Lecture16:
Plasma Physics 1

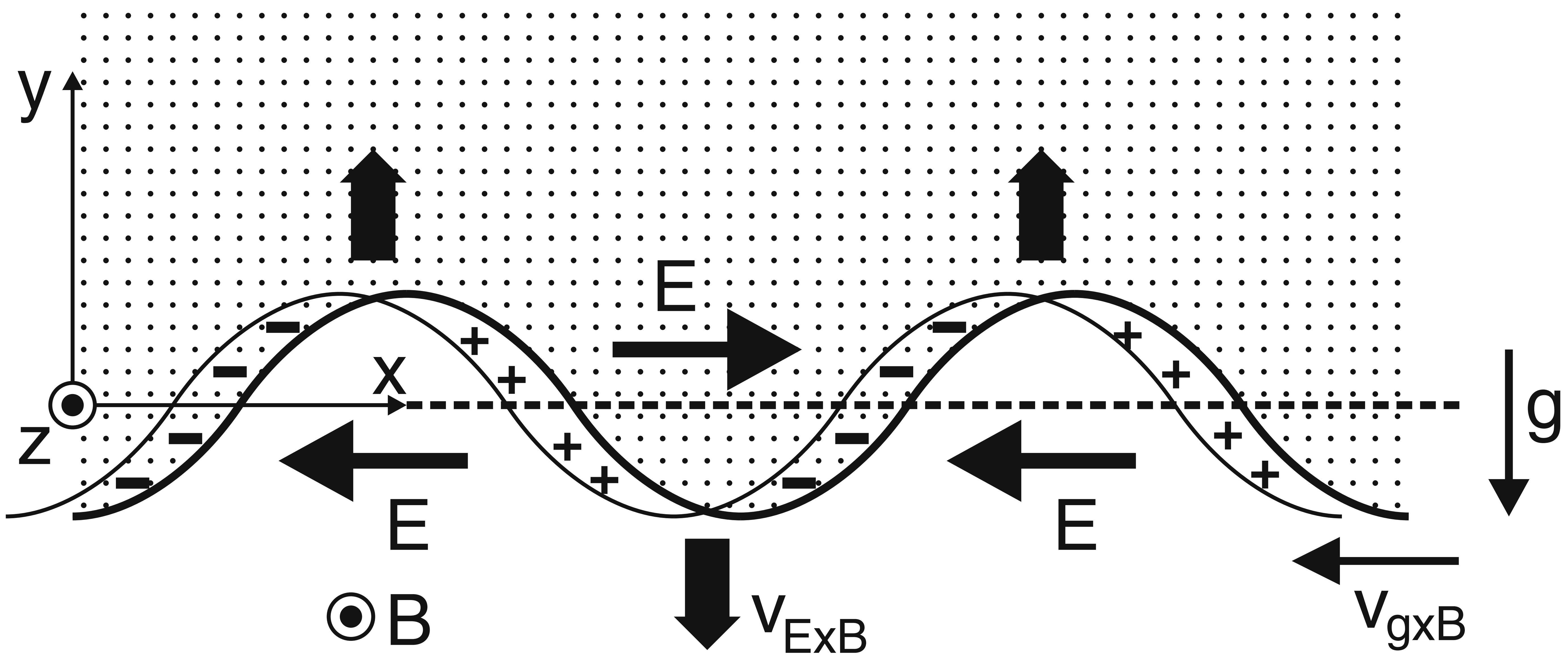
APPH E6101x
Columbia University

Last lecture: Beam-Plasma Instability
This lecture: MHD "Macroinstabilities"

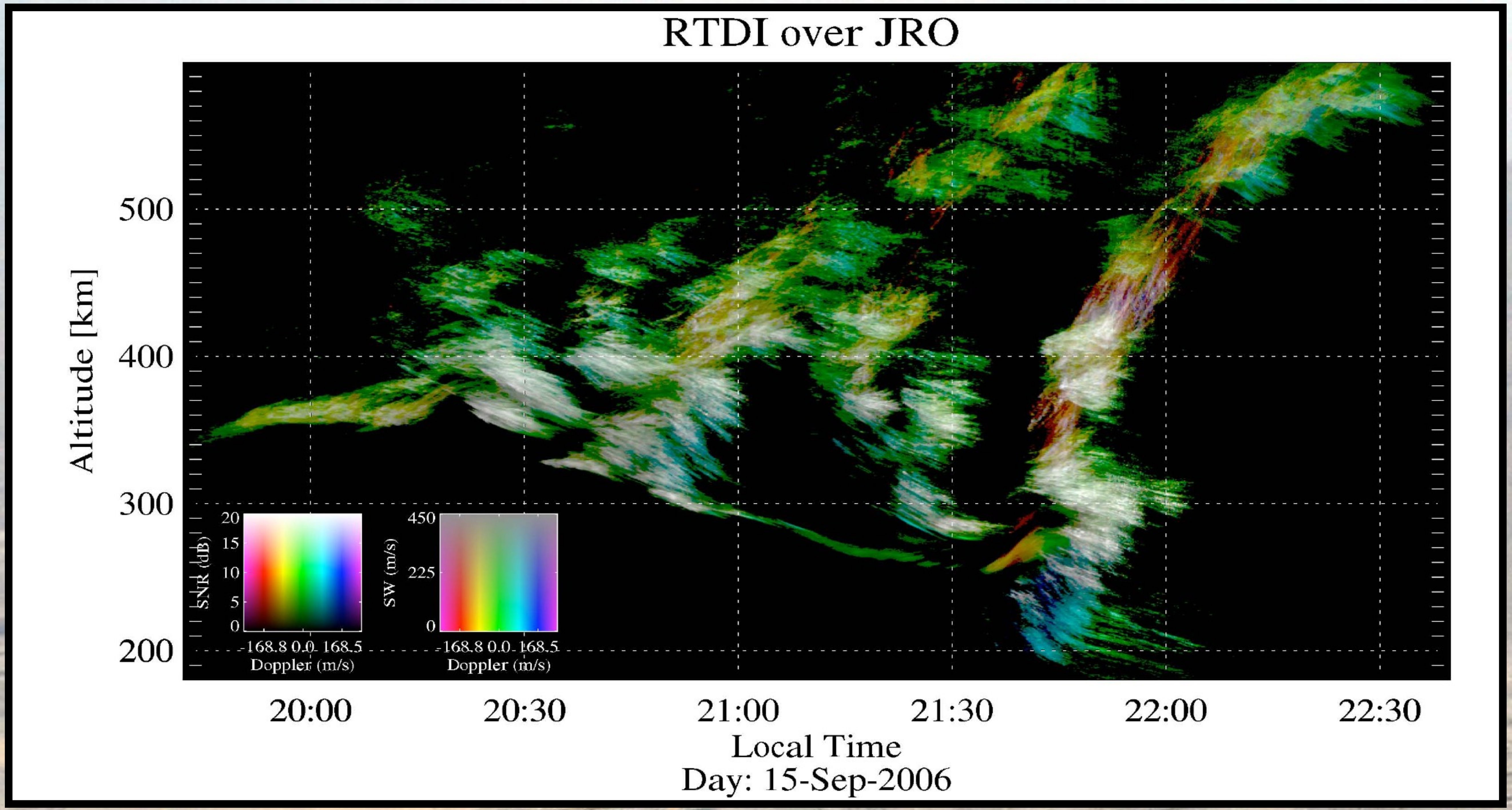
Topics

- Interchange Modes
- Magnetized Plasma Instabilities:
  - Pressure-Curvature Driven
  - Parallel-Current Driven
- Reduced MHD (plasma torus with a strong toroidal field)
- Kink modes

Rayleigh-Taylor Instability



Equatorial Spread-F



Jicamarca Radio Observatory (JRO)

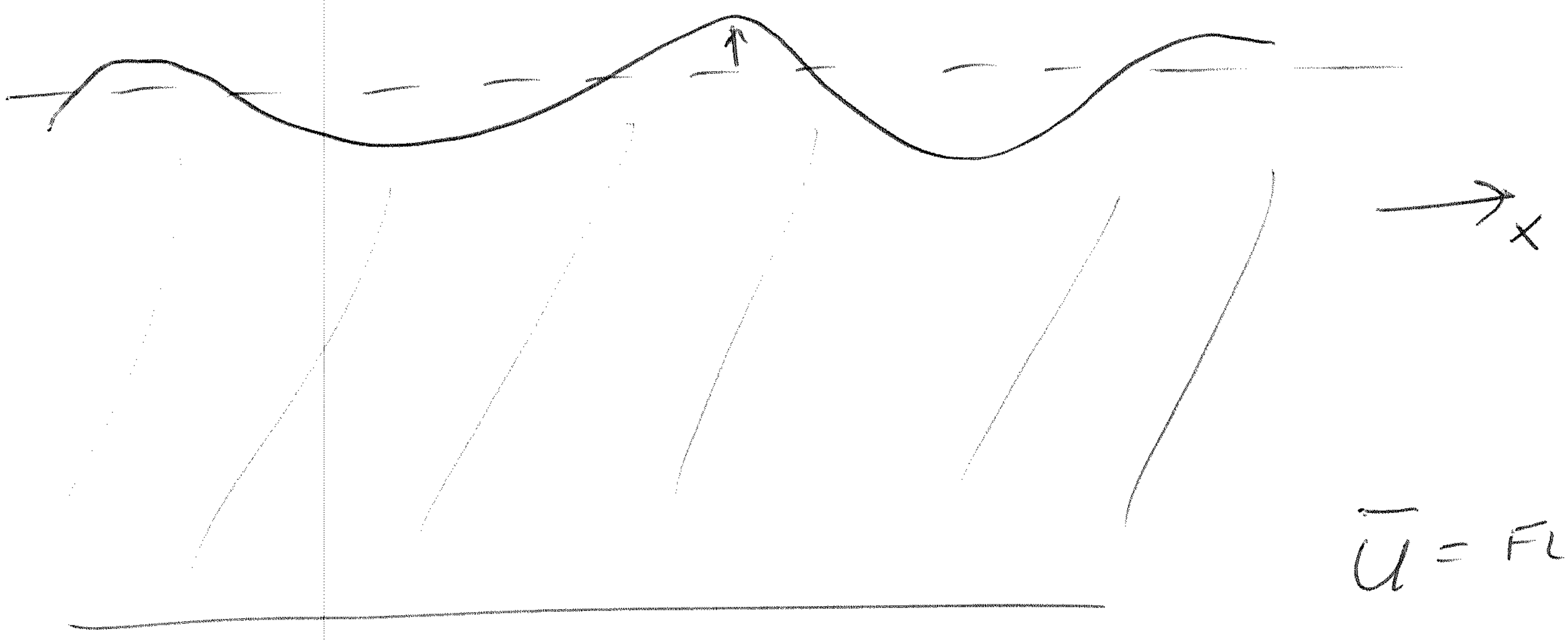
Lima, Peru

http://jro.igp.gob.pe/english/


Ripples in Water

Surface Wave (1): Surface Motion (η)

$\eta(x,t) = \eta_0 \cos(kx - \omega t)$ g



$\bar{u}$ = FLUID VELOCITY

RIPPLES IN WATER

VELOCITY POTENTIAL

INCOMPRESSIBLE $\Rightarrow$ $\nabla \cdot \bar{u}$ $\Rightarrow$ $\bar{u} = \bar{\nabla}\phi$

$\nabla^2 \varphi = 0$ $\Rightarrow$ $\varphi(x,z,t) \approx e^{kz} \sin(kx - \omega t)$ FOR $z < 0$

Surface Wave (2): Bernoulli's Eq

FLUID MOMENTUM IN IRROTATIONAL FLOW

USE "BOUSSINESQ APPROXIMATION"
ρ = FLUID DENSITY
$\simeq$ CONSTANT

$$\frac{\partial \bar{u}}{\partial t} + \bar{u} \cdot \nabla \bar{u} = -\nabla(gz) - \nabla(P/\rho)$$

GRAVITATIONAL POTENTIAL

FLUID PRESSURE

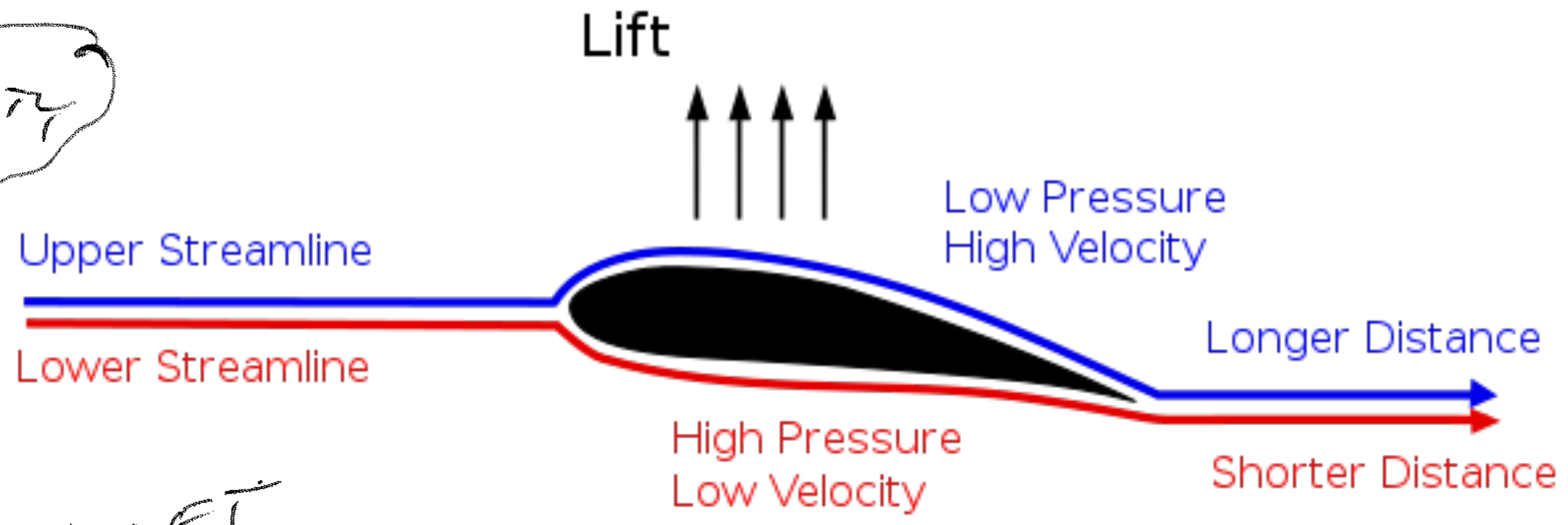
BUT $\bar{u} \cdot \nabla \bar{u} = \nabla(\tfrac{1}{2}u^2) - u \times (\nabla \times u)$

VORTICITY

WITH VORTICITY = 0

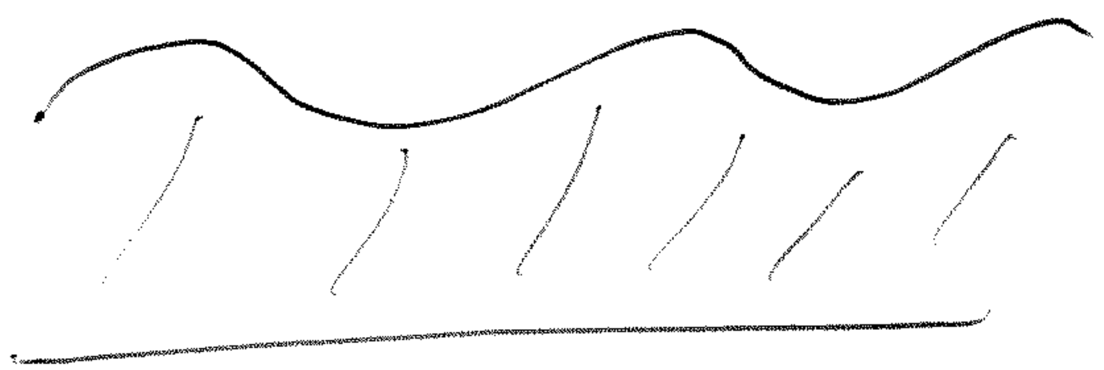
$$\frac{\partial \bar{u}}{\partial t} + \nabla\left(\tfrac{1}{2}u^2 - gz + \frac{P}{\rho}\right) = 0$$

? AIRPLANE LIFT



Surface Wave (3): Linearize

LINEARIZE RIPPLES



$\eta(x,t) = \eta_0 \cos(kx - \omega t)$

← TINY

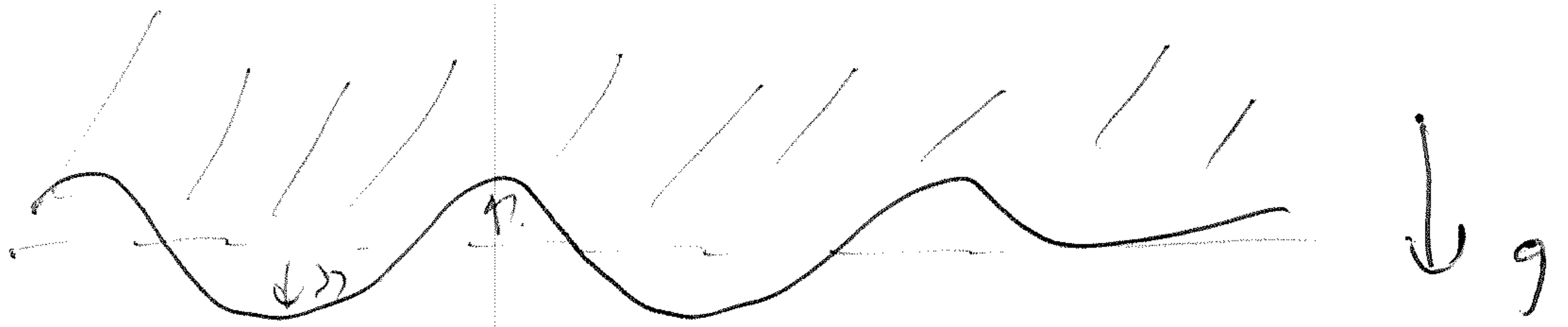
BOUNDARY MOTION: $u_z = \dfrac{\partial \varphi}{\partial z} = \dfrac{\partial \eta}{\partial t}$ AT SURFACE

BERNOULLI EQ: $\dfrac{\partial \varphi}{\partial t} + g\eta \simeq 0$ AT SURFACE

$$\therefore \left.\begin{aligned} -j\omega\hat{\varphi} &= -g\hat{\eta} \\ +k\hat{\varphi} &= -j\omega\hat{\eta} \end{aligned}\right\} \qquad \omega^2 = kg$$

<u>GRAVITY WAVE</u>

Surface Wave (4): Rayleigh-Taylor



RAYLEIGH-TAYLOR INSTABILITY

$\bar{u} = \nabla \varphi \qquad \nabla^2 \varphi = 0$

$\varphi(x, z, t) \simeq e^{-kz} \sin(kx - \omega t)$ FOR $z > 0$

$\therefore \quad \dfrac{\partial \varphi}{\partial t} = +g\eta \qquad -j\omega\hat{\varphi} = -g\tilde{\eta}$

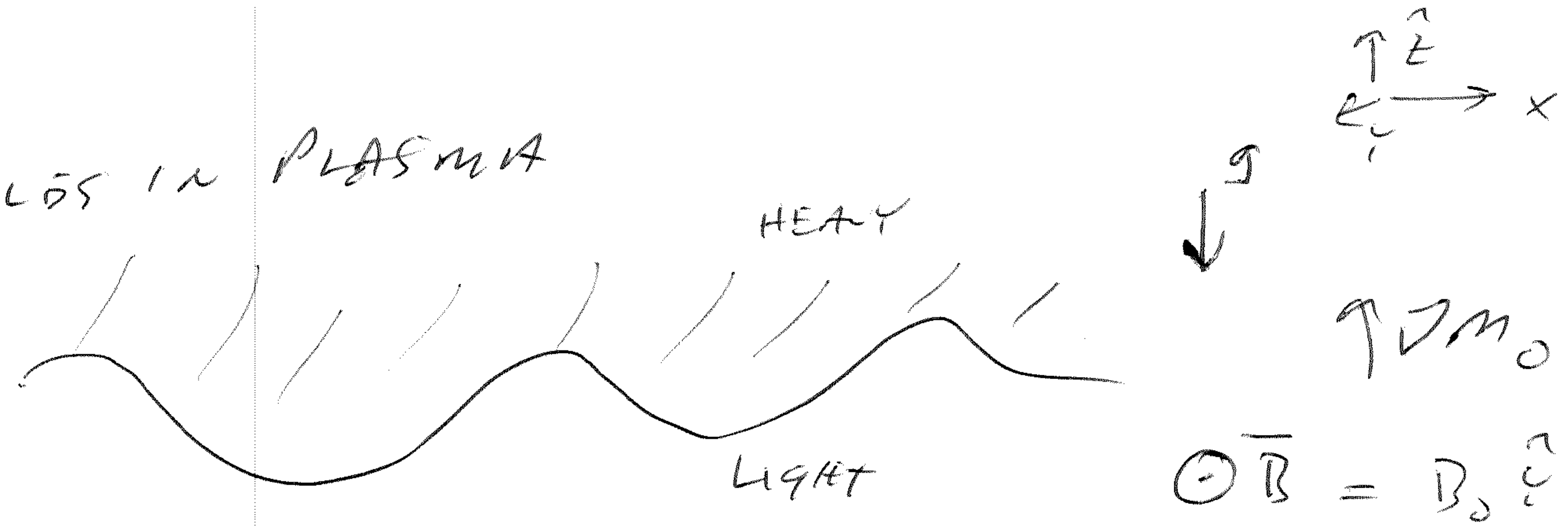
$u_z = \dfrac{\partial \varphi}{\partial z} = -\dfrac{\partial \eta}{\partial t} \qquad +k\hat{\varphi} \simeq +j\omega\tilde{\eta}$

or $\quad \omega^2 = -kg$

RT INSTABILITY

Rayleigh-Taylor in a Plasmas (1)

RIPPLES IN PLASMA



EQUILIBRIUM FROM ION GRAVITATIONAL DRIFT

$$\bar{V}_{ion} = \frac{\tilde{E} \times \bar{B}_0}{B_0^2} + \frac{M}{qB_0^2}\frac{d\tilde{E}}{dt} - \frac{M}{qB_0}\hat{b} \times \bar{g}$$

UP/DOWN

POLARIZATION

GRAVITY

Rayleigh-Taylor in a Plasmas (2): Linearize

$$\frac{\partial \tilde{n}}{\partial t} + \nabla\cdot(\vec{v} n) = 0 \qquad \left(\frac{\partial}{\partial t} + V_0\cdot\nabla\right)\tilde{n} + \tilde{v}_z \frac{\partial n_0}{\partial z} = 0$$

$$\nabla\cdot\bar{J} = 0 \qquad \nabla\cdot\left(\tilde{J} + \epsilon_0 \dot{\tilde{E}}\right) = 0$$

$$\text{OR } \nabla\cdot\left(\underbrace{\left[1 + \frac{nM}{\epsilon_0 B_0^2}\right]\dot{\tilde{E}}}_{\text{POLARIZATION CURRENT}} - \underbrace{\frac{\tilde{n}M}{B}\hat{b}\times\bar{g}}_{\text{PERTURBED GRAVITATIONAL CURRENTS}}\right) = 0$$

$\hat{x}$-DIRECTED CURRENT CONDITION
$$\chi \frac{D\tilde{E}_x}{Dt} - \frac{\tilde{n}M}{B} g = 0 \qquad \chi \approx \frac{n_0 M}{\epsilon_0 B^2}$$

PERTURBED DENSITY
$$-i(\omega - k_x V_0)\tilde{n} + \tilde{v}_z \frac{\partial n_0}{\partial z} = 0$$

$$\text{BUT } \tilde{V}_z = \frac{\tilde{E}_x}{B_0} \Rightarrow -(\omega - k_x V_0)^2 \tilde{n} + \tilde{n}\frac{Mg}{B\chi}\frac{\partial n_0}{\partial z} = 0$$

$$\text{OR} \quad \omega = k_x V_0 \pm i \frac{g}{n_0}\frac{\partial n_0}{\partial z} \quad !!! \quad \text{IF } \frac{\partial n_0}{\partial z} > 0$$

ANNALS OF PHYSICS: 1, 120–140 (1957)

Stability of Plasmas Confined by Magnetic Fields[1]

M. N. ROSENBLUTH* AND C. L. LONGMIRE

Los Alamos Scientific Laboratory, University of California, Los Alamos, New Mexico

In this paper, we examine the question of the stability of plasmas confined by magnetic fields. Whereas previous studies of this problem have started from the magnetohydrodynamic equations, we pay closer attention to the motions of individual particles. Our results are similar to, but more general than, those which follow from the magnetohydrodynamic equations.

that the internal energy of the plasma per unit mass is

$$E_p = \frac{pv}{\gamma - 1} \tag{22}$$

where p is the pressure and v the specific volume. In any adiabatic motion,

$$p \sim v^{-\gamma} \tag{23}$$

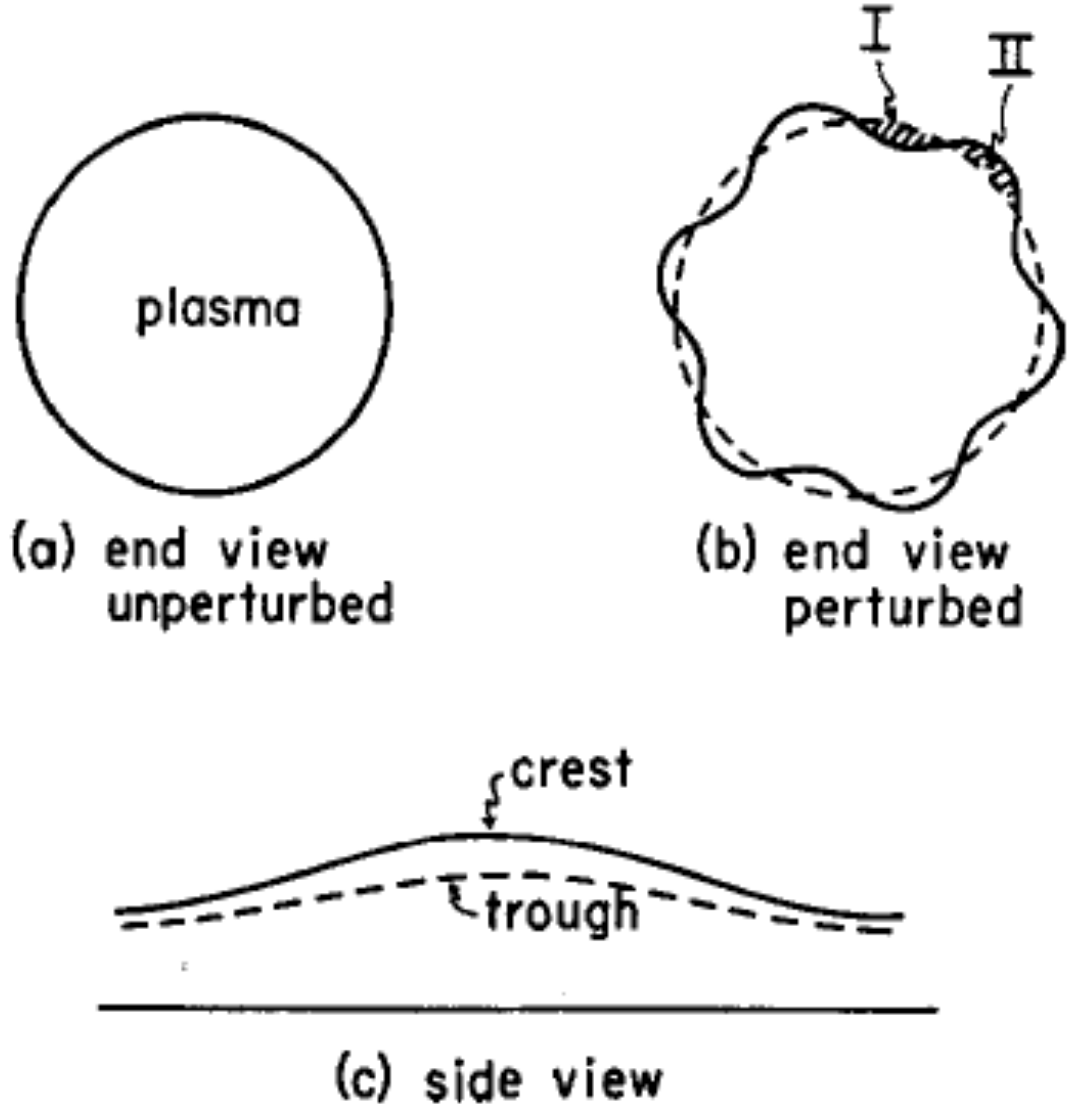


FIG. 6. Illustration of flute-type instability.

The volume V of a flux tube is given by

$$V = \int dl\, A = \phi \int \frac{dl}{B}. \tag{28}$$

Hence, the change in material energy is given by

$$\Delta E_p = \frac{1}{\gamma - 1}\left\{p_{\mathrm{I}} \frac{V_{\mathrm{I}}{}^{\gamma}}{V_{\mathrm{II}}{}^{\gamma}} V_{\mathrm{II}} + p_{\mathrm{II}} \frac{V_{\mathrm{II}}{}^{\gamma}}{V_{\mathrm{I}}{}^{\gamma}} V_{\mathrm{I}} - p_{\mathrm{I}} V_{\mathrm{I}} - p_{\mathrm{II}} V_{\mathrm{II}}\right\}. \tag{29}$$

We have used the scalar pressure result that p is constant along a line. If the flux tubes are nearby, we may expand

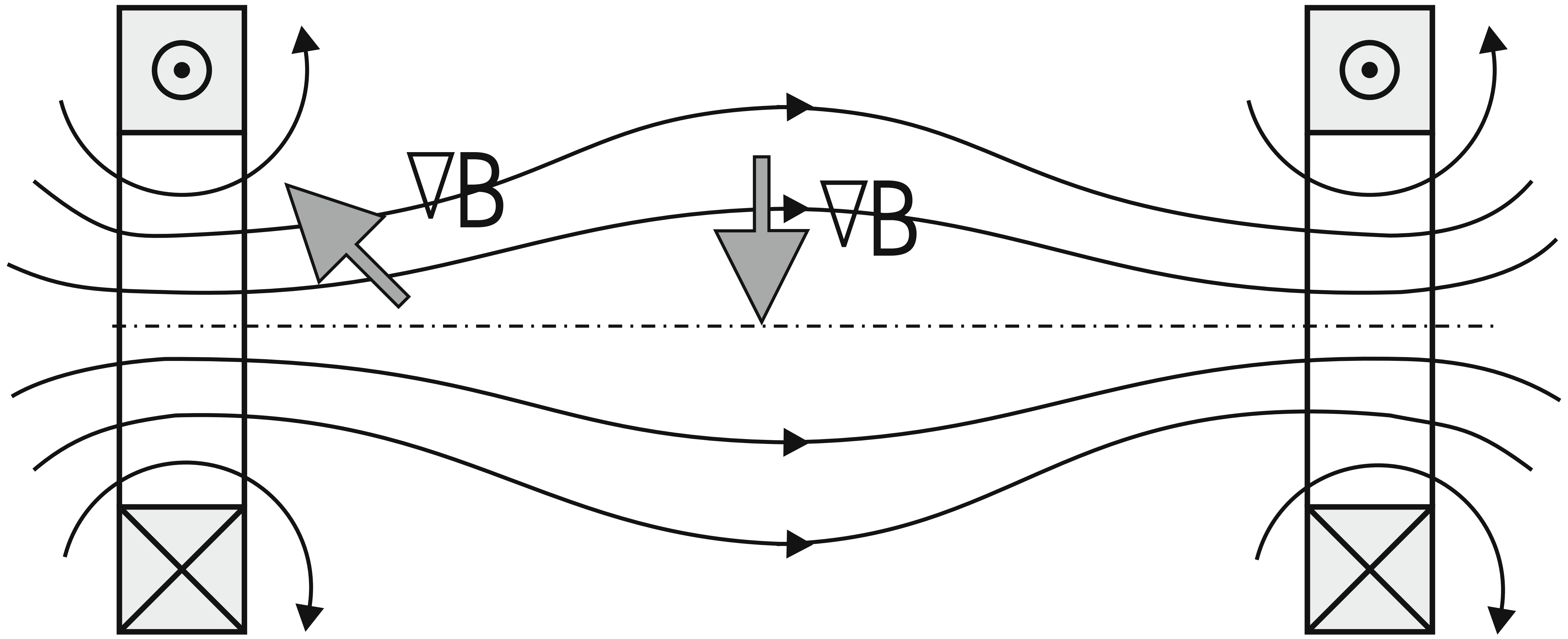
$$\left.\begin{aligned} p_{\mathrm{II}} &= p_{\mathrm{I}} + \delta p \\ V_{\mathrm{II}} &= V_{\mathrm{I}} + \delta V \end{aligned}\right\} \tag{30}$$

and find

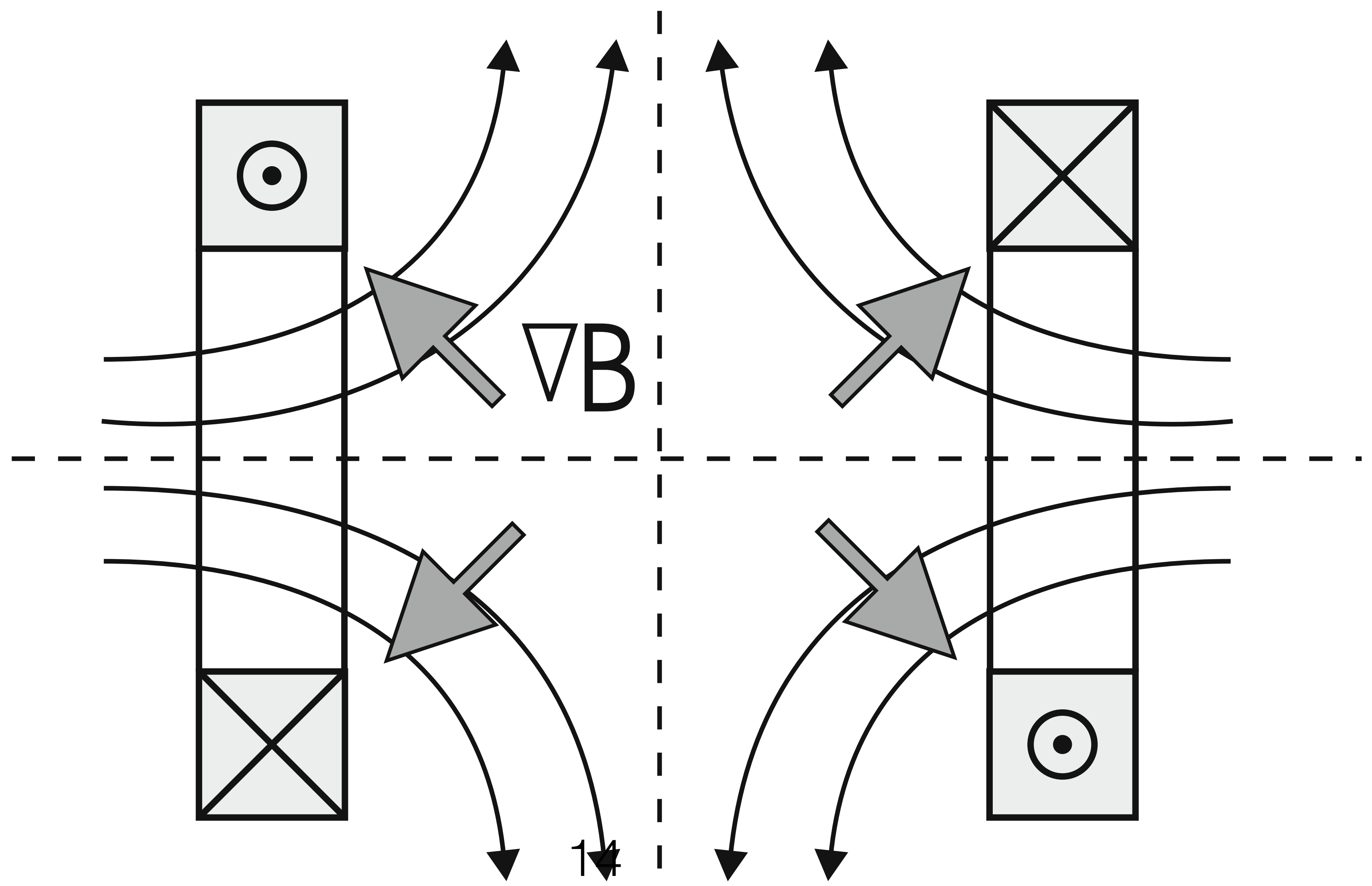
$$\Delta E_p = \delta p \delta V + \gamma p \frac{(\delta V)^2}{V} = V^{-\gamma} \delta(p V^{\gamma}) \delta V. \tag{31}$$

Good/Bad Curvature

(a)

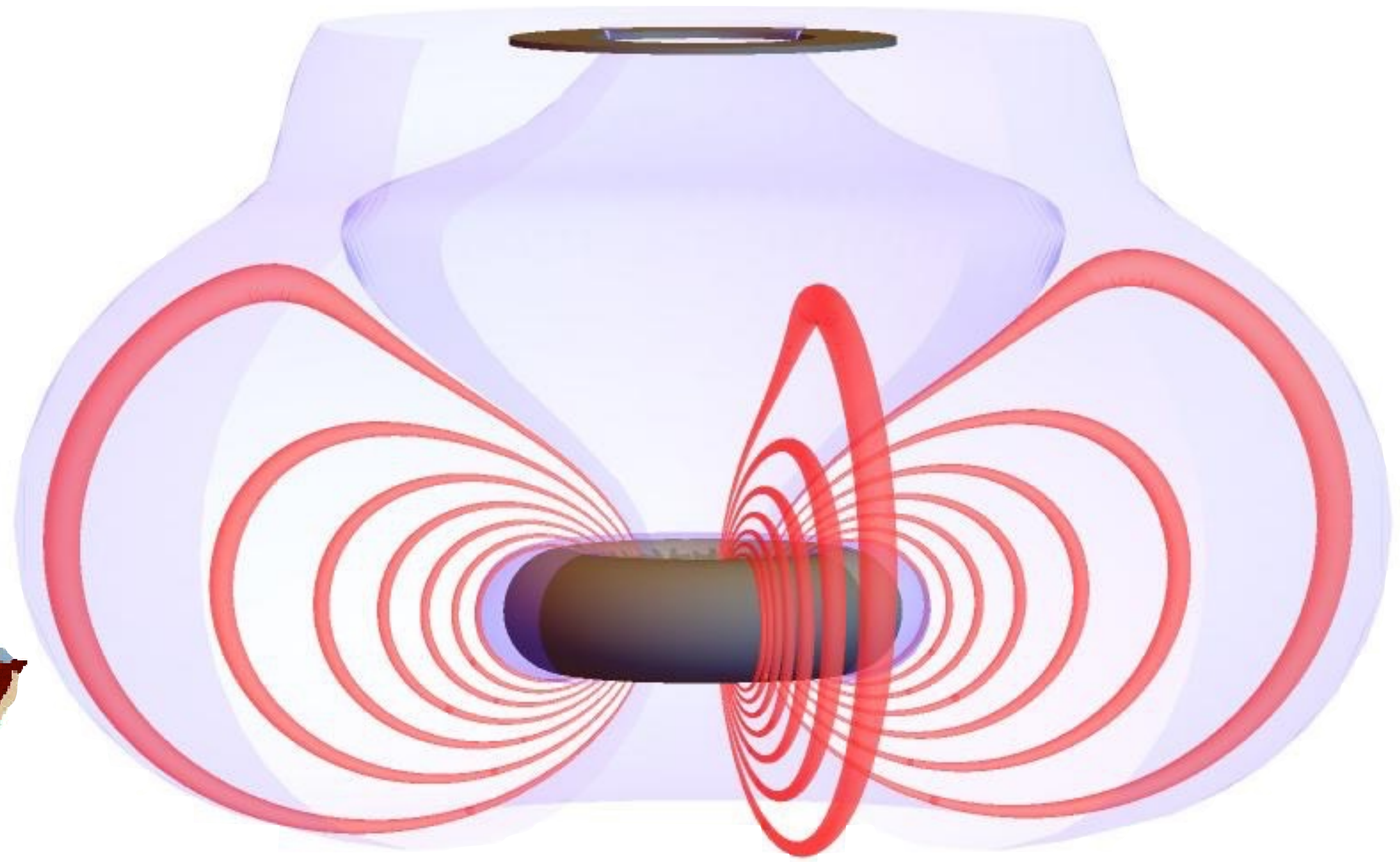


(b)

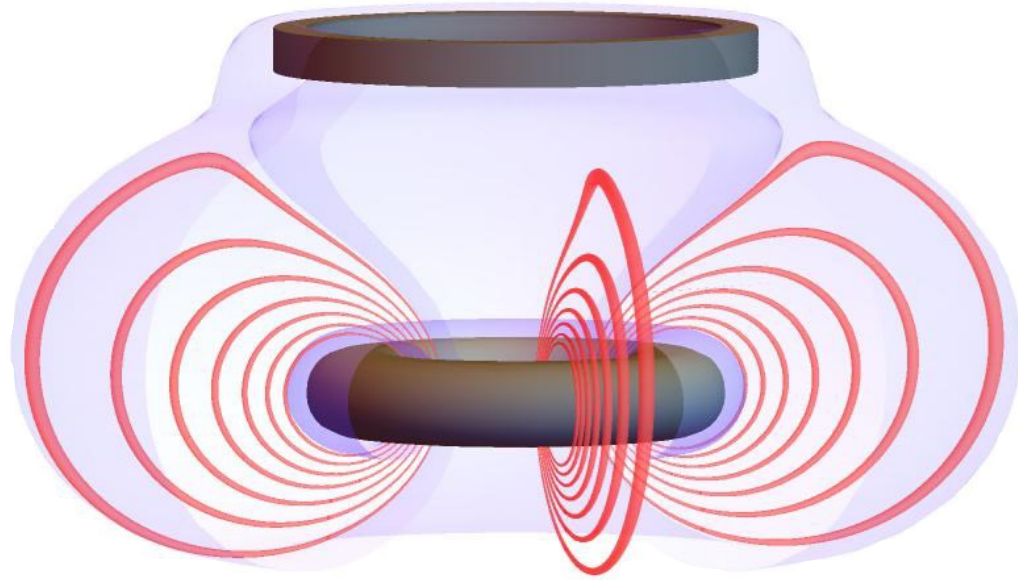


Laboratory Magnetospheres: Designed for Maximum Flux Tube Expansion

$$V = \int \frac{dl}{B} \propto L^4$$



3.6 m



1.8 m

Levitated Dipole Experiment (LDX)

Flux Tube Expansion:

$\delta V(\text{out})/\delta V(\text{in}) = 100$

Ring Trap 1 (RT-1)

Flux Tube Expansion:

$\delta V(\text{out})/\delta V(\text{in}) = 40$

Large Flux Tube Expansion Maximizes Plasma's Stable Pressure Gradient

Ideal MHD interchange instability limits plasma pressure gradient relative to the rate of **flux-tube expansion…**

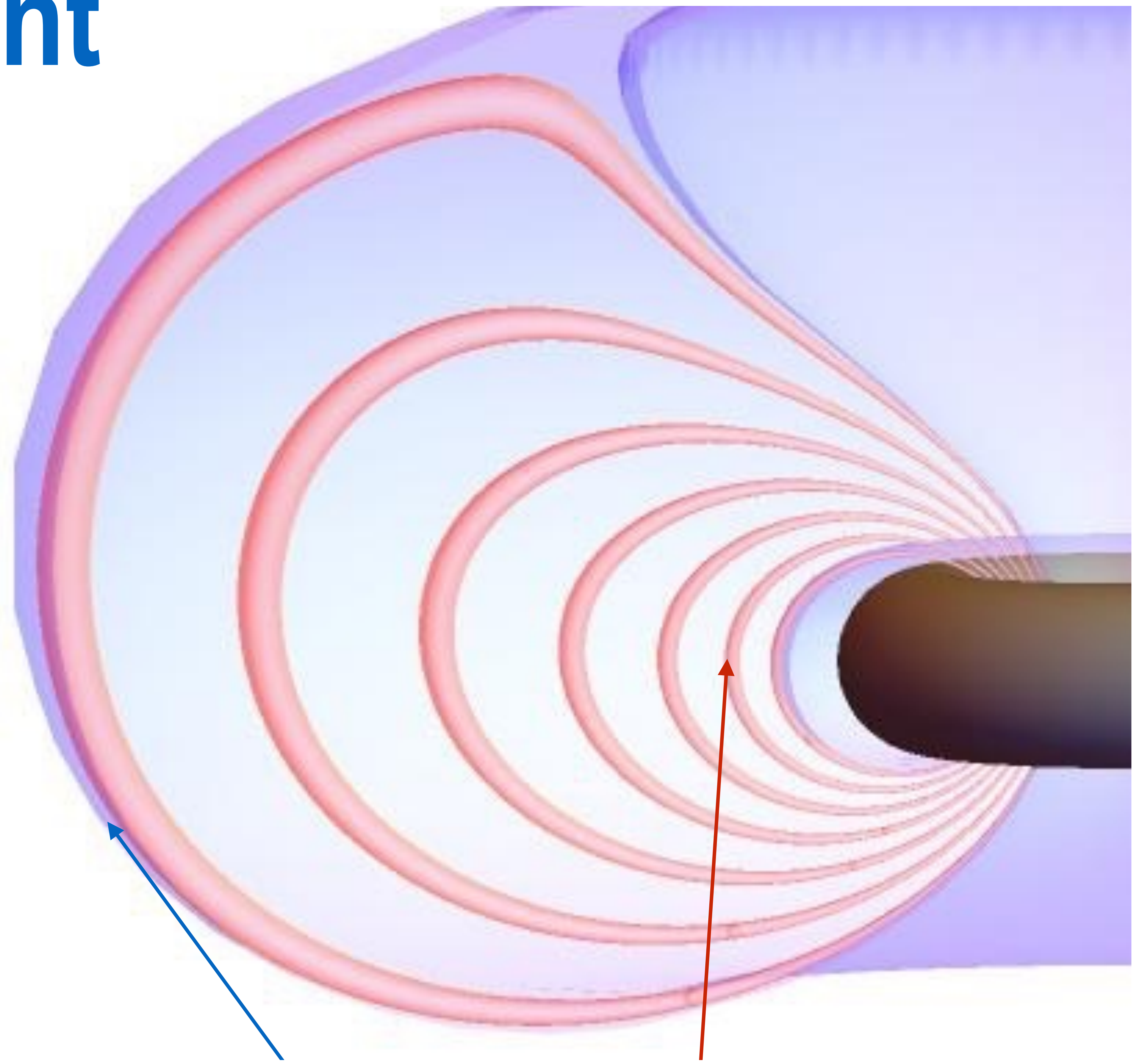
$$\Delta W_p = \Delta\left(PV^{5/3}\right)\frac{\Delta V}{V^{5/3}} > 0$$

and steep pressure gradients are MHD stable, ***even as β >> 1***.

MHD stability ***requires*** finite plasma pressure at edge.

$$\Delta W_p = \underbrace{\Delta P \Delta V}_{\text{Bad Curvature}} + \underbrace{\frac{5}{3}\frac{P}{V}(\Delta V)^2}_{\text{Compressibility}} > 0$$

Magnetosphere: Magnetopause plasma sustained by solar wind
Laboratory: Scrape-off-layer (SOL) maintained by escaping plasma

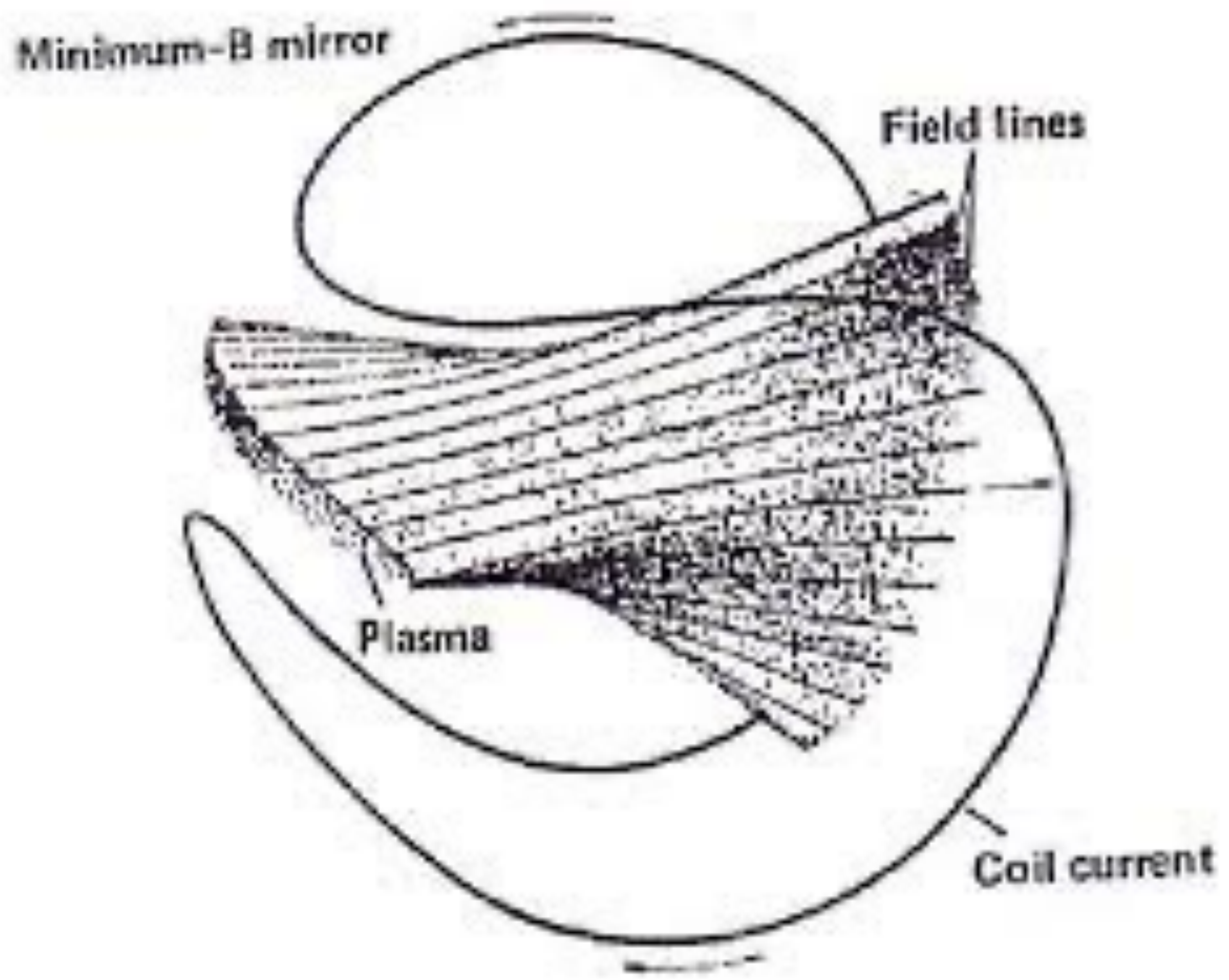


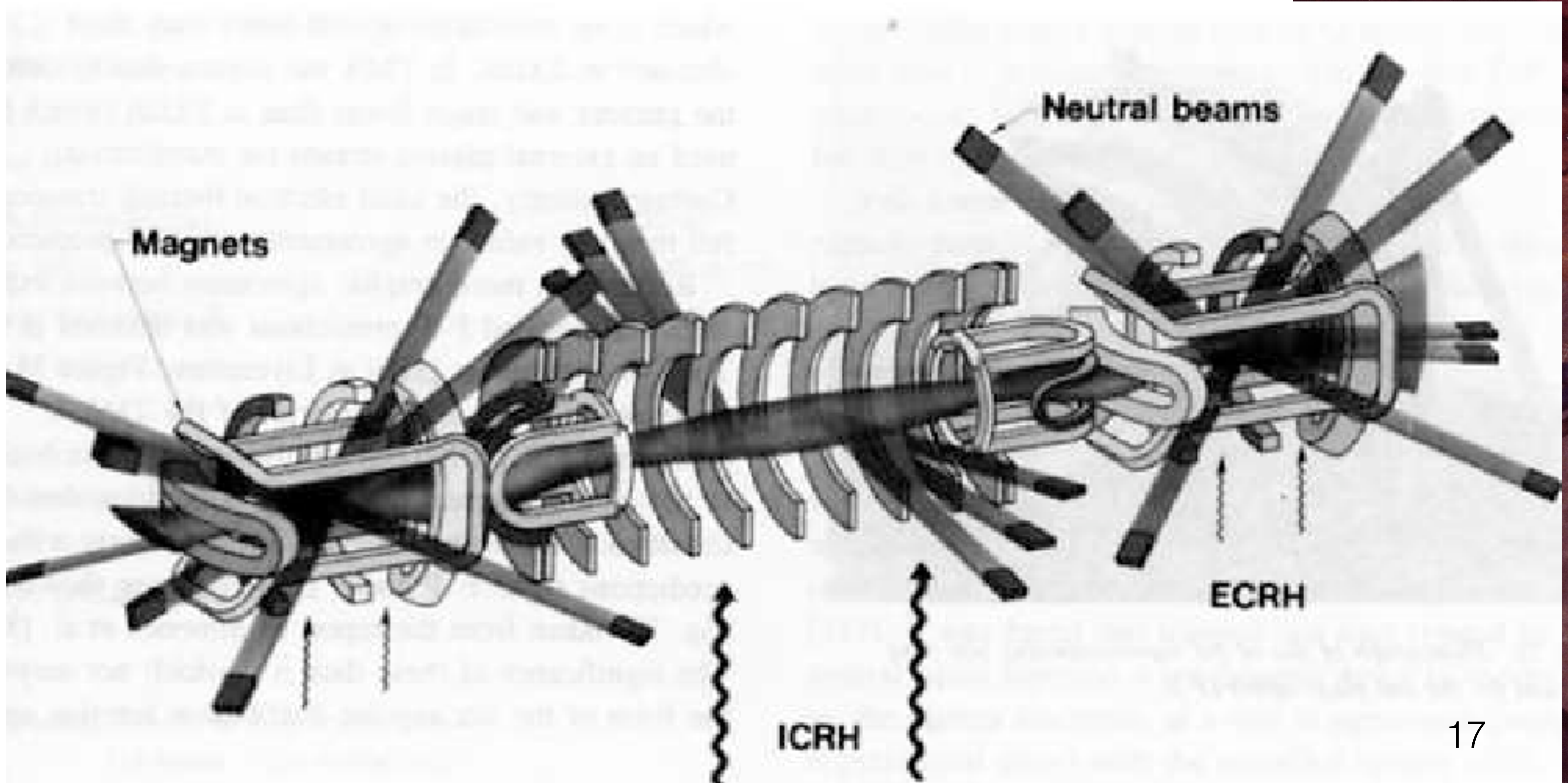
Edge pressure must rise in proportion to core pressure

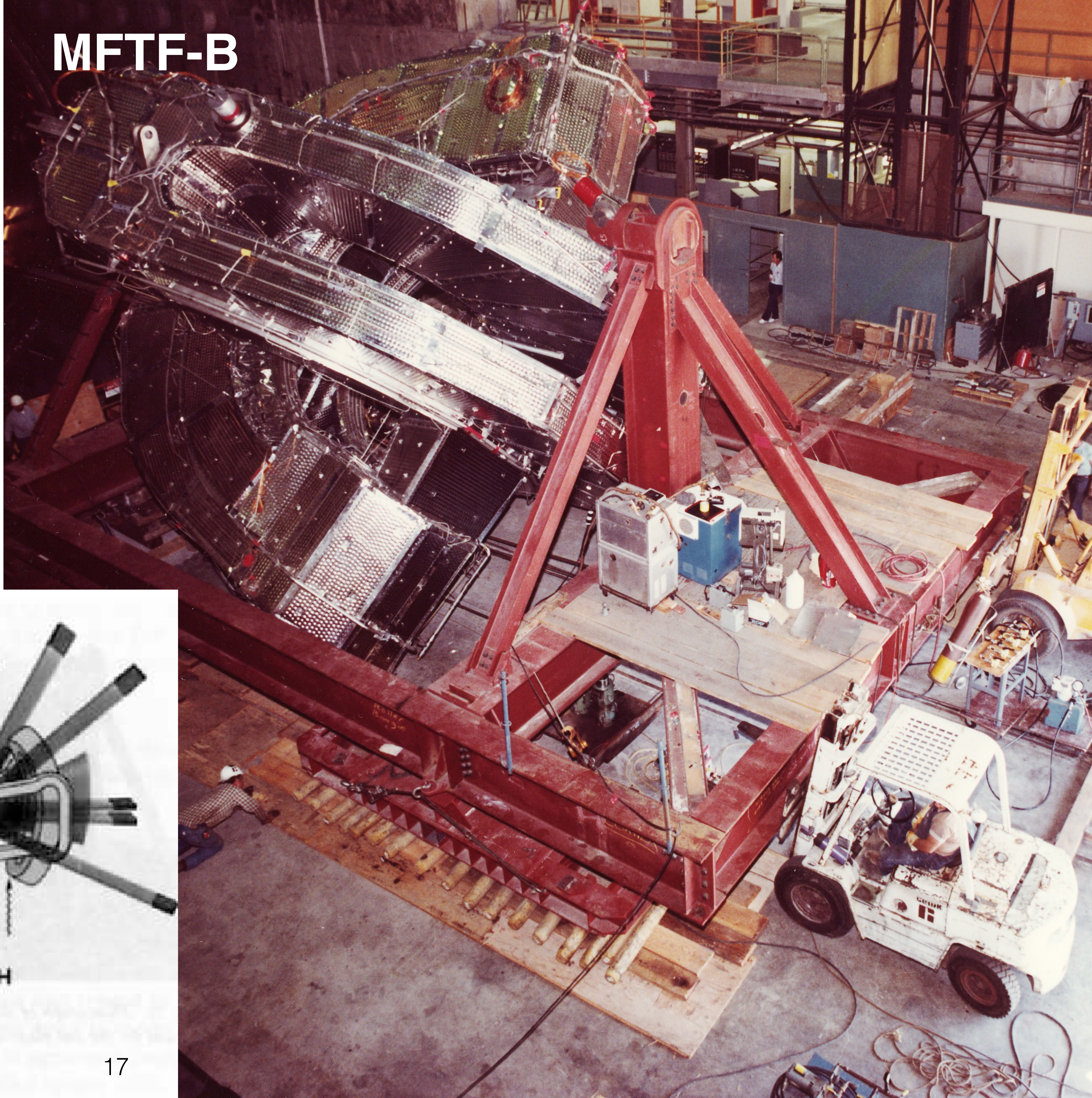
$$\frac{P(core)}{P(edge)} \le \left(\underbrace{\frac{V(edge)}{V(core)}}_{\text{Flux-Tube Expansion}}\right)^{5/3} \sim 2000 \quad (\text{LDX})$$

Rosenbluth and Longmire, "Stability of plasmas confined by magnetic fields," *Ann Phys*, **1**, 120 (1957)
Gold, "Motions in the magnetosphere of the Earth," *JGR*, **64** 1219 (1959)
Garnier, *et al.*, "Magnetohydrodynamic stability in a levitated dipole," *PoP*, **6**, 3431 (1999).
Krasheninnikov, *et al.*, "Magnetic dipole equilibrium solution at finite plasma pressure," *PRL*, **82**, 2689 (1999).

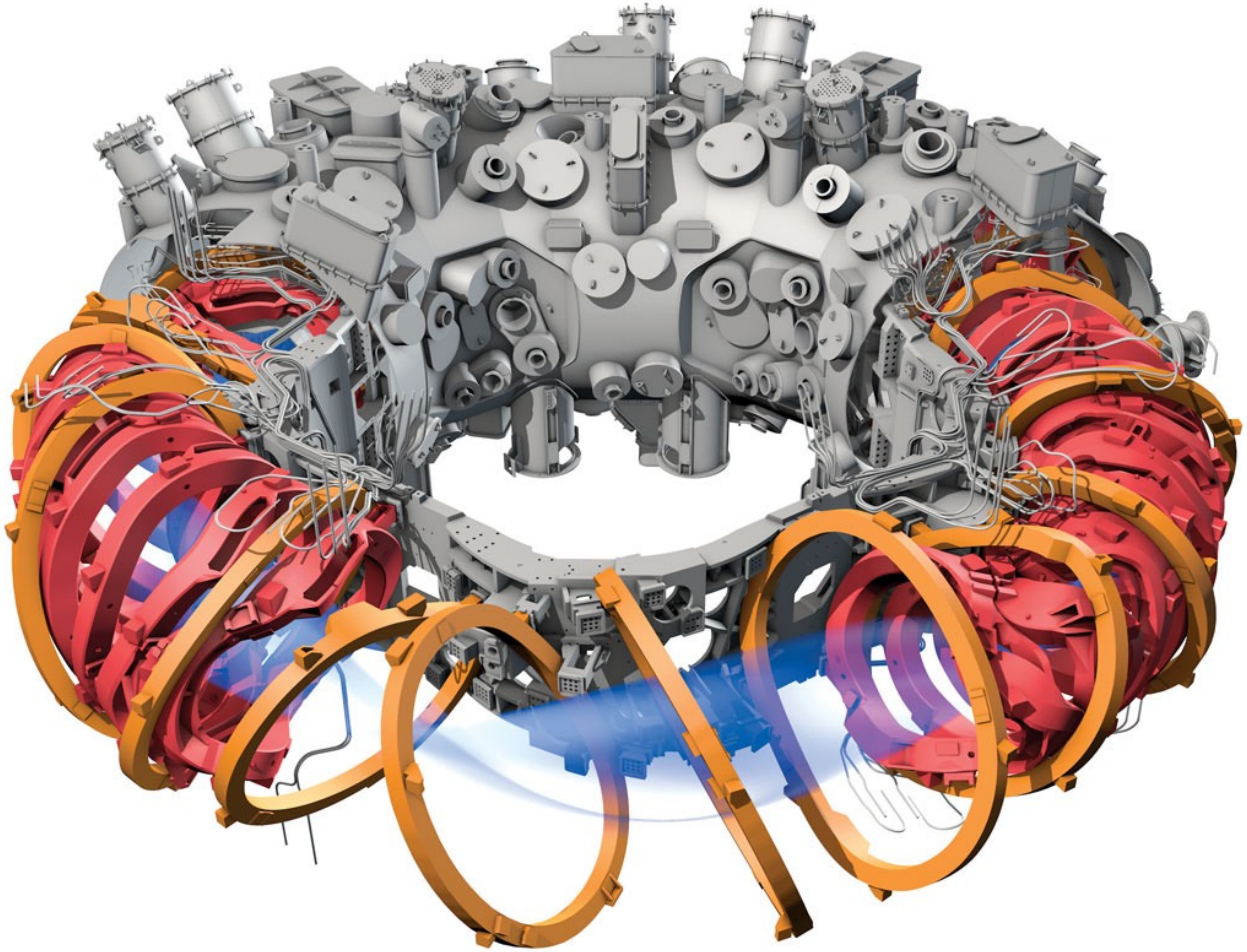
MFTF-B


Minimum-B mirror
Field lines
Plasma
Coil current

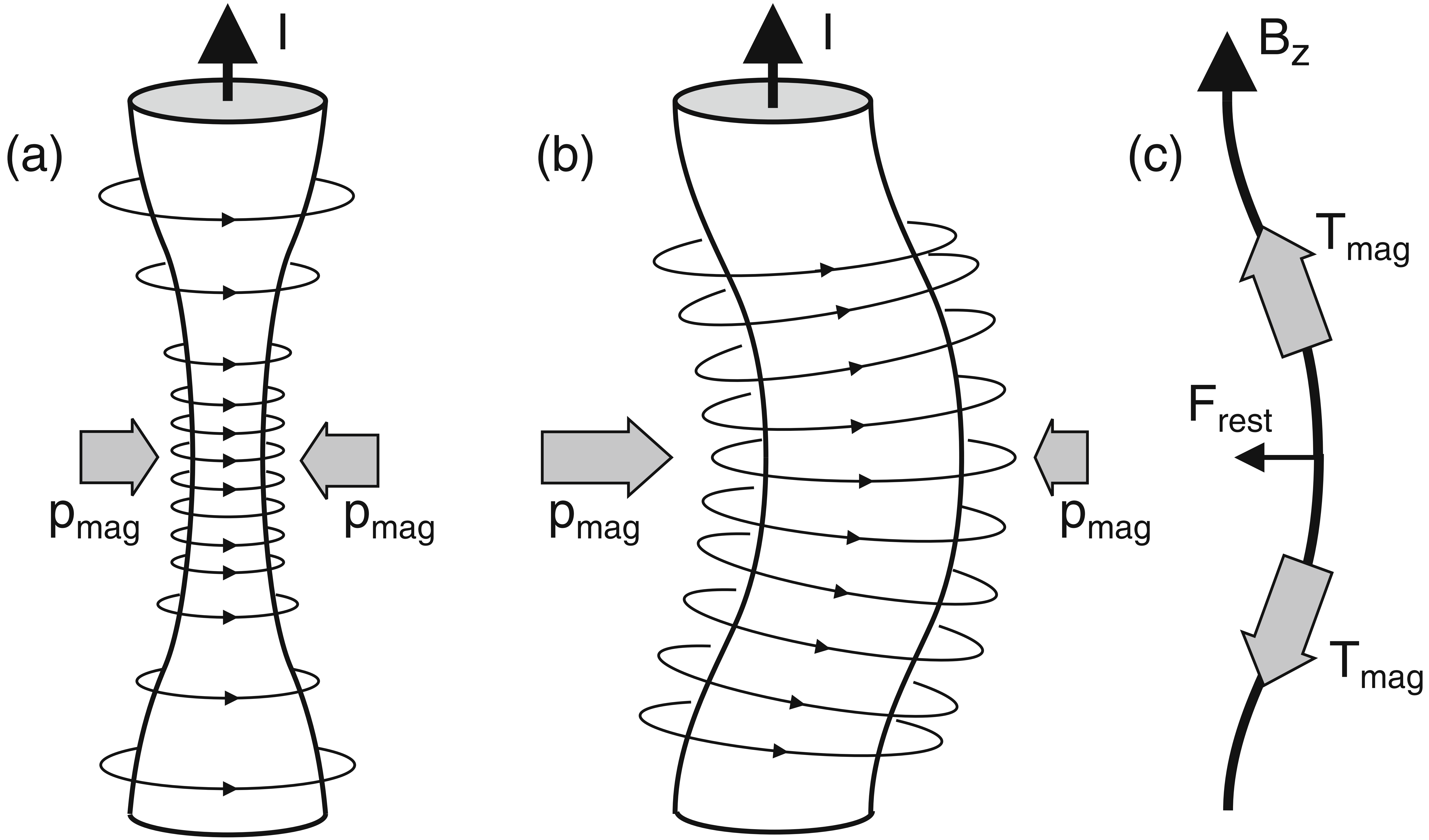

Neutral beams
Magnets
ECRH
ICRH



Wendelstein 7-X



Z-Pinch (stabilized by B_z)



Nonlinear helical perturbations of a tokamak

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(Received 17 May 1976)

An analytic study of small amplitude perturbations of a cylindrical tokamak, using the nonlinear, helically symmetric, reduced magnetohydrodynamic equations of Kadomtsev and Rosenbluth is presented. Results are compared with those of a numerical study of these reduced equations.

I. INTRODUCTION

It has been proposed[1] that the nonlinear evolution of free boundary kink modes in tokamaks could lead to large distortions of the plasma. One purpose of the calculations presented here is to test this hypothesis near the marginally stable point where only one mode is unstable. Our results show that for ratios of plasma radius to wall radius less than 0.65, violent distortions do not occur (a similar result was obtained by Rutherford *et al.*[2] for a more stable equilibrium than the one considered here). For ratios of plasma radius to wall radius greater than 0.65, our results are inconclusive.

The magnetohydrodynamic equations are

$$\rho(d\mathbf{v}/dt) = \mathbf{j}\times\mathbf{B} - \nabla p\ , \qquad (1)$$

$$\mathbf{j} = \nabla\times\mathbf{B}\ , \qquad (2)$$

$$\partial\mathbf{B}/\partial t = -\nabla\times\mathbf{E}\ , \qquad (3)$$

$$\mathbf{E} = -\mathbf{V}\times\mathbf{B}\ , \qquad (4)$$

where

$$d/dt = \partial/\partial t + \mathbf{V}\cdot\nabla$$

is the convective derivative.

H. R. Strauss, D. A. Monticello, Marshall N. Rosenbluth, Roscoe B. White; Nonlinear helical perturbations of a tokamak. *Phys. Fluids* 1 March 1977; 20 (3): 390–395.
https://doi.org/10.1063/1.861901

Reduced MHD

1970's HANK STRAUSS (NYU), MARSHALL ROSENBLUTH (PRINCETON)

THE BASIC SIMPLE ANALYTIC TOOL FOR UNDERSTANDING TOKAMAKS

UNTILL THE "MODERN" AGE OF COMPUTER CODES.

BASIC ASSUMPTIONS :

- LOW BETA $\langle\beta\rangle \sim (a/R)^2 \ll 1 \quad \beta_N \sim (a/R)$
- LARGE ASPECT RATIO $\epsilon \equiv a/R \ll 1$
- WITH $q \gtrsim 1$, THEN $B_P/B_T \sim \epsilon/q \ll 1$
- WITH $\epsilon \ll 1$, THEN $B_T \sim B_0\left(\frac{R_0}{R}\right) \sim B_0(1-\epsilon\cos\theta+\ldots) \approx$ CONSTANT

  BUT 1ST ORDER IN ϵ IS IMPORTANT
- LET $\nabla\cdot\hat{\varphi}$ (OR $\nabla\cdot\hat{z}$) $=0$, ELIMINATING ACOUSTIC MODES (NO "GAMS")
- BASICALLY, A "CYLINDRICAL" TOKAMAK (1D EQUILIBRIUM)

Basic Derivation

REF: KADOMTSEV, "TOKAMAK PLASMA: A COMPLEX PHYSICAL SYSTEM"
IOP (1992)

$\bar{B} = \bar{B}_\perp + \hat{z} B_z$ WITH $B_z \simeq$ CONSTANT

CYLINDRICAL COORDINATES

MHD:

$$\rho \frac{d\bar{v}_\perp}{dt} = -\nabla p + \bar{J} \times \bar{B}$$

$$\bar{E} = -\bar{v} \times \bar{B} \quad \text{(IDEAL)}$$

$$\bar{v} \cdot \hat{z} = 0$$

$$\nabla \cdot \bar{v}_\perp = 0 \quad \text{INCOMPRESSIBLE}$$

MAXWELL'S EM

$$\bar{J} = \frac{1}{\mu_0} \nabla \times \bar{B} \quad \text{(NO DISPLACEMENT CURRENT)}$$

$$\nabla \cdot \bar{B} = 0 = \nabla \cdot \bar{B}_\perp$$

$$\nabla \cdot J = 0$$

$$\frac{\partial \bar{B}}{\partial t} = -\nabla \times \bar{E}$$

IDEAL MHD DESCRIBE PLASMA DYNAMICS AT ALFVEN TIME SCALE: $V \sim V_A \sim \sqrt{B^2/\mu_0 \rho}$ (FAST!)

Stream Function and Poloidal Flux

WITH B_z = CONSTANT, THE REDUCED MHD DYNAMICS IS
$v_z = 0$
DESCRIBED BY FOUR UNKNOWN FUNCTIONS OF (r, θ, z, t):

$$\bar{B}_\perp(r, \theta, z, t) \quad \text{AND} \quad \bar{V}_\perp(r, \theta, z, t)$$

TWO POTENTIALS INSTEAD OF TWO VECTOR FIELDS

WE "GREATLY SIMPLIFY" THE MATH BY INTRODUCING THE STREAM FUNCTION, χ, AND THE POLOIDAL FLUX FUNCTION, ψ

$$\bar{B}_\perp(r, \theta, z, t) = \hat{z} \times \nabla \psi \qquad \bar{V}_\perp(r, \theta, z, t) = \hat{z} \times \nabla \chi$$

$$\nabla \cdot \bar{B} = 0 \qquad \nabla \cdot \bar{V} = 0$$

AMPERE'S LAW

$$\bar{J} = \frac{1}{\mu_0} \nabla \times (\hat{z} \times \psi)$$

$$\mu_0 \bar{J} = \hat{z} \nabla^2 \psi - (\hat{z} \cdot \bar{\nabla}) \bar{\nabla} \psi$$

AXIAL VORTICITY

$$\Omega_z \equiv \hat{z} \cdot \nabla \times \bar{V}_\perp = \nabla^2 \chi$$

TWO UNKNOWN POTENTIALS:
$\psi(r, \theta, z, t) \quad \chi(r, \theta, z, t)$

Simplifying the MHD Equations

$$\rho \frac{d\bar{V}_\perp}{dt} = -\nabla p + \frac{1}{\mu_0}(\nabla \times B) \times B$$

$$= -\nabla\left(p + B^2/2\mu_0\right) + \frac{1}{\mu_0}(\bar{B}\cdot\bar{\nabla})\bar{B}$$

$$\hat{z}\cdot\nabla\times\left[\;'' \quad = \quad ''\;\right]$$

ASSUME $\rho \simeq$ UNIFORM

$$\rho \frac{d}{dt}\left(\hat{z}\cdot\nabla\times V_\perp\right) = \frac{1}{\mu_0}(\bar{B}\cdot\bar{\nabla})\left(\hat{z}\cdot\nabla\times\bar{B}\right)$$

$$\rho \frac{d}{dt}(\nabla_\perp^2 \chi) = \frac{1}{\mu_0}(\bar{B}\cdot\bar{\nabla})\nabla_\perp^2 \psi \quad = (\bar{B}\cdot\bar{\nabla}) J_z$$

$$\boxed{\rho \frac{d}{dt}\Omega_z = (\bar{B}\cdot\bar{\nabla}) J_z}$$

"AXIAL VORTICITY CHANGES ACCORDING TO FIELD-ALIGNED VARIATION OF AXIAL CURRENT"

Simplifying the Induction Equation

$$\frac{\partial \bar{B}}{\partial t} = \nabla \times (\bar{V}_\perp \times \bar{B}) = \bar{V}_\perp \bar{\nabla}\cdot\bar{B} - \bar{B}\bar{\nabla}\cdot\bar{V} + (\bar{B}\cdot\bar{\nabla})\bar{V} - (\bar{V}\cdot\bar{\nabla})\bar{B}$$

$$= \nabla \times (\bar{V}_\perp \times \bar{B}_\perp) + B_z \frac{\partial \bar{V}_\perp}{\partial z}$$

$$= (\bar{B}_\perp \cdot \bar{\nabla})\bar{V}_\perp - (\bar{V}_\perp \cdot \bar{\nabla})\bar{B}_\perp + B_z \frac{\partial \bar{V}_\perp}{\partial z}$$

$$\frac{\partial \bar{B}_\perp}{\partial t} + (\bar{V}_\perp \cdot \bar{\nabla}) B_\perp = \frac{d\bar{B}_\perp}{dt} = (\bar{B}\cdot\bar{\nabla})\bar{V}_\perp$$

SUBSTITUTING FLUX FUNCTIONS

$$\frac{d}{dt}\psi = (\bar{B}\cdot\bar{\nabla})\chi$$

"POLOIDAL FLUX EVOLVES DYNAMICALLY DUE TO FIELD-ALIGNED CHANGES IN THE STREAM FUNCTION"

Summary of Reduced MHD

$$\frac{d}{dt}\psi = (\mathbf{B}\cdot\bar{\nabla})\chi$$

"POLOID FLUX EVOLVES DYNAMICALLY DUE TO FIELD-ALIGNED CHANGES IN THE STREAM FUNCTION"

$$\rho\frac{d}{dt}\Omega_z = (\mathbf{B}\cdot\bar{\nabla})J_z$$

"AXIAL VORTICITY CHANGES ACCORDING TO FIELD-ALIGNED VARIATION OF AXIAL CURRENT"