIAL CURRENT ↓

REDUCED MHD

$$\rho \frac{d\bar{V}}{dt} = -\nabla P + J \times B \longrightarrow \rho \frac{d}{dt} \nabla_\perp^2 \chi = (B \cdot \nabla) \nabla_\perp^2 \psi$$

$\bar{B} \cdot \bar{\nabla}$

$$\bar{E} = -\bar{V} \times \bar{B}$$

INDUCTION

$$\frac{\partial \bar{B}}{\partial t} = -\nabla \times E \longrightarrow \frac{\partial \bar{B}}{\partial t} = \nabla \times (v \times B) - \nabla \times (\eta \bar{J})$$

$$\frac{d\bar{B}}{dt} = (B \cdot \bar{\nabla}) \bar{V} - \boxed{\nabla \times (\eta \bar{J})}$$

$\bar{B} \cdot \bar{\nabla}$

FOR EXTERNAL MODES

$\bar{B} \cdot \bar{\nabla} \neq 0$ ANYWHERE INSIDE PLASMA

$$\bar{B} \cdot \bar{\nabla} \sim \frac{B_\theta}{r} (m - n q(r))$$

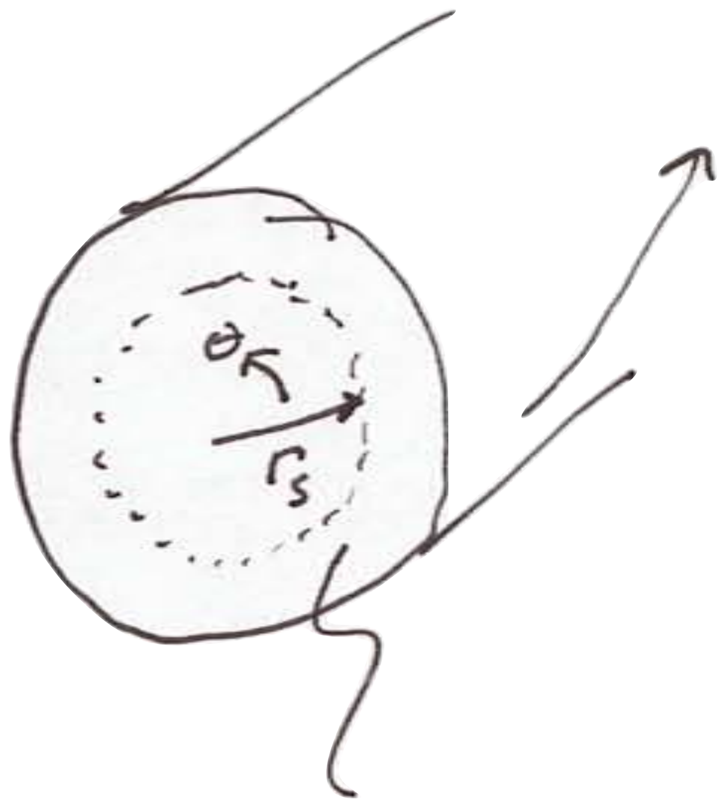
HELICAL INTERNAL RESONANCE

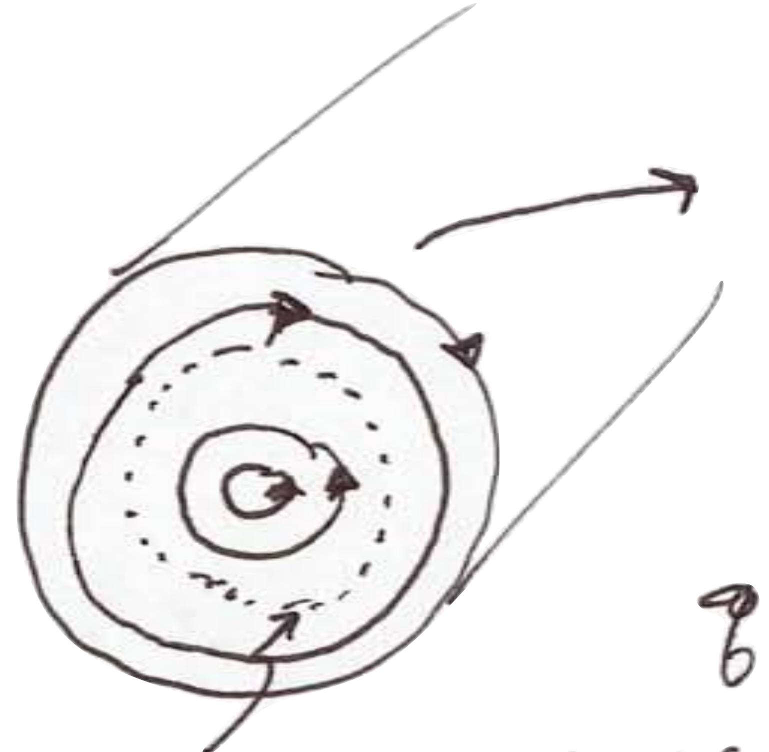
FOR INTERNAL MODES

$\bar{B} \cdot \bar{\nabla} = 0$ INSIDE PLASMA

Magnetic Islands

INTERNAL RESONANT CURRENTS CREATE "ISLANDS"





$$q(0) < q(r_s) < q(a)$$

RESONANT SURFACE $B_\theta^* = 0$

$$m - nq(r_s) = 0$$

$$\bar{B} = \hat{z} B_z + \hat{z} \times \nabla \psi$$

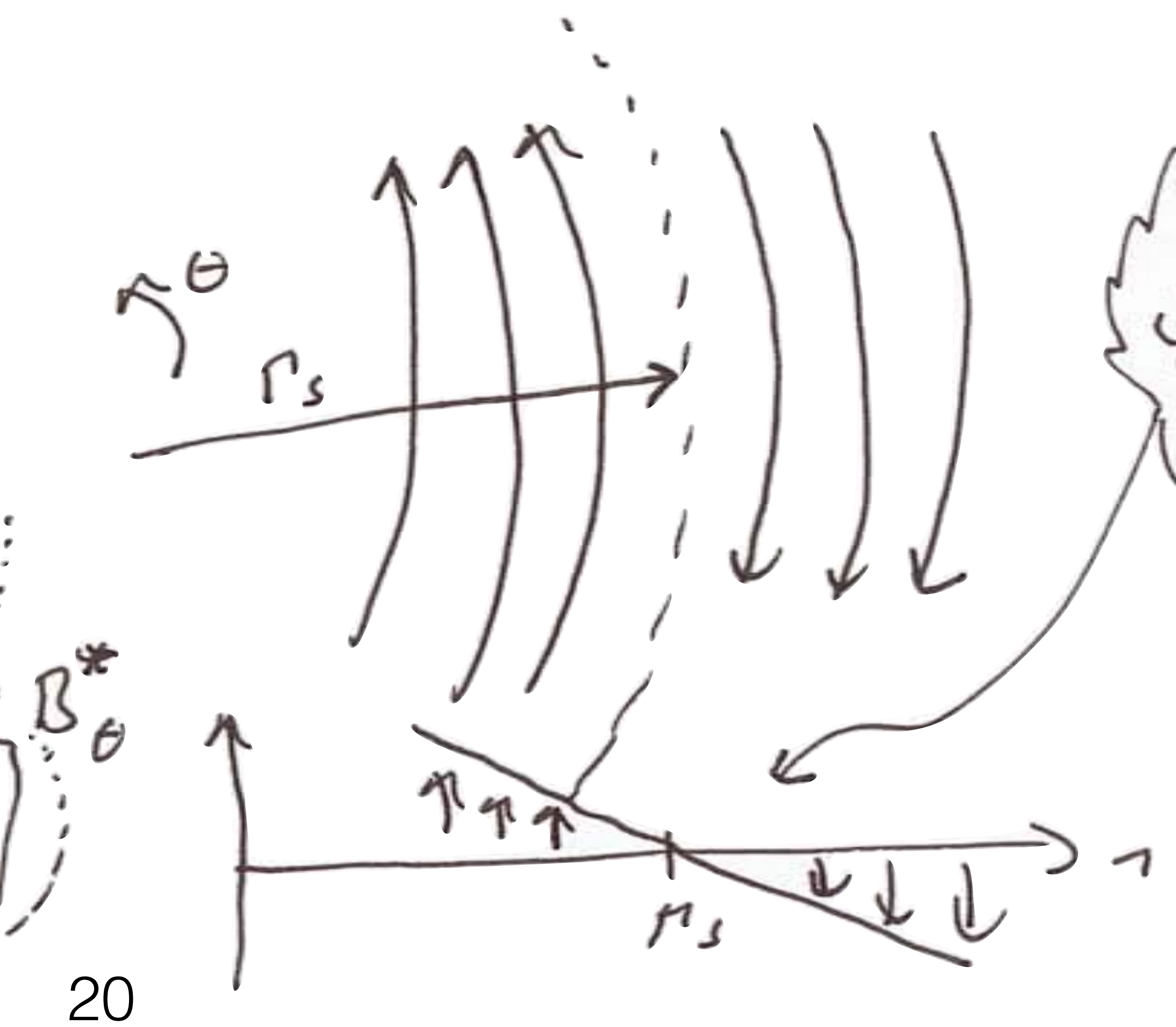
$$B_\theta^* = B_\theta - B_z \frac{n\,r}{mR}$$

$$= \frac{B_\theta}{m}\left(m - nq(r)\right)$$

$$\frac{\partial B_\theta^*}{\partial r} = \frac{\partial B_\theta}{\partial r} - \frac{B_z n}{mR} = \frac{B_z}{Rq}\left[1 - s - \frac{nq}{m}\right]$$

$$= -\frac{B_z}{Rq} s \ @\ r = r_s$$

$$\bar{B}^* = \hat{z} B_z + \hat{z} \times \nabla \psi^*$$



SLOPE: $\frac{\partial B_\theta^*}{\partial r} = -\frac{B_z s}{qR}$

$s \equiv \frac{r}{q}\frac{dq}{dr}$ MAGNETIC SHEAR

Magnetic Islands

Sec. 9.7 Fitzpatrick

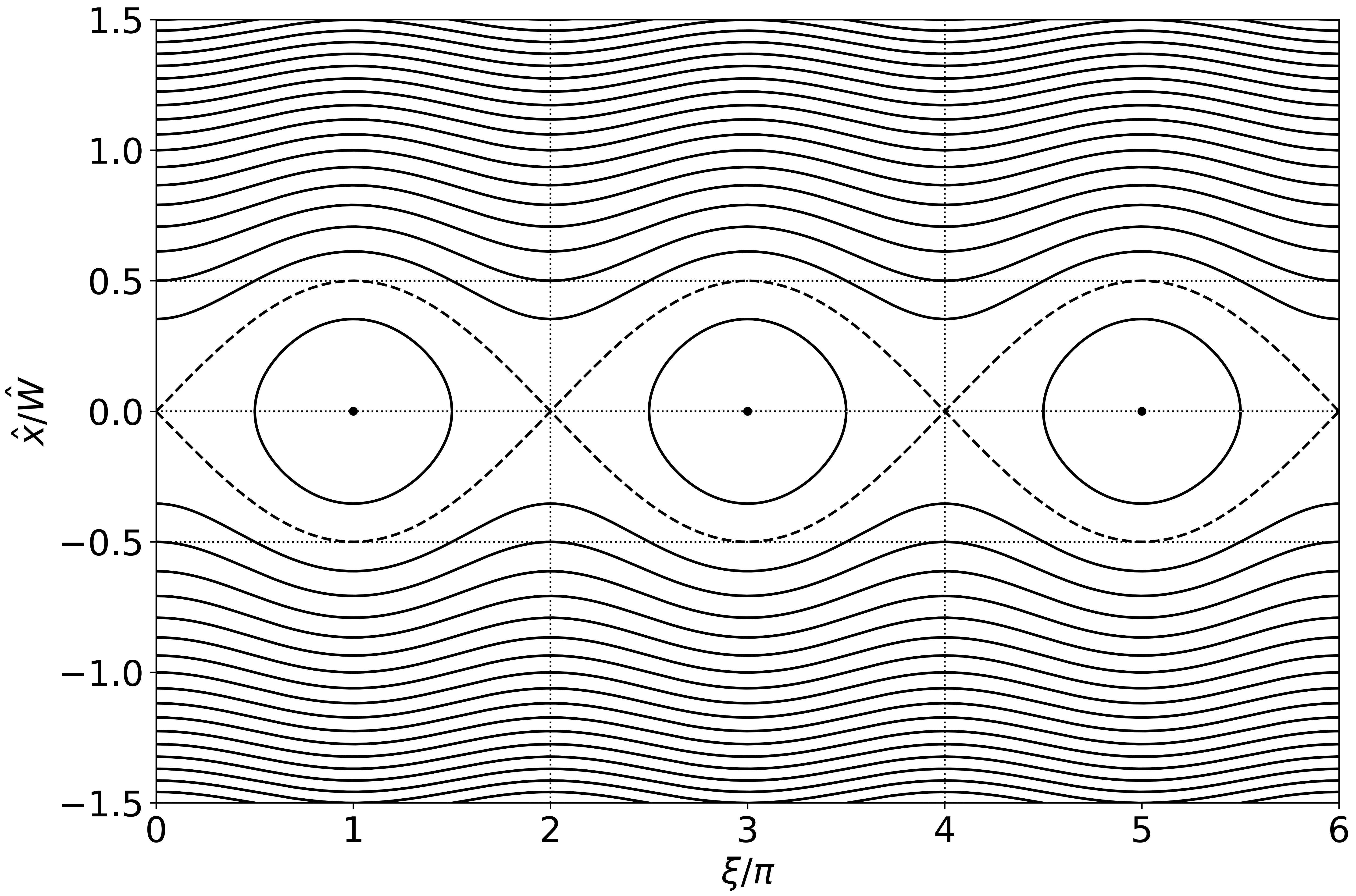


Figure 9.5 Equally-spaced contours of the magnetic flux-function, $\hat{\psi}(\hat{x}, \xi)$, in the vicinity of a constant-ψ magnetic island chain. The magnetic separatrix is shown as a dashed line.

Do Magnetic Islands Grow? Or Shrink?

Sec. 9.8 Fitzpatrick

$$I_1\,\tau_R\,\frac{d(W/a)}{dt} = \Delta'$$

$$\frac{d\tilde\psi}{dt} = \frac{\partial\tilde\psi}{\partial t} + (\mathbf{v}_\perp\cdot\nabla)\tilde\psi \;=\; \frac{\eta}{\mu_0}\nabla_\perp^2\tilde\psi$$

$$\frac{\partial\tilde\psi}{\partial t} \;\approx\; \frac{\eta}{\mu_0}(\Delta'/w)\tilde\psi$$

"Nonlinear growth of the tearing mode"
Phys. Fluids **16**, 1903–1908 (1973)
https://doi.org/10.1063/1.1694232

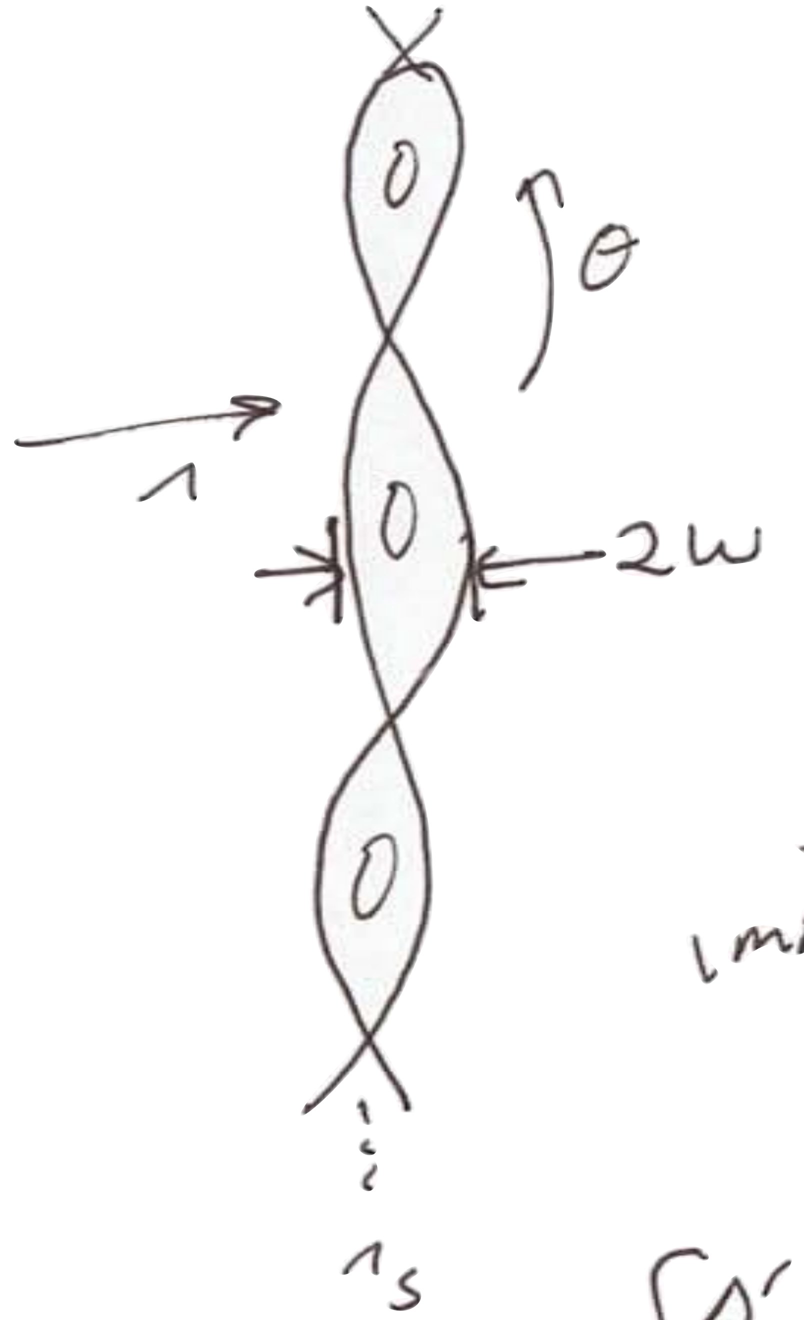
Paul Rutherford
1938 - 2023

IF WE ASSUME THAT THE PLASMA MOTION IS QUASI-STATIC AS THE ISLAND EVOLVES, THEN INDUCTION EQUATION IS

$$\frac{\partial \bar B}{\partial t} = \nabla\times(\underbrace{v\times B}_{\approx 0}) - \nabla\times\left(\frac{\eta}{\mu_0}\nabla\times B\right)$$

OR

$$\frac{d\psi}{dt} \approx \frac{\eta}{\mu_0}\nabla^2\psi$$ ← MAGNETIC DIFFUSION



INTEGRATE INDUCTION ACROSS ISLAND

$$\int_{r_s-w}^{r_s+w} 2\pi r\,dr\,\frac{d\psi}{dt} \approx \frac{\eta}{\mu_0}\int_{r_s-w}^{r_s+w} 2\pi r\,dr\,\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right)$$

$$2\pi r_s\,2w\,\frac{d\bar\psi}{dt} = \frac{\eta}{\mu_0}2\pi r_s\left(\left.\frac{\partial\psi}{\partial r}\right|_{r_s+w} - \left.\frac{\partial\psi}{\partial r}\right|_{r-w}\right)$$

CALLED "RUTHERFORD REGIME"

$$\boxed{\frac{d\psi}{dt} = \frac{\eta}{\mu_0}\frac{\bar\psi\,\Delta'}{2w}}$$

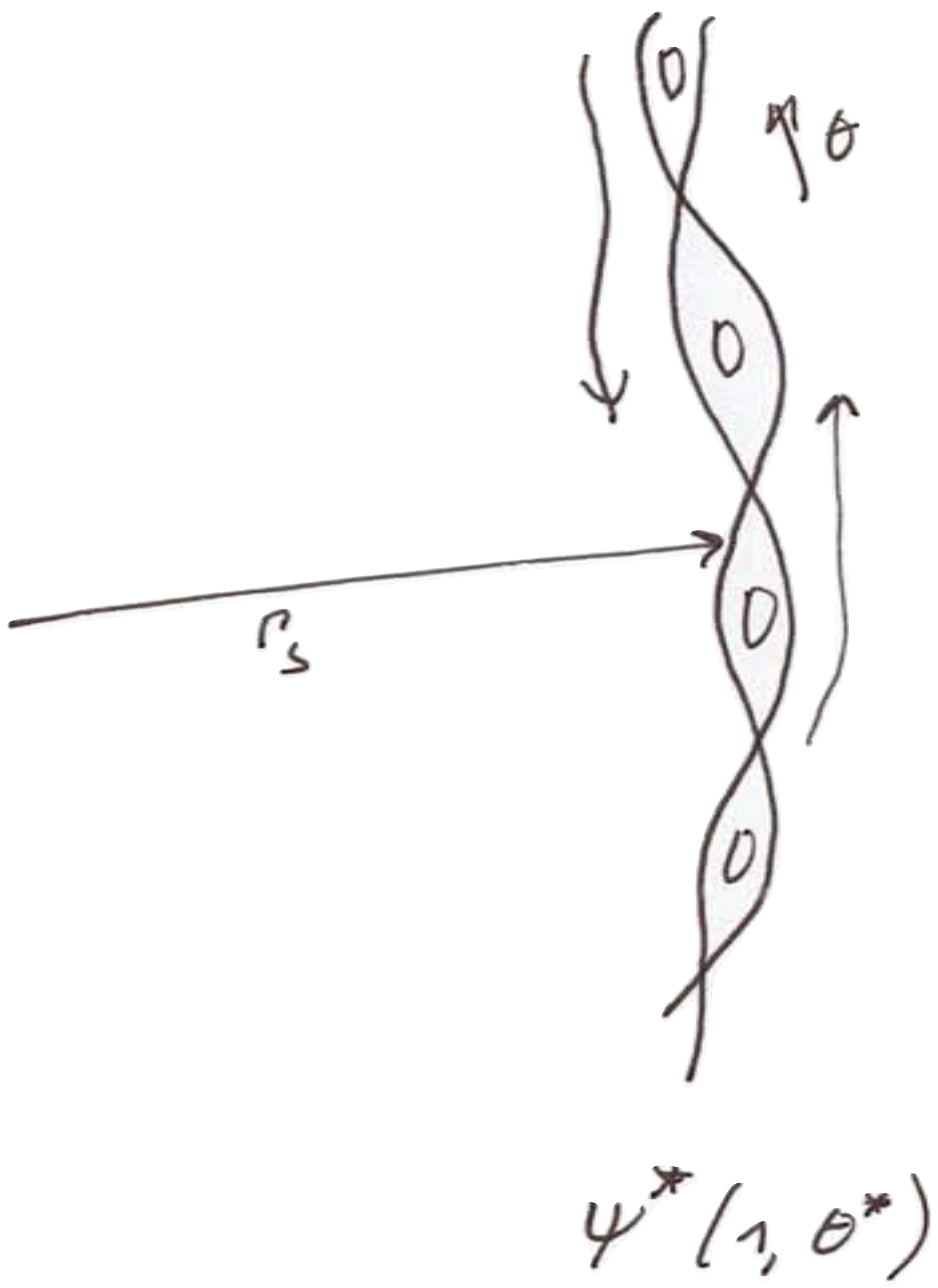
VERY IMPORTANT ↓

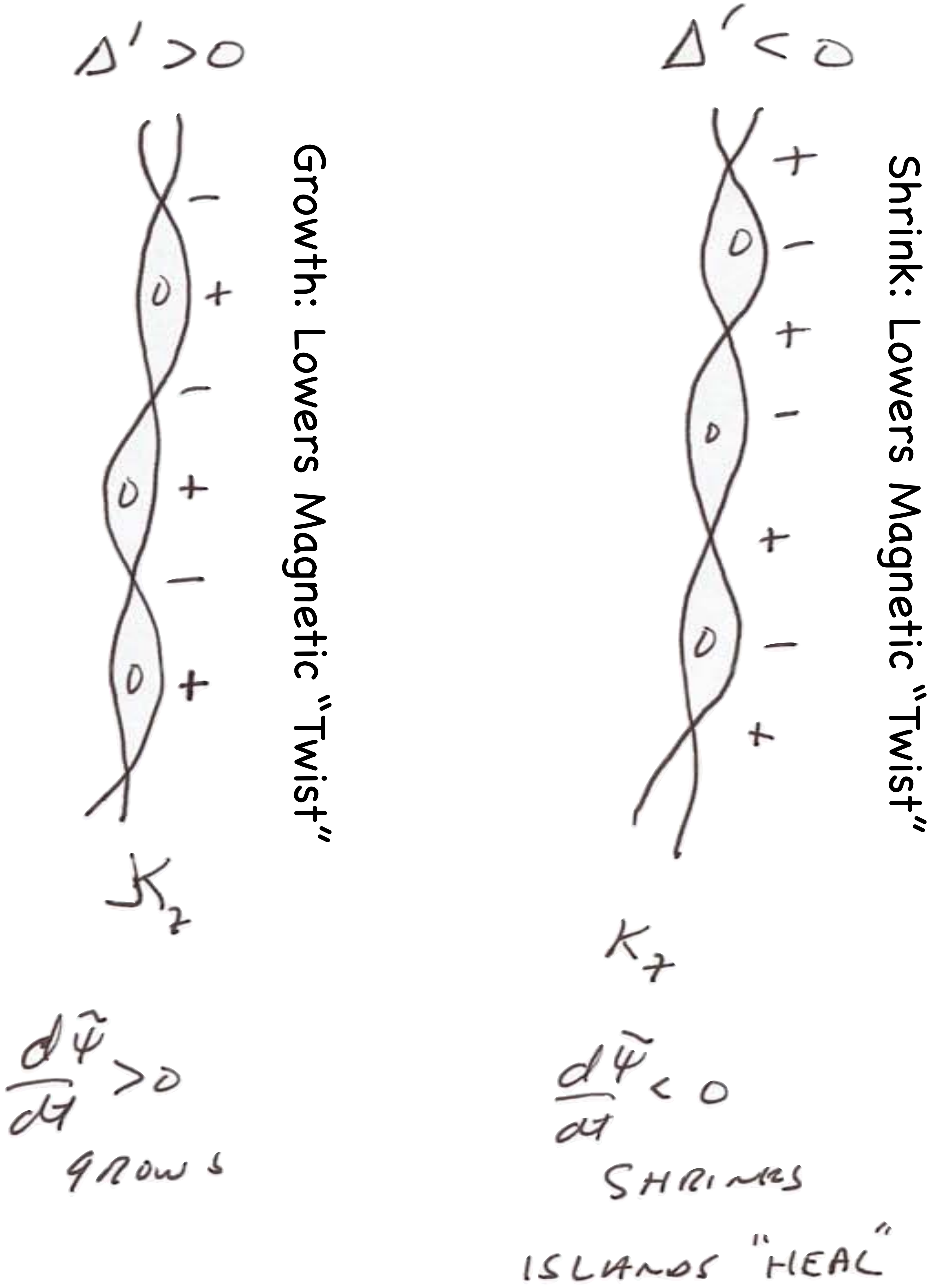
IF $\begin{cases}\Delta' > 0 & \text{GROWS}\\ \Delta' < 0 & \text{SHRINKS}\end{cases}$

RECALL: $\mu_0\tilde K_z(\theta,\varphi) = r_s\Delta'\tilde\psi_s$

$w \propto \sqrt{\psi}$

Do Magnetic Islands Grow? Or Shrink?





Δ' determines structure of surface current, *K*, at resonant surface.

How to calculate Δ'?

ALFVENIC RESPONSE (PLASMA INERTIA) IS IGNORED

A FORCE FREE RESPOND

$$\rho \frac{d}{dt} \nabla_\perp^2 \chi \to 0 \simeq (\bar{B} \cdot \bar{\nabla}) \nabla_\perp^2 \psi \qquad \left(\text{i.e. } \nabla \times (J \times B) \simeq 0 \right)$$

THIS IS NON LINEAR

LINEAR RESPONSE: $(\bar{B} \cdot \bar{\nabla}) \nabla^2 \tilde{\psi} = 0$

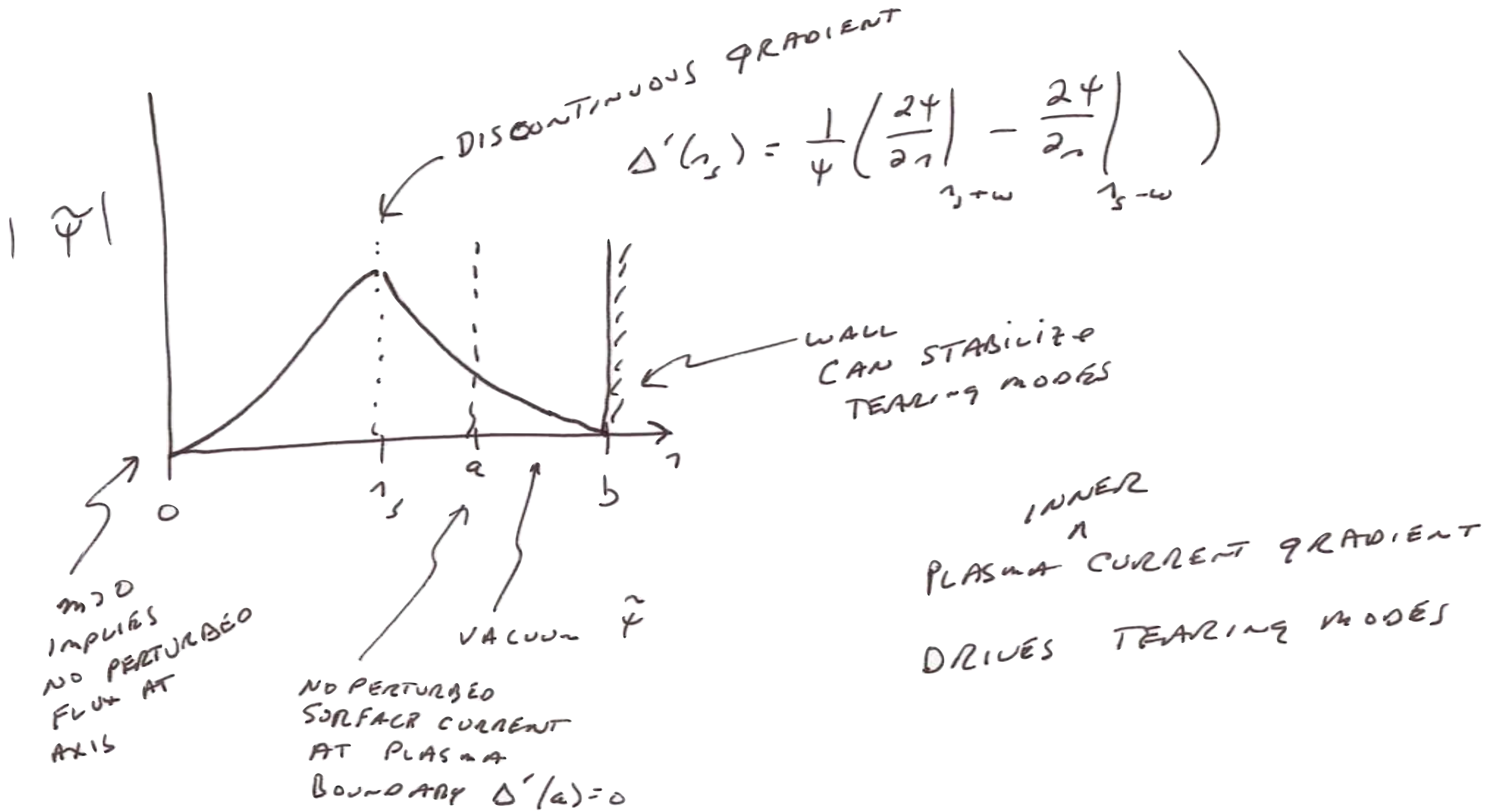
THIS IS LINEARIZED

$$\frac{B_p}{r}(m - nq)\nabla^2 \tilde{\psi} + (\hat{z} \times \nabla \tilde{\psi}) \cdot \bar{\nabla}_\perp J_z(\text{EQUIL}) \simeq 0$$

THIS IS THE SAME EQUATION WE SOLVED FOR KINK MODES!

OR
$$\frac{B_p}{\mu_0 r}(m - nq(r))\nabla_\perp^2 \tilde{\psi} - \frac{m}{r} \frac{\partial J_z}{\partial r} \tilde{\psi} = 0$$

How to calculate Δ’?



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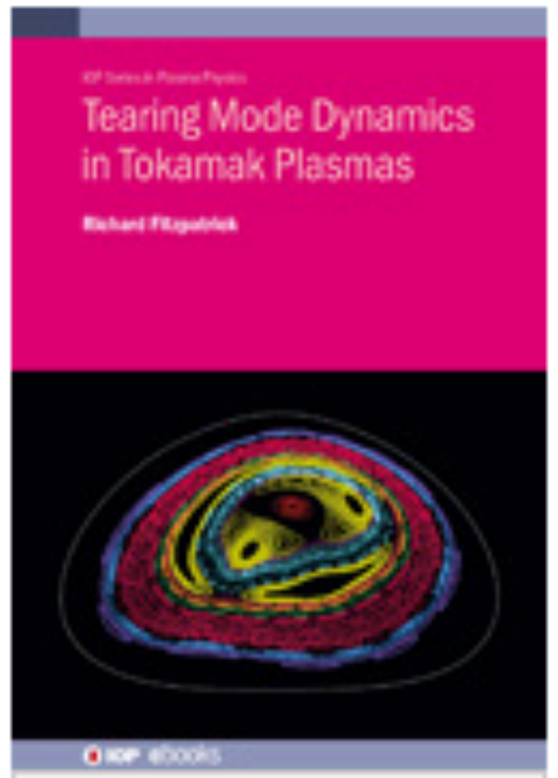
Tearing Mode Dynamics in Tokamak Plasmas



Preview

Author
Richard Fitzpatrick

Published
June 2023

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The development of humankind's ultimate energy source, nuclear fusion, has proceeded slowly but surely over the course of the last 60 years. This comprehensive book aims to outline a realistic, comprehensive, self-consistent, analytic theory of tearing mode dynamics in tokamak plasmas. It discusses a fluid theory of a highly magnetized plasma that treats the electrons and ions as independent fluids, and then proceeds to develop the theory of tearing modes, first approximating the geometry of a tokamak plasma as a periodic cylinder, but eventually considering the toroidal structure of real tokamak plasmas. This book also describes the stability of tearing modes, the saturation of such modes, and the evolution of their phase velocity due to interaction with other tearing modes, as well as the resistive vacuum vessel, and imperfections in the tokamak's magnetic field. This text will appeal to scientists and graduate students engaged in nuclear fusion research, and would make a useful reference for graduate plasma physics courses. Part of IOP Series in Plasma Physics.

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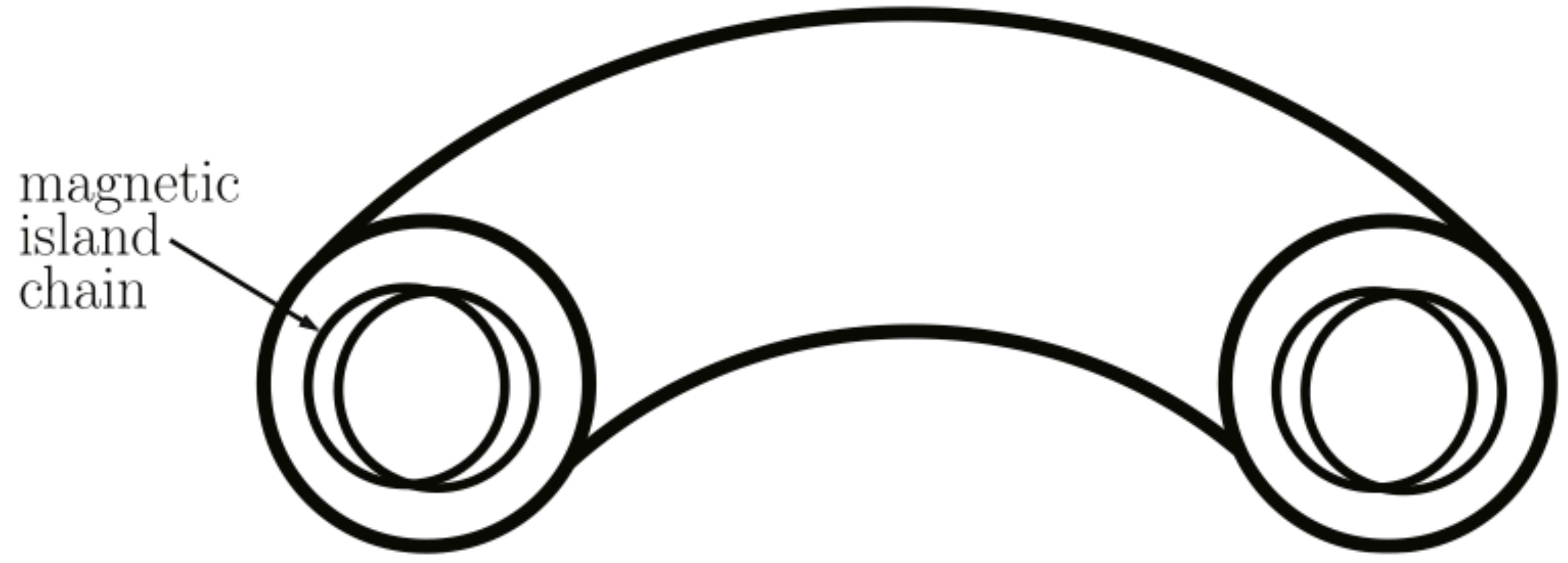

Figure 1.6. Schematic diagram of an $m = 2/n = 1$ magnetic island chain in a tokamak plasma.

Topics *Not* Covered…

- Kinetic theory, Vlasov equation, wave-particle interactions, quasilinear theory, particle trapping, …
- Gyrokinetics, drift waves, drift-wave transport, …
- Nonlinear topics, like shocks, wave-echoes, reconnection, energy-momentum transfer, turbulence, …
- Computational plasma physics, modeling, validation and verification, …
- Relativistic physics, intense laser-plasma, astrophysical objects, particle acceleration, …
- Plasma applications, like surface processing, chemistry, medicine, lighting, electric propulsion. …
- Fusion energy challenges, like plasma-wall interactions, fueling, heating, control, …
- Instrumentation, diagnostics, plasma reactors, …

Next Lecture: Final Exam Review