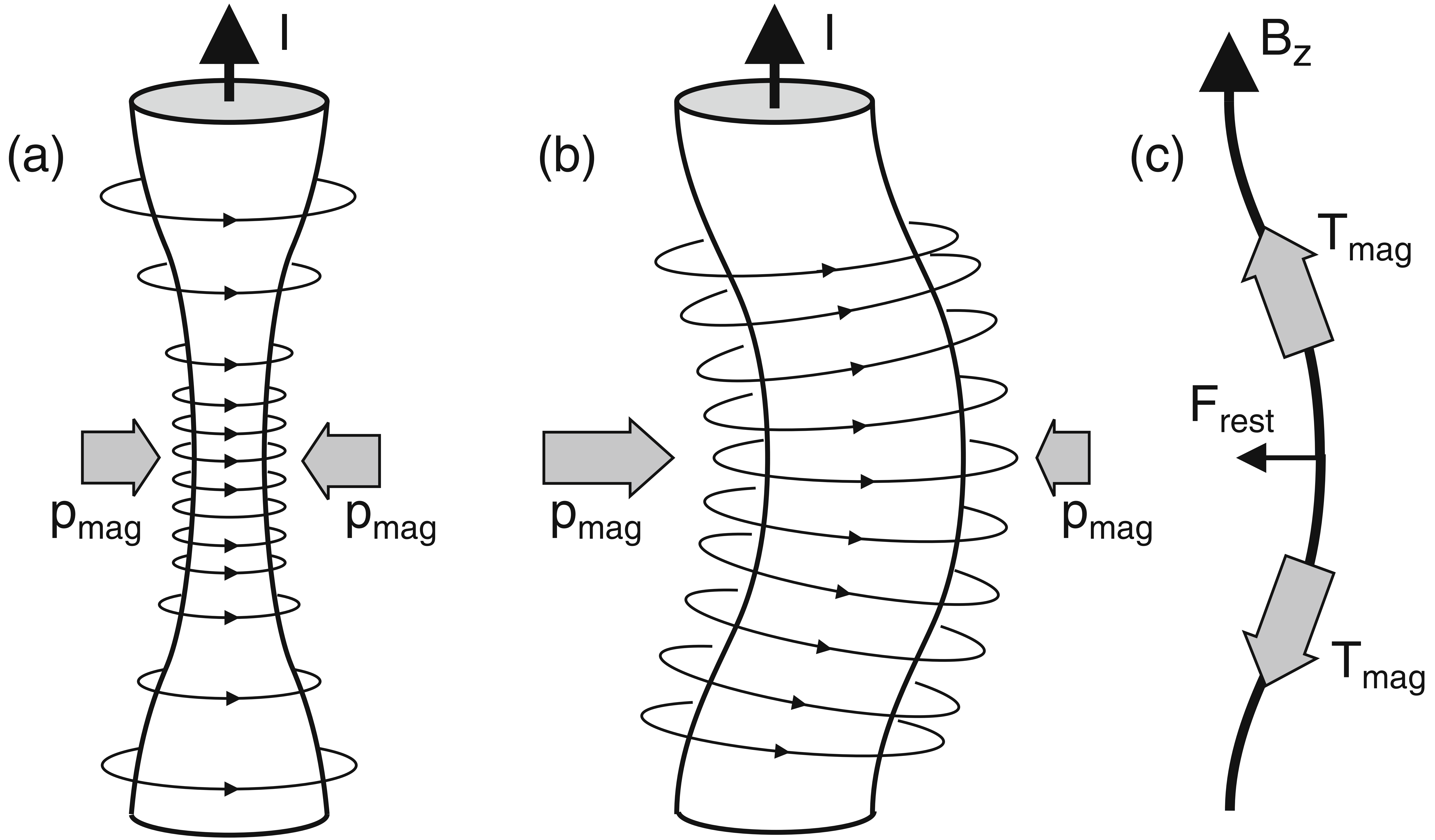
Lecture17: Plasma Physics 1

APPH E6101x
Columbia University

Last lecture: Interchange Instability
This lecture: Reduced MHD and Kink Modes

Z-Pinch (stabilized by B_z)



Nonlinear helical perturbations of a tokamak

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(Received 17 May 1976)

An analytic study of small amplitude perturbations of a cylindrical tokamak, using the nonlinear, helically symmetric, reduced magnetohydrodynamic equations of Kadomtsev and Rosenbluth is presented. Results are compared with those of a numerical study of these reduced equations.

I. INTRODUCTION

It has been proposed[1] that the nonlinear evolution of free boundary kink modes in tokamaks could lead to large distortions of the plasma. One purpose of the calculations presented here is to test this hypothesis near the marginally stable point where only one mode is unstable. Our results show that for ratios of plasma radius to wall radius less than 0.65, violent distortions do not occur (a similar result was obtained by Rutherford *et al.*[2] for a more stable equilibrium than the one considered here). For ratios of plasma radius to wall radius greater than 0.65, our results are inconclusive.

The magnetohydrodynamic equations are

$$\rho(d\mathbf{v}/dt) = \mathbf{j}\times\mathbf{B} - \nabla p\ , \qquad (1)$$

$$\mathbf{j} = \nabla\times\mathbf{B}\ , \qquad (2)$$

$$\partial\mathbf{B}/\partial t = -\nabla\times\mathbf{E}\ , \qquad (3)$$

$$\mathbf{E} = -\mathbf{V}\times\mathbf{B}\ , \qquad (4)$$

where

$$d/dt = \partial/\partial t + \mathbf{V}\cdot\nabla$$

is the convective derivative.

H. R. Strauss, D. A. Monticello, Marshall N. Rosenbluth, Roscoe B. White; Nonlinear helical perturbations of a tokamak. *Phys. Fluids* 1 March 1977; 20 (3): 390–395. https://doi.org/10.1063/1.861901

Ch. 9 Fitzpatrick: Magnetic Reconnection

Magnetic Reynolds Number
Lundquist Number

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{V} \times \mathbf{B}) + \frac{\eta}{\mu_0} \nabla^2 \mathbf{B}.$$

$$S = \frac{\mu_0 \, V \, L}{\eta} \simeq \frac{|\nabla \times (\mathbf{V} \times \mathbf{B})|}{|(\eta/\mu_0)\, \nabla^2 \mathbf{B}|},$$ $\rightarrow \infty$ (“Ideal”)

$$S = \frac{\tau_R}{\tau_H}. \qquad \tau_H = \frac{k^{-1}}{(B_0^2/\mu_0\, \rho)^{1/2}} \qquad \tau_R = \frac{\mu_0\, a^2}{\eta}$$

Reduced MHD: Incompressible

$$\nabla \cdot \mathbf{V} = 0, \qquad \mathbf{V} = \nabla \phi \times \mathbf{e}_z,$$

$$\rho \left[\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} \cdot \nabla) \mathbf{V} \right] + \nabla p - \mathbf{j} \times \mathbf{B} = 0, \qquad \mathbf{B} = \nabla \psi \times \mathbf{e}_z,$$

$$\mathbf{E} + \mathbf{V} \times \mathbf{B} = \eta\, \mathbf{j}.$$ ≈ 0 (“Ideal”)

PHYSICS OF PLASMAS **18**, 111207 (2011)

Phase diagram for magnetic reconnection in heliophysical, astrophysical, and laboratory plasmas

Hantao Ji[1] and William Daughton[2]
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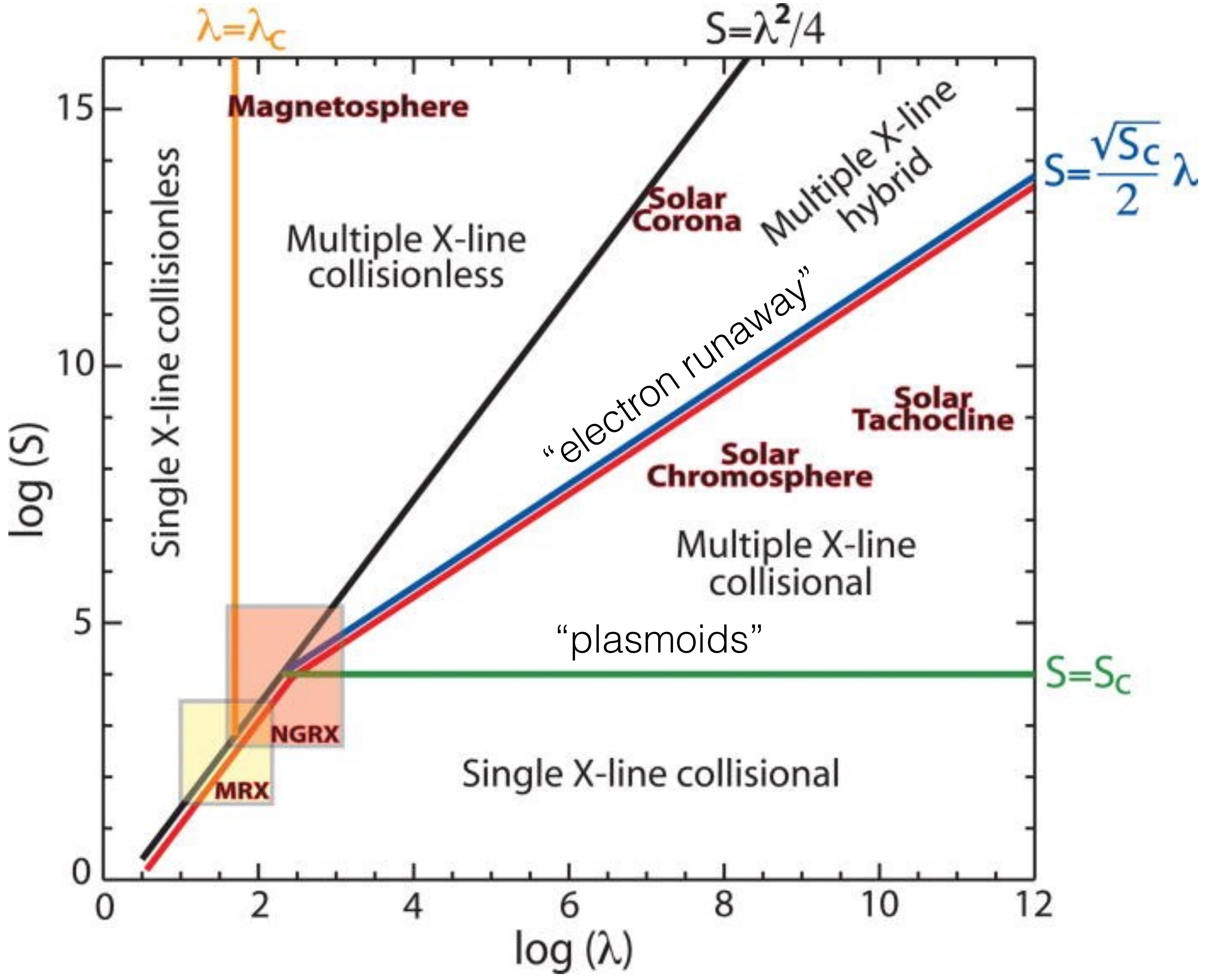
[2]*Los Alamos National Laboratory, Los Alamos, New Mexico 87545, USA*



$$S \equiv \frac{\mu_0 L_{CS} V_A}{\eta}$$ Lundquist Number

$$\lambda \equiv \frac{L}{d_i}$$ $d_i = c/\omega_{pi}$ the ion skin depth

https://doi.org/10.1063/1.3647505

Reduced MHD: Defining a "simple" model…

1970's HANK STRAUSS (NYU), MARSHALL ROSENBLUTH (PRINCETON)

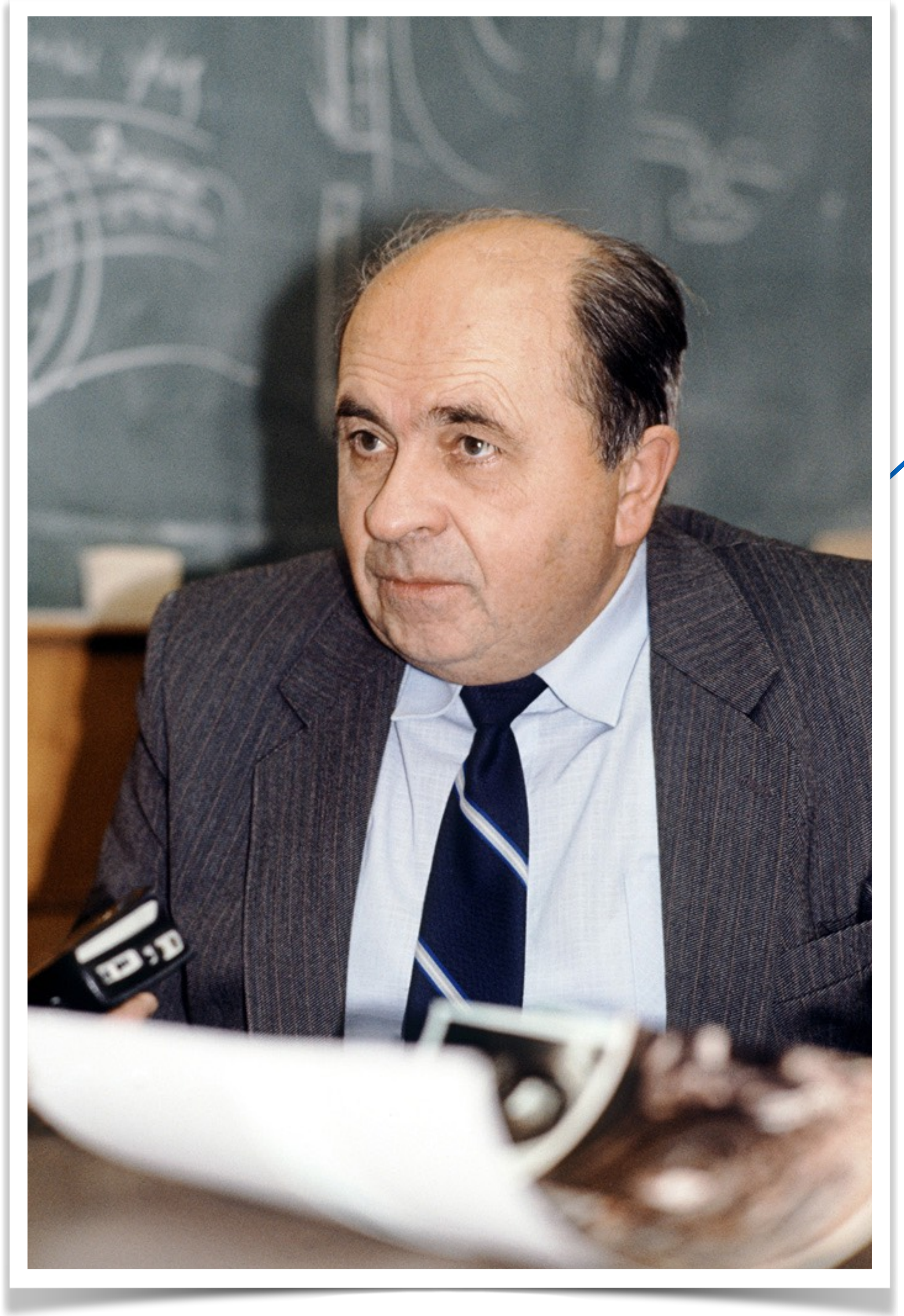
THE BASIC SIMPLE ANALYTIC TOOL FOR UNDERSTANDING TOKAMAKS UNTILL THE "MODERN" AGE OF COMPUTER CODES.

BASIC ASSUMPTIONS :

- LOW BETA $\langle\beta\rangle \sim (a/R)^2 \ll 1$ $\beta_N \sim (a/R)$
- LARGE ASPECT RATIO $\epsilon \equiv a/R \ll 1$
- WITH $q \gtrsim 1$, THEN $B_P/B_T \sim \epsilon/q \ll 1$
- WITH $\epsilon \ll 1$, THEN $B_T \sim B_0\left(\frac{R_0}{R}\right) \sim B_0(1-\epsilon\cos\theta + \dots) \approx$ CONSTANT

  BUT 1ST ORDER IN ϵ IS IMPORTANT
- LET $\nabla\cdot\hat{\varphi}$ (OR $\nabla\cdot\hat{z}$) $=0$, ELIMINATING ACOUSTIC MODES (NO "GAMS")
- BASICALLY, A "CYLINDRICAL" TOKAMAK (1D EQUILIBRIUM)

Basic Derivation ("*Ideal*")



Boris Kadomtsev
1928 - 1998

REF: KADOMTSEV, "TOKAMAK PLASMA: A COMPLEX PHYSICAL SYSTEM"
IOP (1992)

$\bar{B} = \bar{B}_\perp + \hat{z} B_z$ WITH B_z = CONSTANT

CYLINDRICAL COORDINATES

MHD:

$$\rho \frac{d\bar{v}_\perp}{dt} = -\nabla p + \bar{J} \times \bar{B}$$

$$\bar{E} = -\bar{v} \times \bar{B} \quad \text{(IDEAL)}$$

$$\bar{v} \cdot \hat{z} = 0$$
$$\nabla \cdot \bar{v}_\perp = 0 \quad \text{INCOMPRESSIBLE}$$

MAXWELL'S EM

$$\bar{J} = \frac{1}{\mu_0} \nabla \times \bar{B} \quad \text{(NO DISPLACEMENT CURRENT)}$$

$$\nabla \cdot \bar{B} = 0 = \nabla \cdot \bar{B}_\perp \qquad \downarrow \; \nabla \cdot J = 0$$

$$\frac{\partial \bar{B}}{\partial t} = -\nabla \times E$$

IDEAL MHD DESCRIBE PLASMA DYNAMICS AT ALFVEN TIME SCALE: $V \sim V_A \sim \sqrt{B^2/\mu_0 \rho}$ (FAST!)

Stream Function and Poloidal Flux

WITH B_z = CONSTANT,
$V_z = 0$
THE REDUCED MHD DYNAMICS IS DESCRIBED BY FOUR UNKNOWN FUNCTIONS OF (r, θ, z, t):

$$\bar{B}_\perp(r, \theta, z, t) \quad \text{AND} \quad \bar{V}_\perp(r, \theta, z, t)$$

WE "GREATLY SIMPLIFY" THE MATH BY INTRODUCING TWO POTENTIALS INSTEAD OF TWO VECTOR FIELDS
THE STREAM FUNCTION, χ, AND THE POLOIDAL FLUX FUNCTION, ψ

$$\bar{B}_\perp(r, \theta, z, t) = \hat{z} \times \nabla \psi \qquad \bar{V}_\perp(r, \theta, z, t) = \hat{z} \times \nabla \chi$$

$$\nabla \cdot \bar{B} = 0 \qquad \nabla \cdot \bar{V} = 0$$

AMPERE'S LAW

$$\bar{J} = \frac{1}{\mu_0} \nabla \times (\hat{z} \times \psi)$$

$$\mu_0 \bar{J} = \hat{z} \nabla^2 \psi - (\hat{z} \cdot \bar{\nabla}) \bar{\nabla} \psi$$

AXIAL VORTICITY

$$\Omega_z \equiv \hat{z} \cdot \nabla \times \bar{V}_\perp = \nabla^2 \chi$$

TWO UNKNOWN POTENTIALS:
$\psi(r, \theta, z, t) \quad \chi(r, \theta, z, t)$

Simplifying the MHD Momentum Equation

$$\rho \frac{d\bar{V}_\perp}{dt} = -\nabla p + \frac{1}{\mu_0}(\nabla \times B) \times B$$

$$= -\nabla\left(p + B^2/2\mu_0\right) + \frac{1}{\mu_0}(\bar{B}\cdot\bar{\nabla})\bar{B}$$

$$\hat{z}\cdot\nabla\times\left[\;'' \quad = \quad ''\;\right]$$

ASSUME $\rho \simeq$ UNIFORM "Boussinesq Approximation"

$$\rho \frac{d}{dt}\left(\hat{z}\cdot\nabla\times V_\perp\right) = \frac{1}{\mu_0}(\bar{B}\cdot\bar{\nabla})\left(\hat{z}\cdot\nabla\times\bar{B}\right)$$

$$\boxed{\mu_0 J_z = \nabla^2_\perp \psi}$$

$$\rho \frac{d}{dt}\left(\nabla_\perp^2 \chi\right) = \frac{1}{\mu_0}(\bar{B}\cdot\bar{\nabla})\nabla_\perp^2 \psi = (\bar{B}\cdot\bar{\nabla}) J_z$$

$$\rho \frac{d}{dt}\Omega_z = (\bar{B}\cdot\bar{\nabla}) J_z$$

"AXIAL VORTICITY CHANGES ACCORDING TO FIELD-ALIGNED VARIATION OF AXIAL CURRENT"

Simplifying the Induction Equation

$$\frac{\partial \bar{B}}{\partial t} = \nabla \times (\bar{V}_\perp \times \bar{B}) = \bar{V}_\perp \bar{\nabla}\cdot\bar{B} - \bar{B}\bar{\nabla}\cdot\bar{V} + (\bar{B}\cdot\bar{\nabla})\bar{V} - (\bar{V}\cdot\bar{\nabla})\bar{B}$$

$$= \nabla \times (\bar{V}_\perp \times \bar{B}_\perp) + B_z \frac{\partial \bar{V}_\perp}{\partial z}$$

$$= (\bar{B}_\perp\cdot\bar{\nabla})\bar{V}_\perp - (\bar{V}_\perp\cdot\bar{\nabla})\bar{B}_\perp + B_z \frac{\partial \bar{V}_\perp}{\partial z}$$

$$\frac{\partial \bar{B}_\perp}{\partial t} + (\bar{V}_\perp\cdot\bar{\nabla})B_\perp = \frac{d\bar{B}_\perp}{dt} = (\bar{B}\cdot\bar{\nabla})\bar{V}_\perp$$

SUBSTITUTING FLUX FUNCTIONS

$$\frac{d}{dt}\psi = (\bar{B}\cdot\bar{\nabla})\chi$$

"POLOIDAL FLUX EVOLVES DYNAMICALLY DUE TO FIELD-ALIGNED CHANGES IN THE STREAM FUNCTION"

Summary of Reduced MHD

$$\frac{d}{dt}\psi = (\bar{B}\cdot\bar{\nabla})\chi$$

"POLOID FLUX EVOLVES DYNAMICALLY DUE TO FIELD-ALIGNED CHANGES IN THE STREAM FUNCTION"

$$\rho \frac{d}{dt}\Omega_z = (\bar{B}\cdot\bar{\nabla}) J_z$$

"AXIAL VORTICITY CHANGES ACCORDING TO FIELD-ALIGNED VARIATION OF AXIAL CURRENT"

$\mu_0 J_z = \nabla^2_\perp \psi$

Importance of B · ∇ (!!)

$$\rho \frac{d}{dt} \nabla_\perp^2 \chi = \frac{1}{\mu_0}(B \cdot \bar{\nabla}) \nabla_\perp^2 \psi \qquad \text{(MHD)}$$

$$\frac{d}{dt} \psi = (\bar{B} \cdot \bar{\nabla}) \chi \qquad \text{(INDUCTION)}$$

$$= i \bar{k} \cdot \bar{B}_0 \qquad \bar{k} \equiv -\frac{n}{R} \hat{z} + \frac{m}{r} \hat{\theta} \quad \text{(PLUS RADIAL TERMS)}$$

LINEAR

$$\bar{B} \cdot \bar{\nabla} = B_z \frac{\partial}{\partial z} + B_\perp \cdot \bar{\nabla}_\perp = -i \frac{n}{R} B_z + i \frac{m}{r} B_p = i \frac{B_p(r)}{r} (m - n q(r))$$

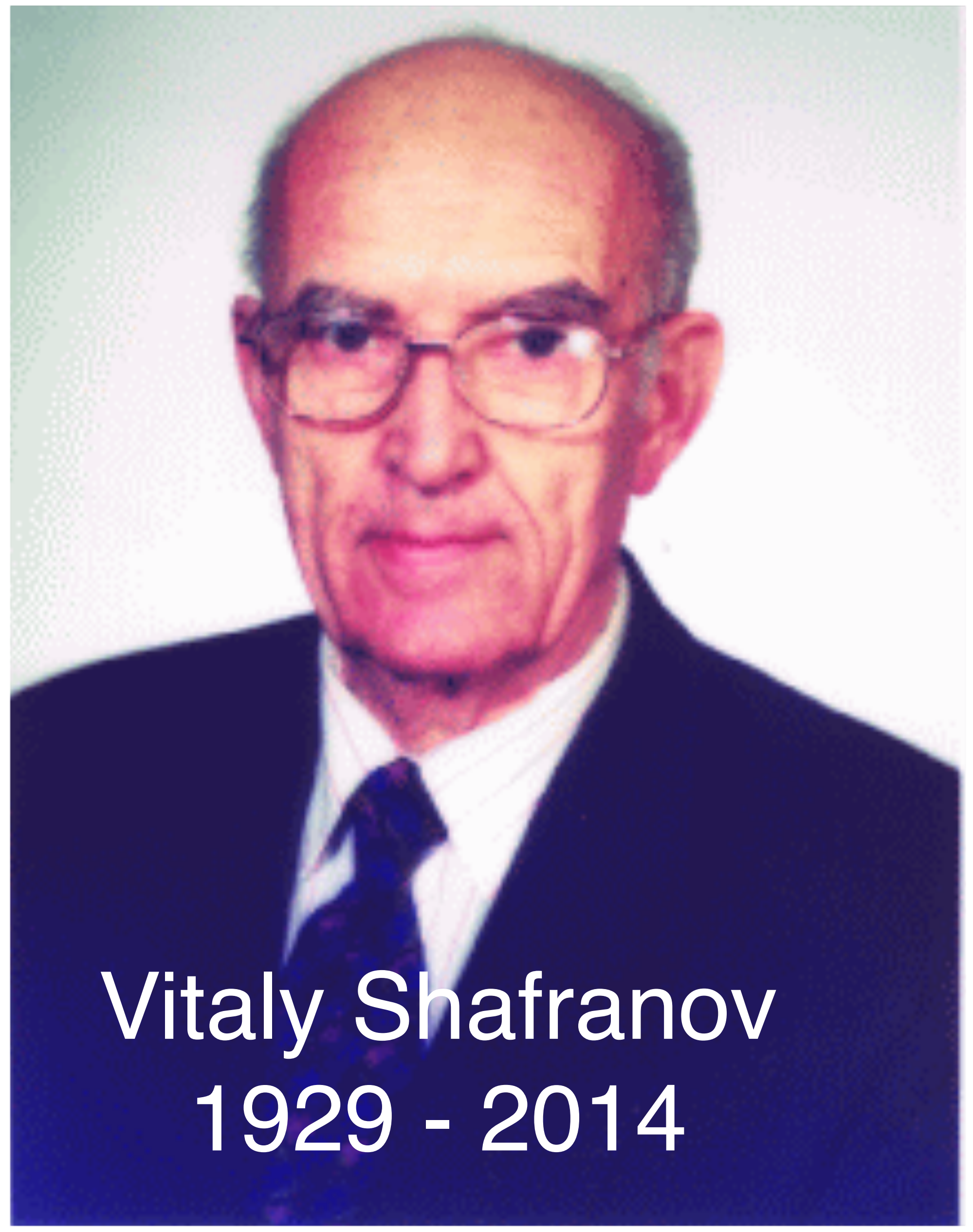
$$\text{WITH } q(r) = \frac{r B_z}{R B_p(r)} = \text{SAFETY FACTOR}$$

$\bar{B} \cdot \bar{\nabla} \to 0$ WHEN $m/n = q(r)$ (RESONANCE)

WHEN $\bar{B} \cdot \bar{\nabla} \neq 0$, THEN IDEAL REDUCED MHD MAKE SENSE

$B \cdot \bar{\nabla} = 0$, THEN REDUCED MHD DOES NOT DESCRIBE DYNAMICS

(SIDE BAR! $\bar{B} \cdot \bar{\nabla} = 0$ DEFINES "INTERCHANGE" MODES. THESE ARE THE DOMINANT MODES IN MAGNETOSPHERES AND DIPOLES, ETC)



Vitaly Shafranov

Professor Vitaly D Shafranov is widely known as one of the acknowledged leaders of the world's scientific community in plasma physics and controlled fusion. His theoretical research on plasma equilibrium and stability made an outstanding contribution to the physics of magnetically confined toroidal plasmas which plays a decisive role for the problem of magnetic fusion.

The first work by V D Shafranov, in co-authorship with M A Leontovich, *On the stability of a flexible conductor in the presence of a magnetic field* (1952) determined the goal for the experiments initiated by A D Sakharov's proposal on the confinement of a plasma by both the toroidal magnetic field and the inductive electric current, which later led to tokamaks.

He was the first to explain the stability of the tokamak plasma against perturbations with high azimuthal modes (1970) and to give a theory for the determination of certain internal equilibrium parameters in tokamaks by external magnetic measurements. Shafranov's review, written in co-authorship with V S Mukhovatov (*Nuclear Fusion* 1971), which became a handbook on tokamak systems, was very important for world tokamak research.

https://iopscience.iop.org/article/10.1088/0741-3335/43/12A/002

First: Equilibrium

$$V_\perp = 0, \quad \frac{\partial}{\partial t} = 0, \qquad 0 = -\nabla P + J \times B$$
$$= -\nabla P + \bar{J} \times (\hat{z} \times \nabla \psi)$$
$$= -\nabla P - J_z \nabla \psi$$

ALL EQUILIBRIUM VARIATION IS RADIAL, IN $\bar{\nabla}\psi$ DIRECTION

SO

$$\frac{\nabla\psi \cdot \nabla P}{(\nabla \psi)^2} = -J_z \quad \Longrightarrow \quad \frac{\partial P}{\partial \psi} = -J_z$$

($=$ CONSTANT) WHEN J_z = CONSTANT SHAFRANOV'S

$$q(r) = \frac{r B_z}{R B_p(R)}$$

$$\mu_0 J_z = \nabla \times B_p = \nabla_\perp^2 \psi = -\mu_0 \frac{\partial P}{\partial \psi}$$

EQUILIBRIUM EQUATION FOR $\psi(r)$

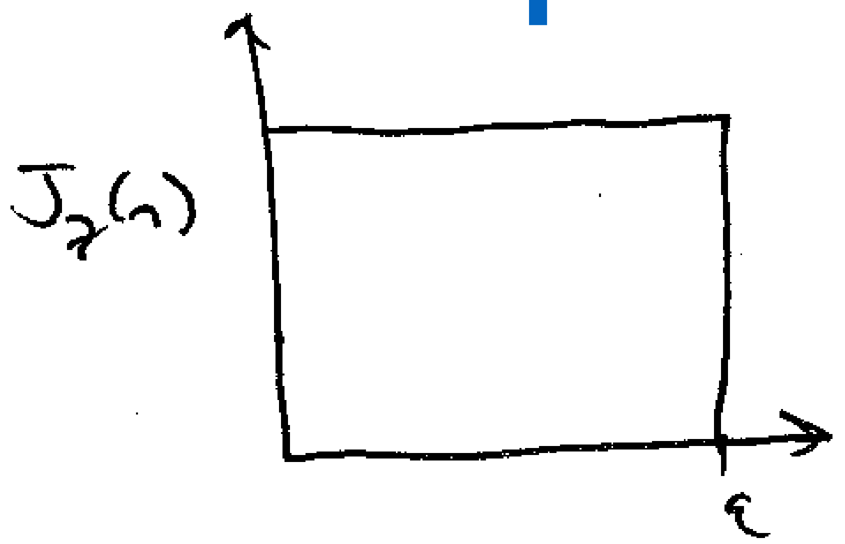
$$= \frac{1}{r}\frac{\partial}{\partial r}(r B_p)$$
$$= \frac{B_z}{rR}\frac{\partial}{\partial r}\left(r^2/q(r)\right)$$

EQUILIBRIUM FUNCTIONS

$P(\psi)$, $J_z(\psi)$, $q(\psi)$

$P(r)$, $J_z(r)$, $q(r)$

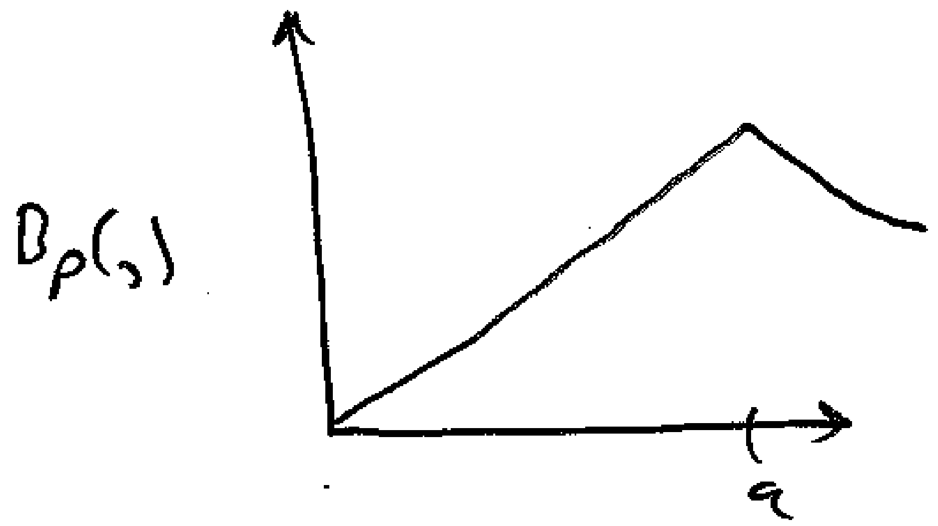
Step 1: Equilibrium (Shafranov's Simplest Case)



CONSTANT J_z

$\ell_i = \frac{1}{2}$

$B_p(a) = \frac{a\mu_0 J_z}{2}$ $\qquad B_p(r) = \frac{r\mu_0 J_z}{2}$



$q(r) = q(a) = $ CONSTANT **

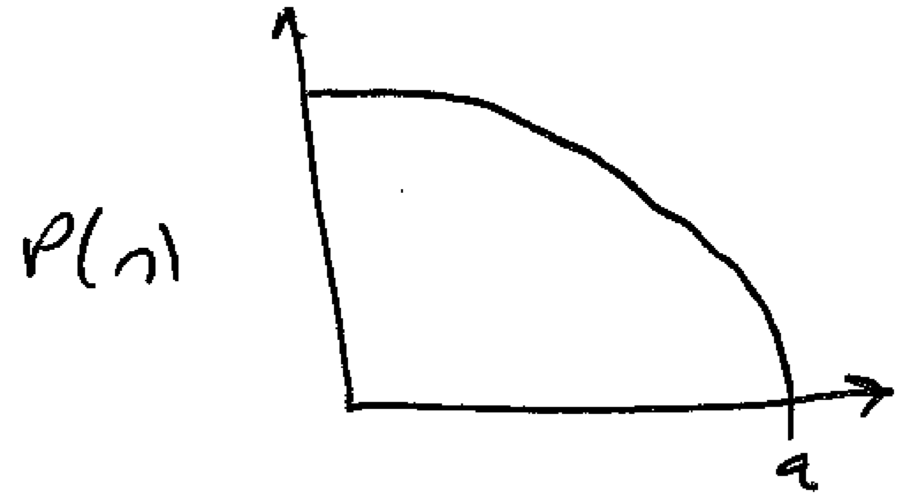
$= \frac{2B_z}{\mu_0 R J_z}$

$B_p(r) = \hat{z} \times \nabla\psi$

OR

$\psi(r) = \frac{r^2 B_p(a)}{2a} = \frac{1}{2} B_p(r)$

$= \frac{r^2}{a^2}\left(\frac{aB_p}{2}\right)$



$\frac{\partial P}{\partial\psi} = -J_z = -\frac{2B_p(a)}{a\mu_0}$

OR

$P(\psi) = (\psi(a) - \psi(r))\frac{2B_p(a)}{a\mu_0}$

$= (1 - r^2/a^2)\frac{B_p^2(a)}{\mu_0}$

$\langle\beta\rangle = \frac{2\pi R\int_0^a 2\pi r\,dr\,P(r)}{2\pi R\,\pi a^2\,B_z^2/2\mu_0}$

$= \frac{B_p^2(a)}{B_z^2} = \epsilon^2/q^2(a) \ll 1$

$\ell_i \equiv \frac{\text{POLOIDAL FIELD ENERGY}}{\frac{1}{4}\mu_0 R I_p^2}$

$= \frac{2\pi R\int_0^a 2\pi r\,dr\,\frac{B_p^2(r)}{2\mu_0}}{\frac{1}{4}\mu_0 R\left(\pi^2 a^4 J_z^2\right)} = \frac{1}{2}$

$\beta_p = \langle\beta\rangle\frac{B_z^2}{B_p^2} = 1 \quad (**)$ $\qquad \langle\beta_N\rangle \equiv \frac{\langle\beta\rangle\,\%}{I/aB\ (\text{MKS})}\ (\text{MA}) = \left(20/q(a)\right)\left(\frac{a}{R}\right)$

PLASMA IS NOT DIAMAGNETIC

IS NOT PARAMAGNETIC

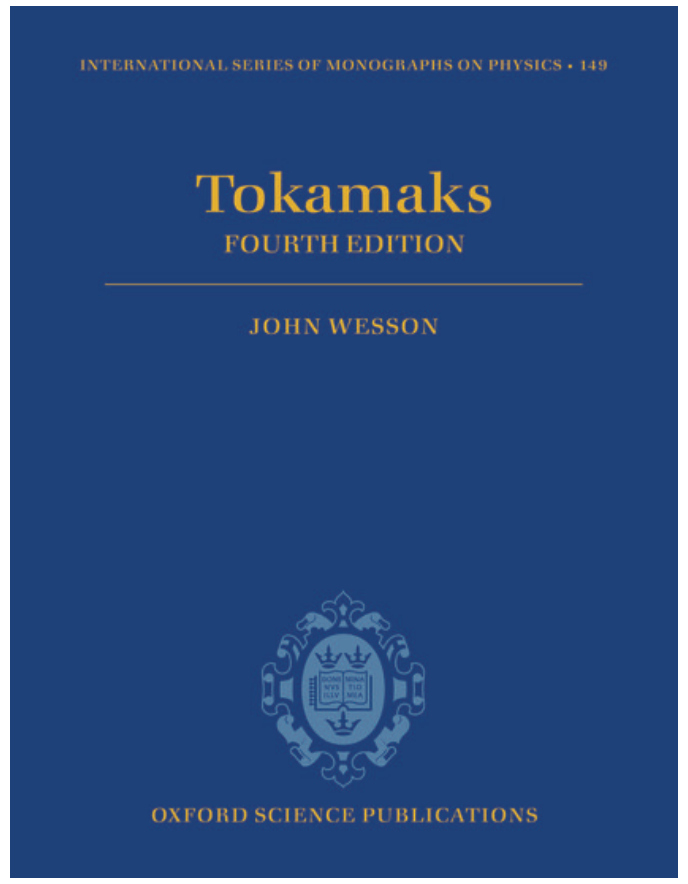
NO CHANGE IN B_z

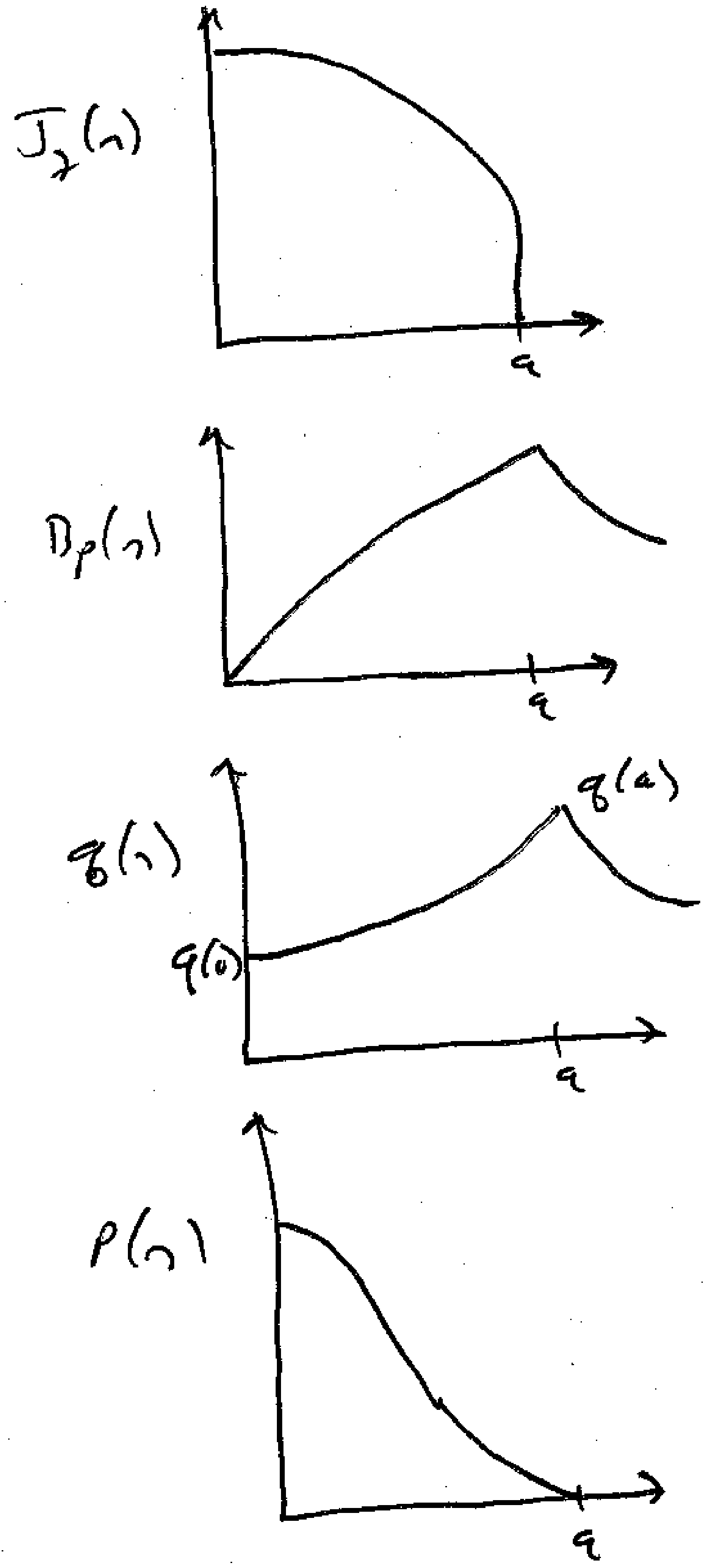
NOT TINY (OF ORDER 1)

Wesson's Cylindrical Equilibrium ($dJ_0/dr \neq 0$)

J.A. Wesson 1978 *Nucl. Fusion* 18 87; http://doi.org/10.1088/0029-5515/18/1/010

John Weston
1932 - 2020





$J_z(r) = J_0\left(1 - r^2/a^2\right)^{\nu}$

J_0 = CENTRAL CURRENT DENSITY

$q(0) = \dfrac{2B_z}{\mu_0 R J_0}$ $\quad \mu_0 I_p = \dfrac{2\pi^2 a^2 B_z}{\mu_0 q(0) R(1+\nu)}$

$\nu = \dfrac{q(a)}{q(0)} - 1$

$\ell_i \neq \frac{1}{2}$

$\langle\beta\rangle \sim \epsilon^2/q_a^2$ $\quad \beta_p \sim 1$ $\quad \langle B_r\rangle \sim 20\,\epsilon/q_a$

EQUILIBRIUM SET BY $q(0), q(a)$
<u>TWO</u> PARAMETERS

$\dfrac{r}{q}\dfrac{dq}{dr} = s =$ MAGNETIC SHEAR ** $\neq 0$

PRESSURE PROFILE CAN BE MORE PEAKED

$B\cdot\nabla \to 0$ AT <u>INTERNAL</u> RESONANT LAYER

$\left(m - n/q(r)\right) \approx 0$

Linearized Reduced ("*Ideal*") MHD

MHD: $\rho \frac{d}{dt} \nabla_\perp^2 \chi = \frac{1}{\mu_0} (\bar{B}\cdot\nabla) \nabla_\perp^2 \psi$

$\frac{d}{dt} = -i\omega$

INDUCTION: $\frac{d\psi}{dt} = (B\cdot\nabla)\chi$

$B_0\cdot\nabla \to i \frac{B_p(r)}{r}(m - n q(r))$

$$\boxed{(\bar{B}\cdot\bar{\nabla})\nabla_\perp^2\psi = (\tilde{B}\cdot\nabla)\underbrace{\nabla_\perp^2\psi_0}_{\mu_0 J_z(r)} + (\bar{B}_0\cdot\bar{\nabla})\nabla_\perp^2\tilde{\psi}}$$

$+$ NONLINEAR TERMS

$\mu_0 J_z(r)$ ← WHEN EQUILIBRIUM CURRENT DENSITY VARIES WITHIN PLASMA

$\tilde{B}\cdot\nabla J_z(r) = (\hat{z}\times\nabla\psi)\cdot_r \frac{\partial J_z}{\partial r} = -\hat{\theta}\cdot\nabla\psi \frac{\partial J_z^0}{\partial r}$

LINEAR MHD:

$$\boxed{-\underbrace{\rho\,\omega\,\nabla_\perp^2\tilde{\chi}}_{\text{PLASMA INERTIA}} = -\frac{m}{r}\frac{\partial J_z}{\partial r}\tilde{\psi} + \frac{B_p}{\mu_0 r}(m - nq(r))\nabla_\perp^2\tilde{\psi}}$$

LINEAR INDUCTION: $-\omega\tilde{\psi} = \frac{B_p}{r}(m - nq(r))\tilde{\chi}$

Linearized Reduced ("*Ideal*") MHD

MHD: $\rho \frac{d}{dt} \nabla_\perp^2 \chi = \frac{1}{\mu_0} (\vec{B}\cdot\nabla)\nabla_\perp^2 \psi$

INDUCTION: $\frac{d\psi}{dt} = (B\cdot\nabla)\chi$

$\frac{d}{dt} = -i\omega$

$B_0\cdot\nabla \to i \frac{B_p(r)}{r}(m - n q(r))$

$$
\begin{aligned}
-\rho\omega\nabla_\perp^2\chi &= -\frac{m}{r}\frac{\partial J_{z,0}}{\partial r}\tilde{\psi} + \frac{B_p}{\mu_0 r}(m - nq)\nabla_\perp^2\tilde{\psi} \\
-\omega\tilde{\psi} &= \frac{B_p}{r}(m - nq)\chi\,.
\end{aligned}
$$

If we continue to assume that ρ is constant except for a sharp discontinuity at the edge, then the jump condition is

$$
\omega\rho \left.\frac{\partial\chi}{\partial r}\right|_{a^-} = \frac{B_p(a)}{\mu_o a}(m - nq_a)\tilde{\psi}_a \boxed{\Delta'(a)},
$$

since $J_{z,0}(a) = 0$ at the edge.

Alfvén Waves in Shafranov's Equilibrium ($dJ_0/dr \sim 0$)

WITHIN PLASMA

$$-\rho\omega \nabla_\perp^2 \hat{\chi} = \frac{B_p}{\mu_0 r}(m - nq)\nabla_\perp^2 \hat{\varphi}$$

$$-\omega \hat{\varphi} = \frac{B_p}{\mu_0 r}(m - nq)\chi$$

NOTE! $\frac{B_p}{r}$ = CONSTANT = $\frac{B_p(a)}{a}$

$$\omega_A^2 \equiv \frac{B_p^2/\mu_0\rho}{a^2} = \frac{B_z^2/\mu_0\rho}{(q_a R)^2}$$

$$= V_A^2/(q R)^2$$ ALFVEN TRANSIT TIME!

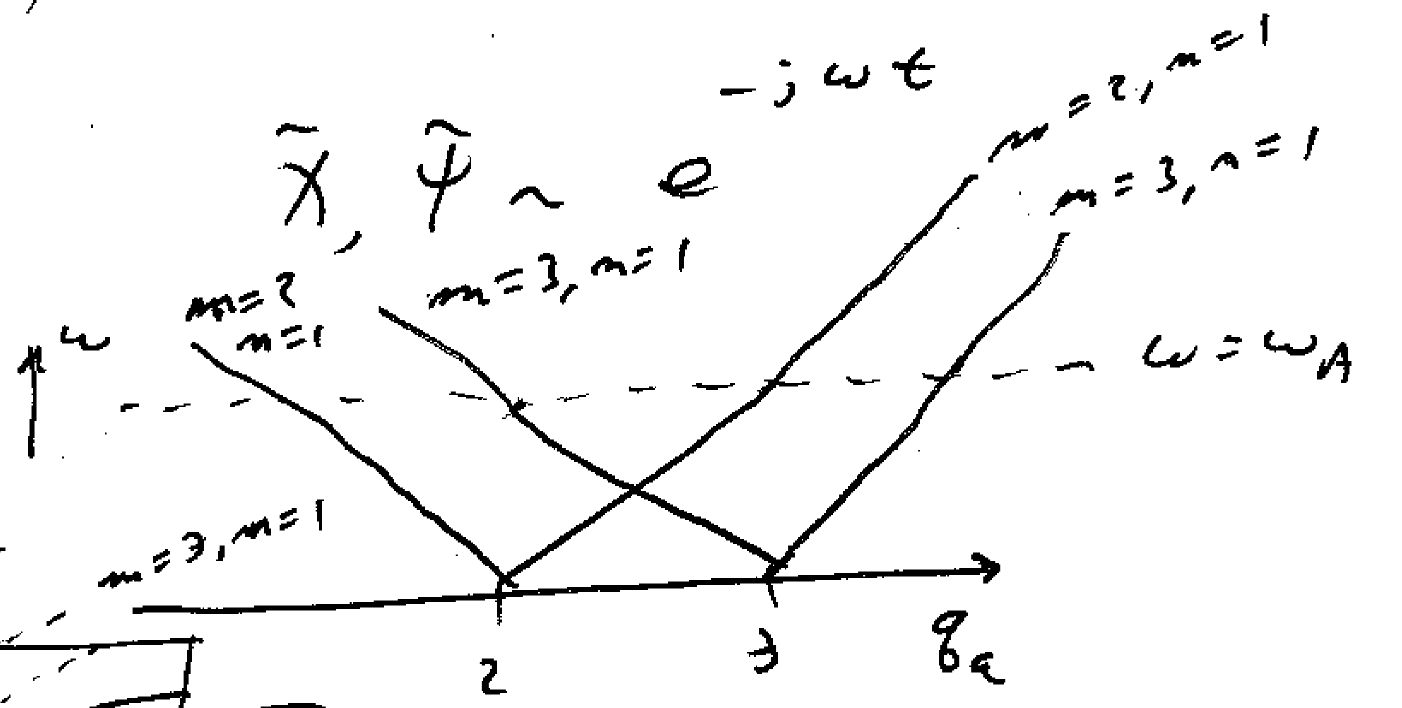
NORMAL MODES (ALFVEN WAVES)

$$\begin{pmatrix} \rho\omega & \frac{B_p}{\mu_0 a}(m-nq) \\ \frac{B_p}{\mu_0 a}(m-nq) & \omega \end{pmatrix} \cdot \begin{pmatrix} \hat{\chi}(r) \\ \tilde{\varphi}(r) \end{pmatrix}$$

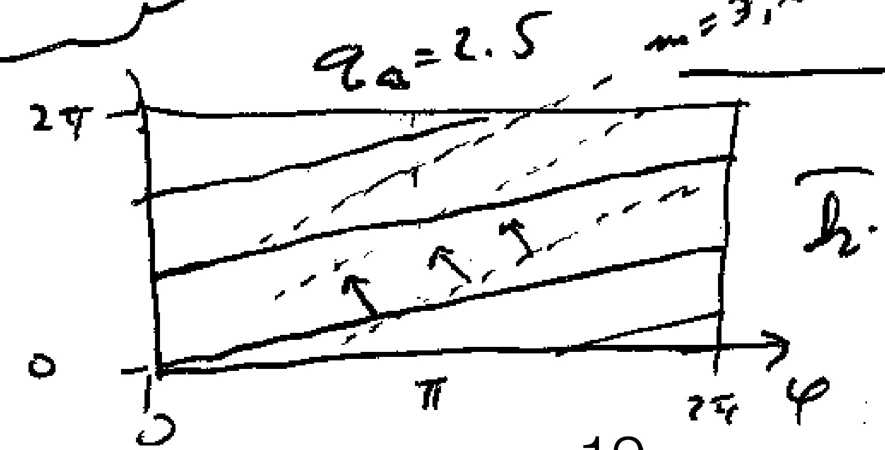
with $\rho \simeq$ CONSTANT

$$\tilde{\chi}, \tilde{\varphi} \sim e^{-j\omega t}$$

$$\omega^2 = \omega_A^2 (m - nq_a)^2$$



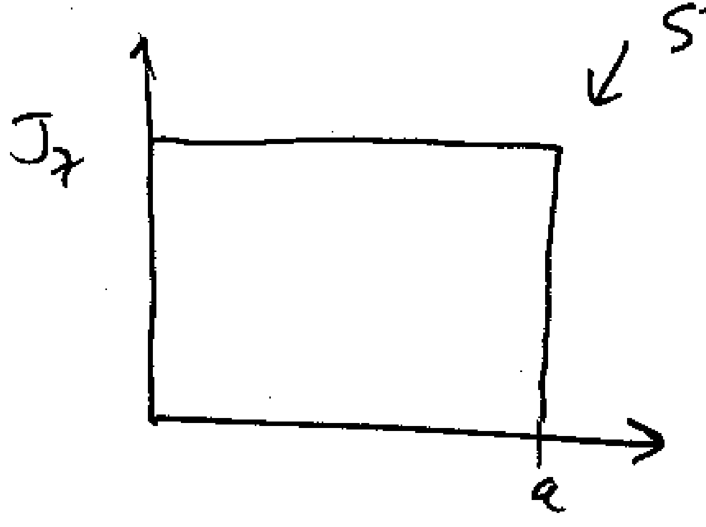
RADIAL STRUCTURE NOT SPECIFIED



$\vec{k} \cdot \vec{B}_0 \neq 0$

Global Kink Eigenmodes ($dJ_0/dr \sim 0$, except edge!)

$$\nabla_\perp \cdot \vec{A} = \frac{1}{r}\frac{\partial}{\partial r}(r A_r) + \cdots$$



STEP DISCONTINUITY AT PLASMA EDGE

$$\frac{d}{dt}\nabla_\perp\cdot(\rho \nabla_\perp \chi) = (\hat z \times \nabla_\perp \tilde\psi)\cdot \hat r \frac{\partial J_z}{\partial r} + i\frac{B_\theta}{\mu_0 r}(m-nq)\nabla_\perp^2\tilde\psi$$

VERY BIG AT EDGE

$$\int_{a^-}^{a^+} 2\pi r\,dr \left\{ \text{" MHD "} \right\} \Rightarrow$$

$$\omega\rho \frac{\partial\chi}{\partial r}\Big|_{a^-} = \frac{2m B_\theta(a)}{\mu_0 a^2}\tilde\psi_a + \frac{B_\theta(a)}{\mu_0 a}(m-nq)\tilde\psi_a\underbrace{\left[\frac{1}{\tilde\psi}\frac{\partial\tilde\psi}{\partial r}\Big|_{a^+} - \frac{1}{\tilde\psi}\frac{\partial\tilde\psi}{\partial r}\Big|_{a^-}\right]}$$

$$\boxed{\Delta'(a) \equiv \frac{1}{\psi}\left(\frac{\partial\tilde\psi}{\partial r}\Big|_{a^+} - \frac{\partial\psi}{\partial r}\Big|_{a^-}\right)}$$

THIS MEASURES PERTURBED SURFACE CURRENT ON PLASMA

IMPORTANT:

$$\omega\rho\frac{\partial\chi}{\partial r}\Big|_{a^-} = \frac{2mB_\theta(a)}{\mu_0 a^2}\tilde\psi_a\left[(m-nq)\left(\frac{-\Delta'(a)}{2m/a}\right) - 1\right]$$

$$\Delta'(a)\tilde\psi_a = \mu_0 \tilde K_z(\theta,\phi)/a$$

(K = Surface current)

Global Kink Eigenmodes ($dJ_0/dr \sim 0$, except edge!)

WHAT ARE $(\psi(r), \chi(r))$ <u>INSIDE</u> AND <u>OUTSIDE</u> PLASMA?

Boundary Matching Conditions

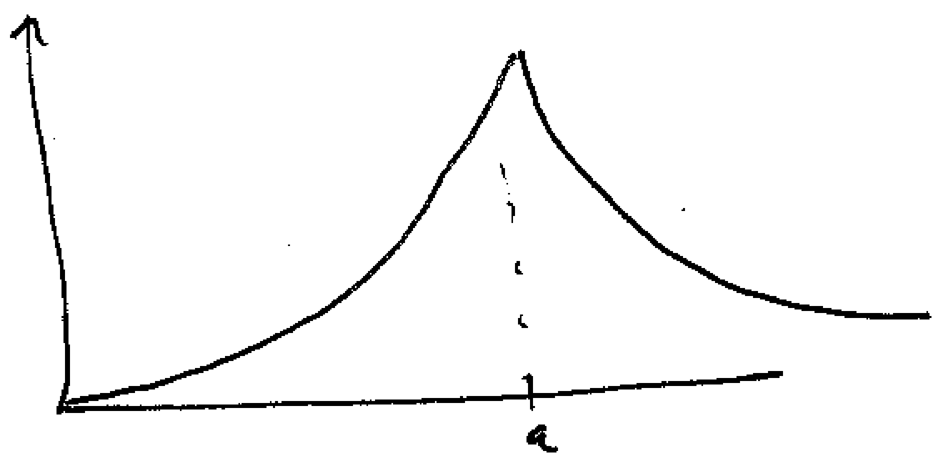
$\nabla^2 \psi = 0$ (NO CURRENT) OUTSIDE PLASMA

$\nabla^2 \chi = 0$ (NO FLOW, VORTICITY, NO SLIP)

$\nabla^2 \psi = 0$ (NO CURRENTS INSIDE PLASMA TOO)

$\nabla^2 \chi = 0$ (NO VORTICITY <u>WITHIN</u> PLASMA)

PERTURBED FIELDS + PLASMA MOTION
BUT <u>NO</u> CURRENTS OR VORTICITY



<u>NO WALL</u>

$\psi(r) \sim \left(\frac{r}{a}\right)^m \quad r<a$

$\sim \left(\frac{a}{r}\right)^m \quad r>a$

WITH WALL

$\sim \left(\frac{r}{a}\right)^m \quad r<a$

$\sim \dfrac{\left(\frac{b}{r}\right)^m - \left(\frac{r}{b}\right)^m}{\left(\frac{b}{a}\right)^m - \left(\frac{a}{b}\right)^m} \quad (a<r<b)$

$V_\perp = \hat{z} \times \nabla \chi = \hat{\theta} \frac{m}{r}\left(\frac{r}{a}\right)^m - \hat{r} \frac{im}{r}\left(\frac{r}{a}\right)^m$ INSIDE

$= -\hat{\theta} \frac{m}{r}\left(\frac{a}{r}\right)^m - \hat{r} \frac{im}{r}\left(\frac{a}{r}\right)^m$ OUTSIDE

Kink Mode ($dJ_0/dr \sim 0$)

$$-\omega \tilde{\varphi}_a = \frac{B_p}{a}(m-nq)\tilde{\chi}_a$$

$$\omega\rho \frac{\partial \chi}{\partial r}\Big|_{a^-} = \frac{2mB_p}{\mu_s a^2}\tilde{\varphi}_a\left[(m-nq)\left(\frac{-\Delta'(a)}{2m/a}\right)-1\right]$$

$$\Delta'(a) = -\frac{2m}{a}\frac{(b/a)^m}{(b/a)^m-(a/b)^m} \equiv -\frac{m}{a}(\Lambda+1)$$

$$\Delta'(a) = -\frac{2m}{a}\frac{(b/a)^m}{(b/a)^m-(a/b)^m}$$ FOR A WALL AT $r=b$

$$\frac{\tilde{\chi}_a}{\tilde{\varphi}_a} = -\frac{\omega q_a R}{B_0(m-nq_a)}$$

EIGENVECTOR

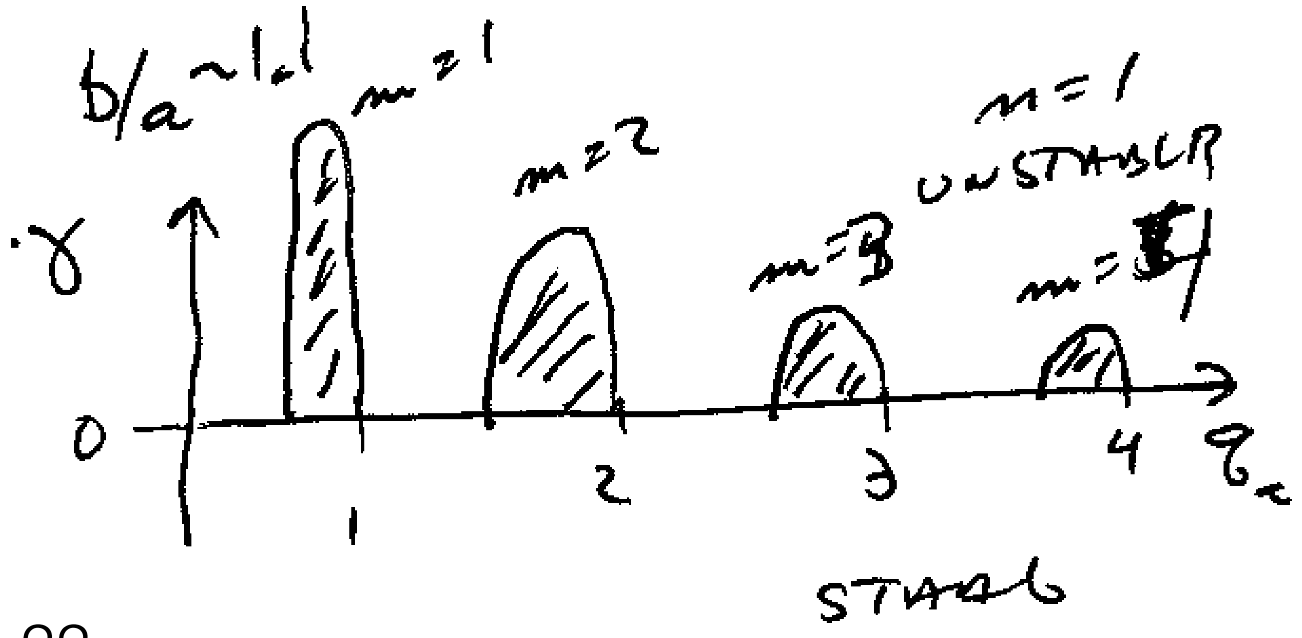
$$\frac{\partial \chi}{\partial r}\Big|_{a^-} = -\frac{m}{a}\tilde{\chi}_a$$

EIGENVALUE

$$\omega^2 = 2\omega_A^2(m-nq) \times \left[(m-nq)\frac{(\Lambda+1)}{2}-1\right]$$

GLOBAL KINK MODES

$$\begin{pmatrix} \omega & \frac{B_p}{a}(m-nq) \\ \frac{2mB_p}{\mu_0 a}\left[(m-nq)\frac{(\Lambda+1)}{2}-1\right] & \omega\rho \end{pmatrix}\begin{pmatrix} \tilde{\varphi}_a \\ \tilde{\chi}_a \end{pmatrix} = 0$$



$$\boxed{\Delta'(a) \equiv -\frac{m}{a}(\Lambda+1)}$$

SHAFRANOV'S FORMULA

Kink Mode

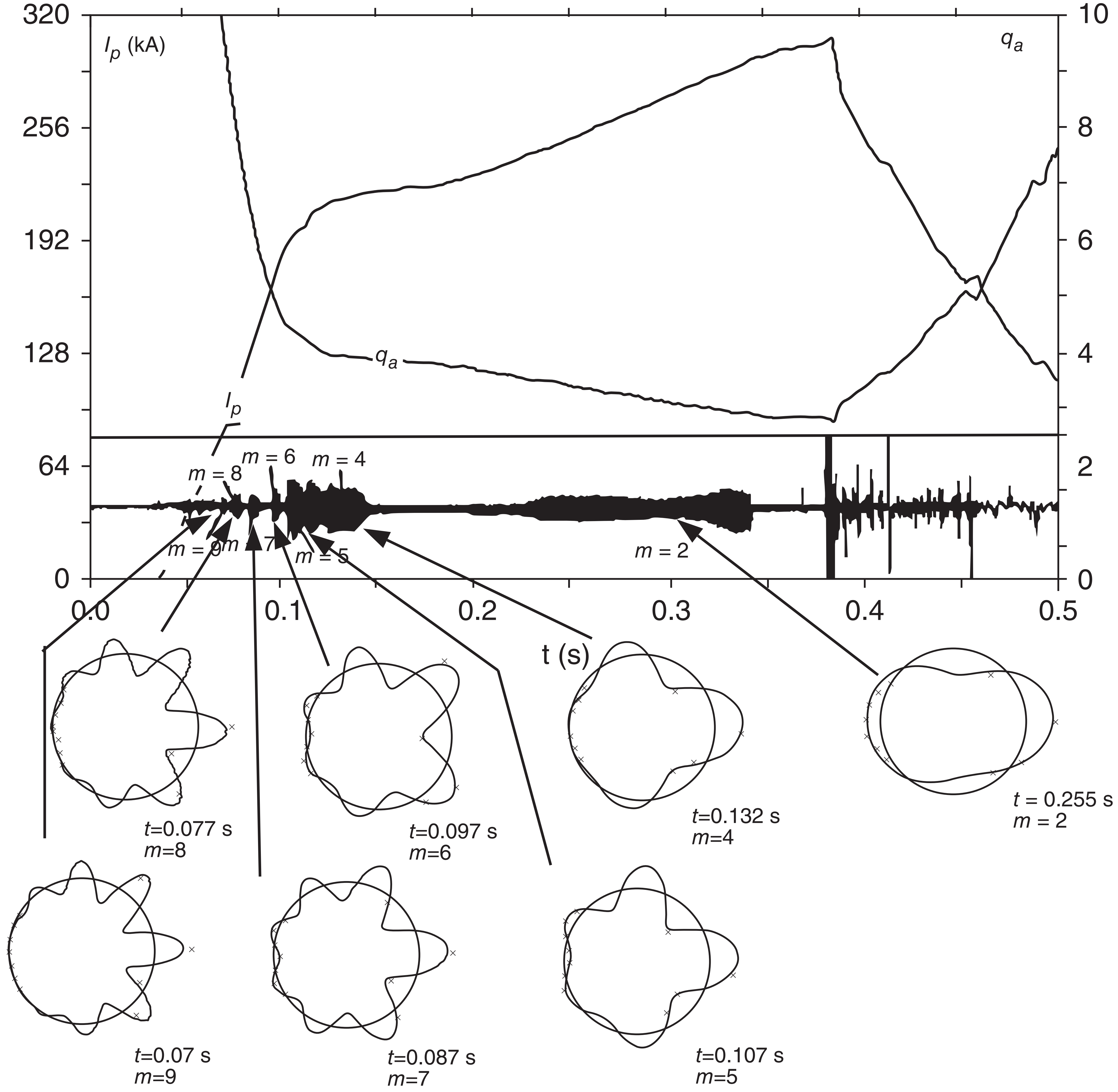
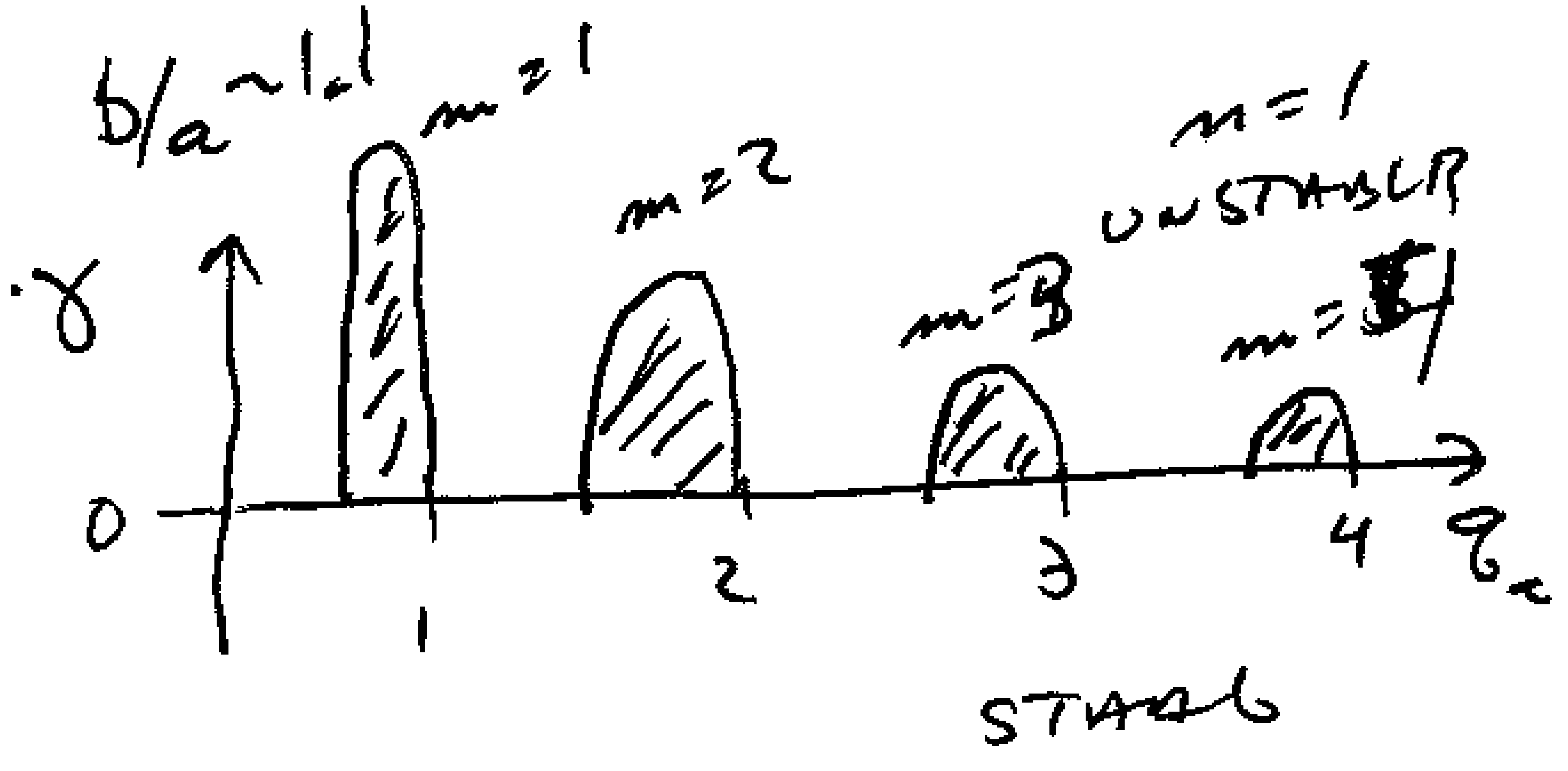


Figure 4.4 Occurrence of ideal external kinks in the current ramp up of a tokamak discharge in the ASDEX tokamak. The (2,1) mode occurring at reduced ramp rate is a resistive mode that terminates the discharge disruptively. This instability is treated in Chapter 10.

EIGENVALUE

$$\omega^2 = 2\omega_A^2 (m - nq) \times \left[(m-nq)\frac{(\Lambda+1)}{2} - 1 \right]$$

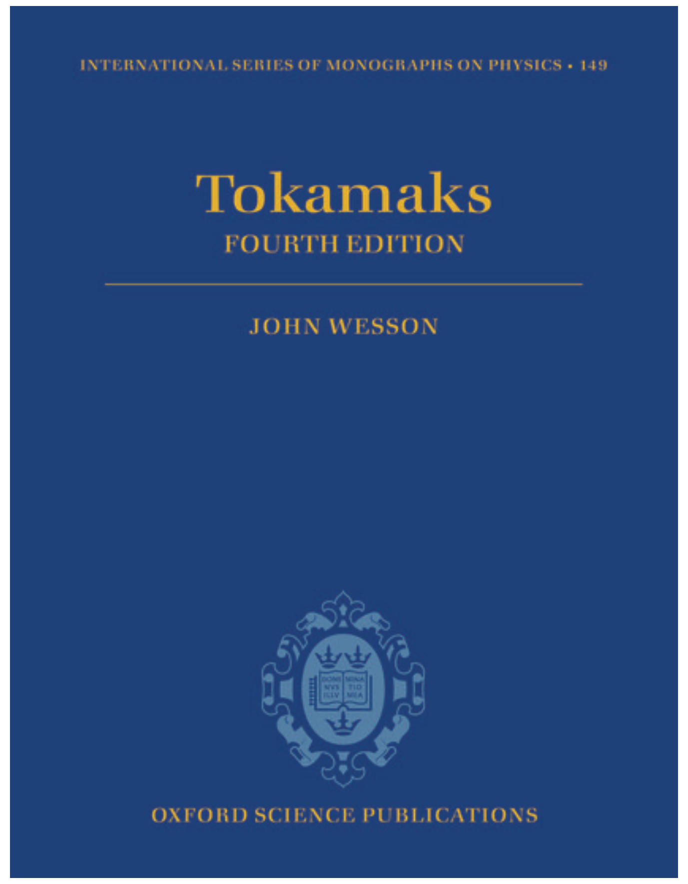


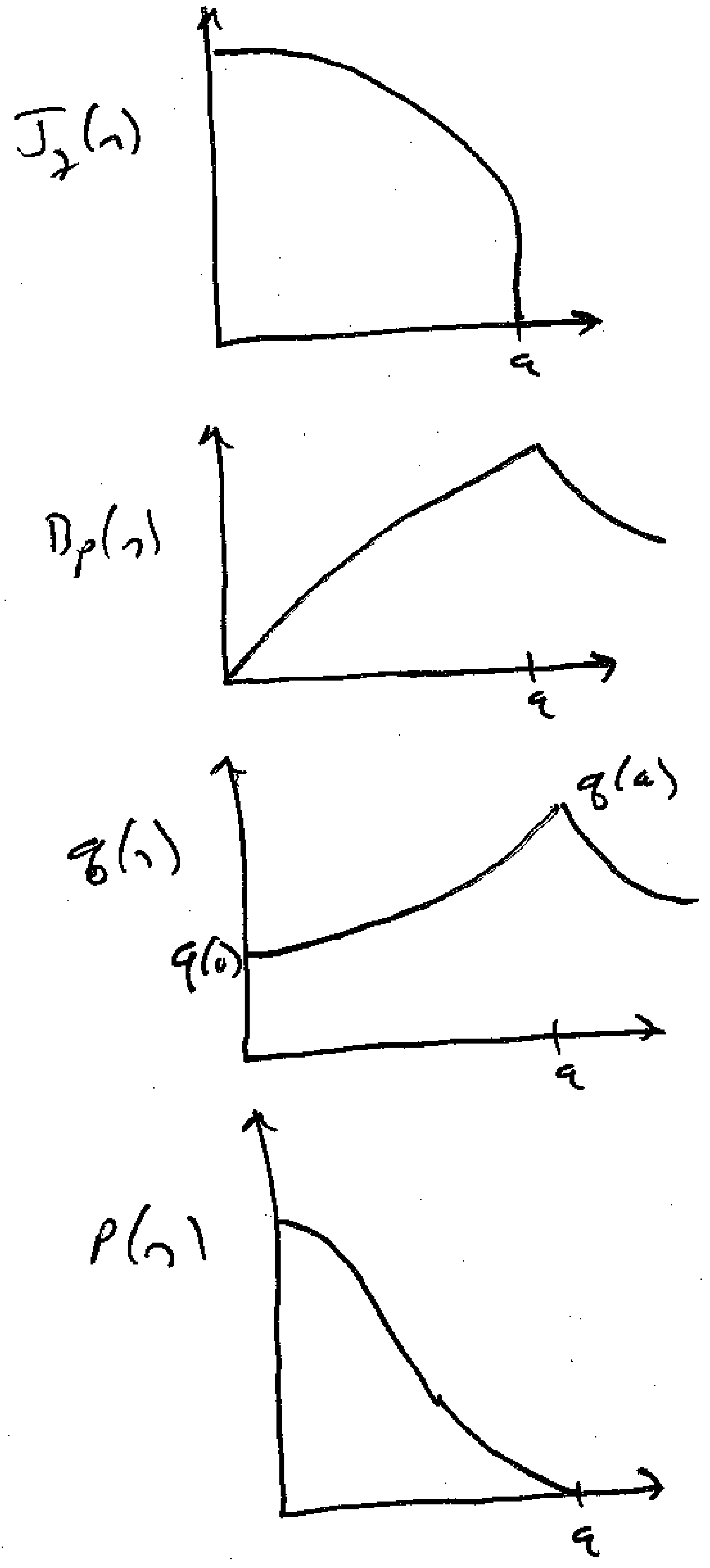
Wall position is imporant: Why?

Wesson’s Cylindrical Equilibrium ($dJ_0/dr \neq 0$)

J.A. Wesson 1978 *Nucl. Fusion* 18 87; http://doi.org/10.1088/0029-5515/18/1/010

John Weston
1932 - 2020





$J_z(r) = J_0 (1 - r^2/a^2)^\nu$

J_0 = CENTRAL CURRENT DENSITY

$q(0) = \frac{2 B_z}{\mu_0 R J_0}$ $\quad \mu_0 I_p = \frac{2\pi^2 a^2 B_z}{\mu_0 q(a) R (1+\nu)}$

$\nu = \frac{q(a)}{q(0)} - 1$

$\langle\beta\rangle \sim \epsilon^2/q_a^2 \quad \beta_p \sim 1 \quad \langle\beta_N\rangle \sim 20\,\epsilon/q_a$

$\ell_i \neq \frac{1}{2}$

EQUILIBRIUM SET BY $q(0), q(a)$
<u>TWO</u> PARAMETERS

$\frac{r}{q}\frac{dq}{dr} = s =$ MAGNETIC SHEAR
$\neq 0$

PRESSURE PROFILE CAN BE MORE PEAKED

$B \cdot \nabla \to 0$ AT INTERNAL RESONANT LAYER
$(m - n/q(r)) \approx 0$

Wesson's Kink Modes

LINEARIZED EQUATIONS FOR PERTURBED STREAM FUNCTION (χ)
AND PERTURBED POLOIDAL FLUX (ψ)

$$-\rho\,\omega\,\nabla_\perp^2\tilde{\chi} = -\frac{m}{r}\frac{\partial J_z}{\partial r}\tilde{\psi} + \frac{B_p}{\mu_0 r}(m - nq)\nabla_\perp^2\tilde{\psi} \qquad \text{(MHD)}$$

$$-\omega\tilde{\psi} = \frac{B_p}{r}(m - nq)\tilde{\chi} \qquad \text{(INDUCTION)}$$

LET'S TAKE $\rho \simeq$ UNIFORM, WITH A SHARP JUMP AT THE PLASMA'S EDGE:

$$\omega\,\rho\left.\frac{\partial\chi}{\partial r}\right|_{a^-} = \frac{B_p(a)}{\mu_0 a}(m - nq_a)\,\tilde{\psi}_a\,\Delta'(a)$$

$\Delta'(a)$ ← PERTURBED SURFACE CURRENT AT PLASMA'S EDGE

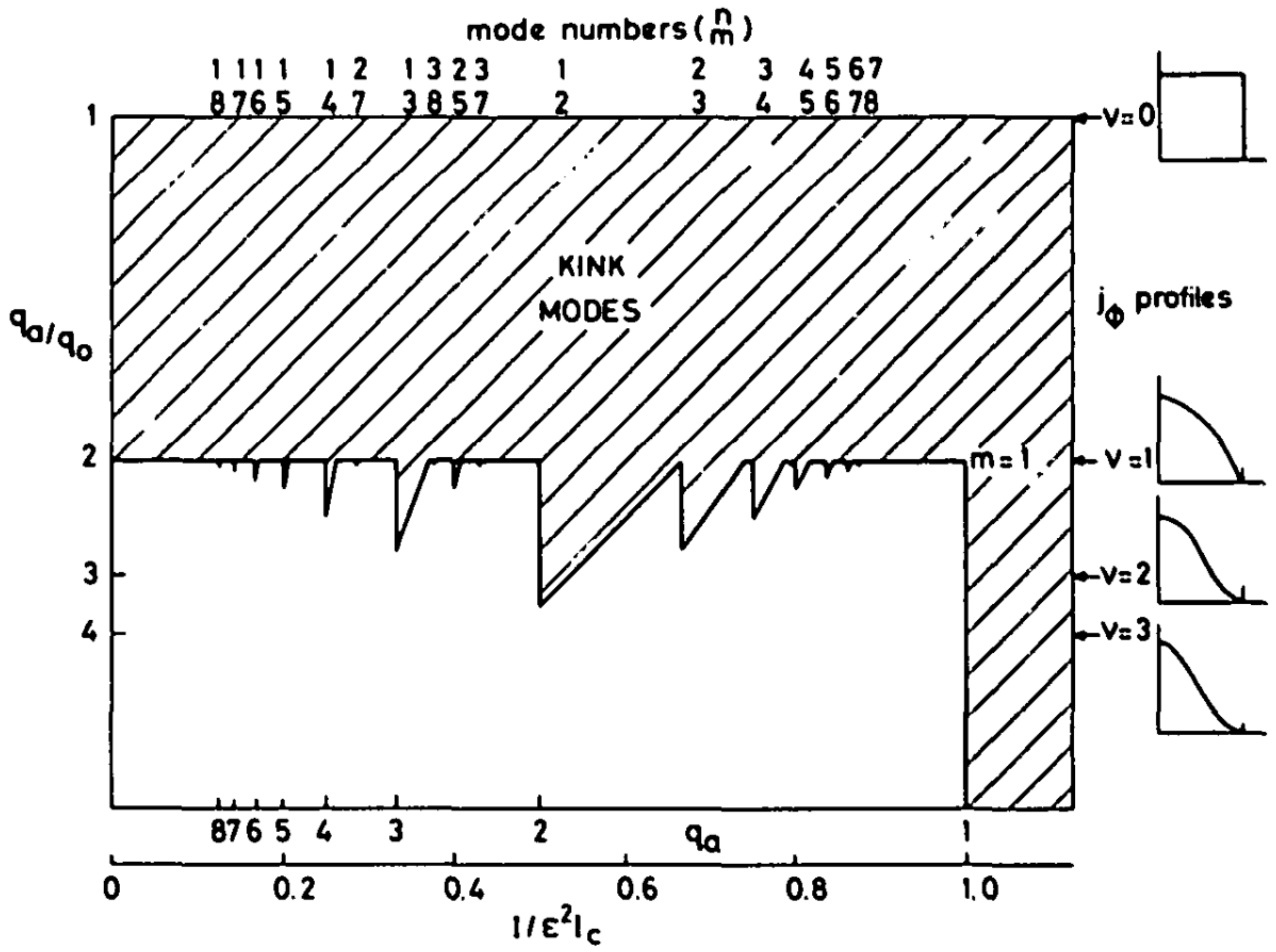
WITH INDUCTION EQUATION:

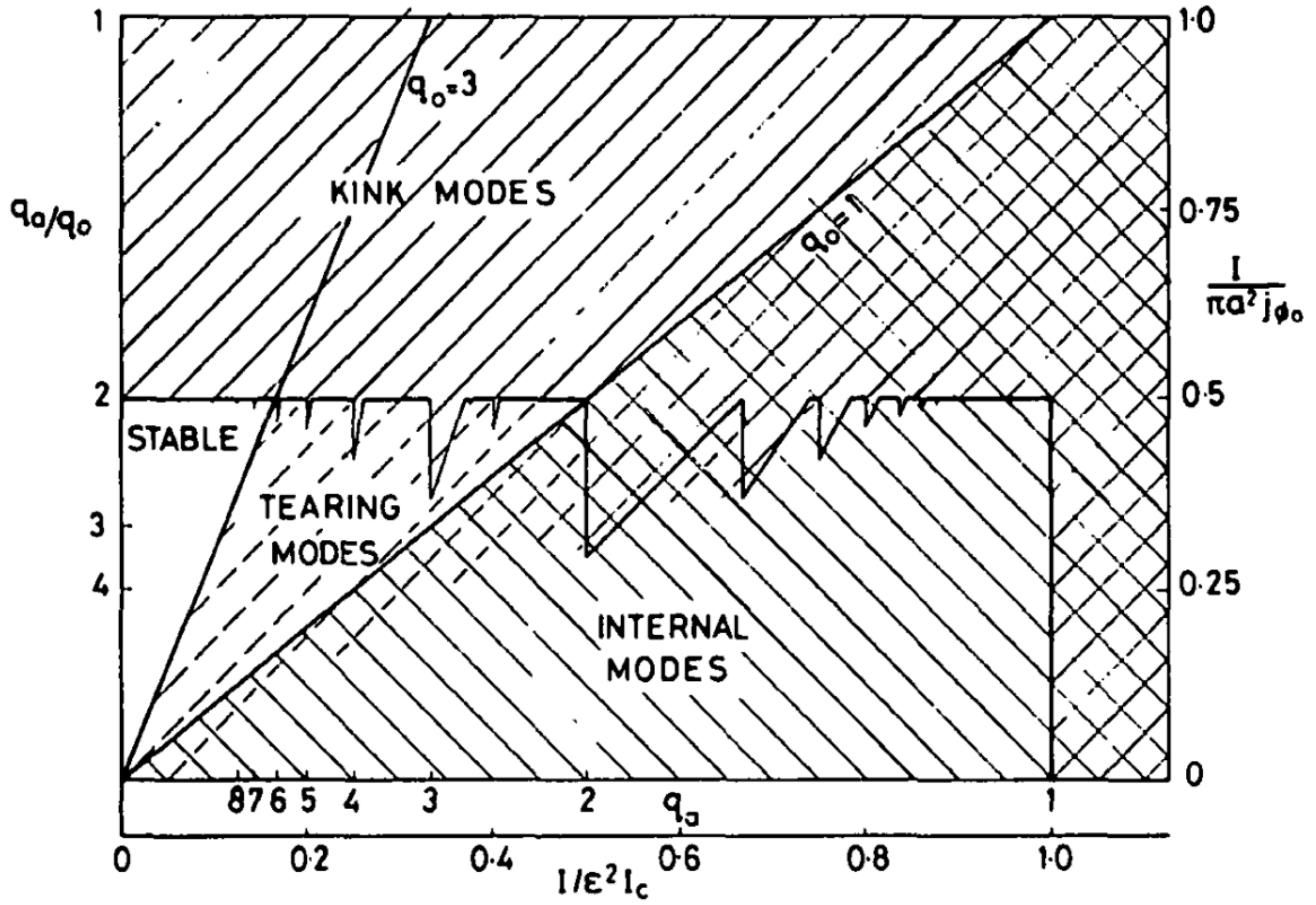
$$\omega^2 = -\omega_A^2(m - nq_a)^2\,\Delta'(a)\,\frac{\chi_a}{(\partial\chi/\partial r)_{a^-}}$$

INSTABILITY REQUIRES $\Delta'(a) > 0$

BUT, HOW TO FIGURE OUT $\tilde{\psi}(r)$?

Wesson's Kink Modes





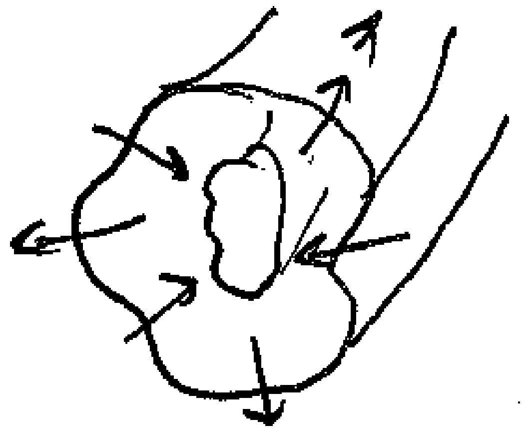
Since $|\omega| < \omega_A$, the kink mode causes the "internal" plasma to respond "quickly", so quickly that we can ignore the time it takes to form a distorted, 3D, quasi-equilibrium.

Inside, the plasma is a "forces" equilibrium

$$0 \simeq -\frac{m}{r}\frac{\partial J_z}{\partial r}\tilde{\Psi} + \frac{B_\theta}{\mu_0 r}(n - mq)\nabla_\perp^2 \tilde{\Psi}$$

Outside, the response is the "vacuum" response.

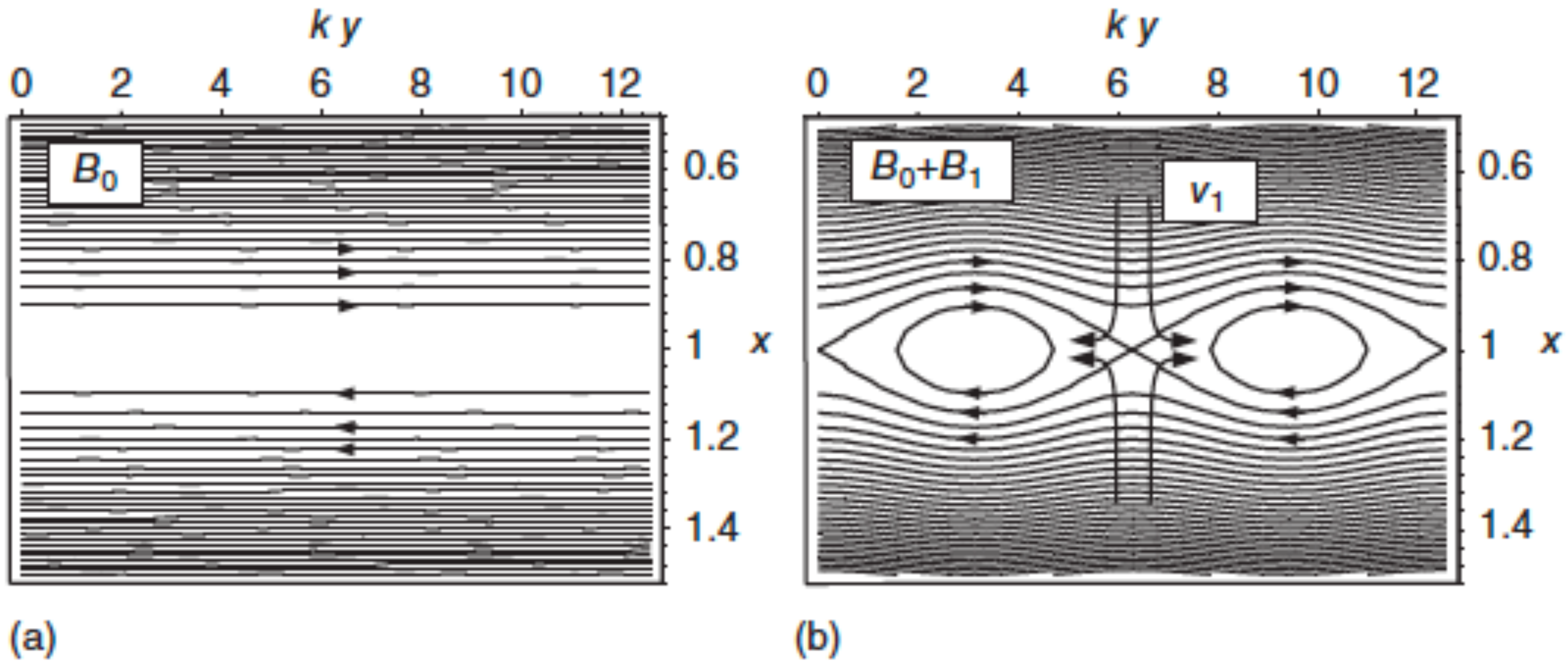
With $J_z(r)$, we have to solve for $\tilde{\Psi}$ using a computer. (This is very easy for the cylindrical "tokamak")



← The surface current "pushes"/"pulls" plasma, and the "distorted" plasma is measured by $\tilde{\Psi}(r, \theta, z)$

Next Lecture

Resistive MHD (and tearing modes)



Magnetohydrodynamic Stability of Tokamaks by Hartmut Zohm (Wiley 2014) and *Plasma physics : An introduction* by Richard Fitzpatrick (2nd ed. 2022; CRC Press).
(both online at Columbia University)