Last Lecture

• Beam-plasma instability
This Lecture

- Magnetized Plasma Instabilities:
  - Pressure-Curvature Driven
  - Parallel-Current Driven
- Interchange Modes
- Reduced MHD (plasma torus with a strong toroidal field)
- Kink modes
In this Section, we are interested in plasma instabilities occurring in real space, called macroinstabilities. These instabilities are characterized by a displacement of the plasma relative to a magnetic field. Here, the energy principle can be used to determine the stability of the system. Nevertheless, normal mode analysis will be the tool to detect the wavelength and growth rate of the unstable modes.

8.4.1 Stable Magnetic Configurations

Consider the magnetic field topologies for a magnetic mirror and a magnetic cusp in Fig. 8.12. We had seen in (3.27) that the gradient of the magnetic field intensity points always towards the center of field line curvature. In the center of the mirror field the gradient points inwards whereas, near the magnets, the gradient points outwards.

Let us now consider the total energy of a small volume of plasma, which consists of kinetic and magnetic energy, which represents the potential energy for this case,

\[ W_{\text{tot}} = W_{\text{kin}} + B^2/\mu_0. \]

(8.40)

When this plasma volume is shifted into a region of weaker magnetic field, the magnetic energy will decrease accordingly. The existence of such a state of lower potential energy makes the situation unstable. However, we cannot give the detailed mechanism, how the plasma manages to get to this lower energy state. We can only say that the plasma in the center of a mirror field has no stable confinement against radial displacements. Consider now the field line topology of a magnetic cusp in Fig. 8.12b. There, the magnetic field increases in all directions and the plasma is in a stable confinement. Such situations are called minimum \( B \) configuration.

Fig. 8.12

(a) A magnetic mirror field is generated by currents of the same polarity in the magnets, (b) A magnetic cusp forms when the current in one magnet is reverted. Note that the direction of the gradient in magnetic field strength always points towards the center of the local field line curvature.
Rayleigh-Taylor Instability

The plasma is the heavy fluid that rests on the horizontal magnetic field. Under the action of the gravitational force, the ions experience a drift with velocity given by (3.14) when neglecting collisions. The (opposing) drift velocity of the electrons is smaller by a factor $m_e/m_i$ and will be omitted here. The initial homogeneous equilibrium of the boundary can be understood from the force balance

\[ j_i \times B + n_i m_i g = 0, \]

in which $j_i = e n_i v_g$.

For understanding the instability mechanism, we consider an initial sinusoidal perturbation of the boundary, as indicated by the heavy line. The effect of the $g \times B$ drift is to shift the ions slightly in $-x$ direction, as indicated by the light line. This generates positive surplus-charges at the surface by an overshoot of ions on the leading edge and a lack of ions on the trailing edge. These surface charges generate an $E \times B$ motion of the perturbed plasma region, as indicated by the arrows.

Remember that the $E \times B$ drift is the same for electrons and ions and does not lead to further charge separation. The effect of this secondary drift is to amplify the original perturbation. This is the mechanism of the gravitational Rayleigh-Taylor instability.

The Rayleigh-Taylor instability originally described the interface between a heavy fluid (e.g., water) resting on a lighter fluid (e.g., oil). There, a sinusoidal perturbation of the interface leads to rising oil blobs and descending water blobs. In the equatorial ionosphere, the magnetic field is horizontal and the ionospheric plasma rests on the magnetic field, which represents the lighter fluid. After sunset, the lower parts of the ionosphere (E-region) rapidly disappear by recombination. At the bottom of the F-layer ($\approx 270$ km altitude) a steep density gradient forms, which can become Rayleigh-Taylor unstable and leads to rising bubbles of low-density plasma into the high-density F-layer [189–193]. An example of such plasma bubbles is shown in Fig. 8.15. The bubbles appear as reduced plasma density in a comparison of the density profile during the upleg and downleg of the rocket trajectory. The upleg intersected the bubble region whereas the downleg traversed unperturbed plasma. This result was obtained during the DEOS (Dynamics of the Equatorial Ionosphere Over Shar) rocket campaign [193].

Magnetized plasmas are generally susceptible to Rayleigh-Taylor-like instabilities. Here, the role of gravity can be taken over by the internal kinetic pressure of the plasma particles, as shown in Fig. 8.16a,b. The plasma surface develops a periodic perturbation $\propto \exp(i m \phi)$. The azimuthal number $m$ gives the number of grooves in the plasma column. This pattern resembles the fluted columns in ancient Greece, as shown in Fig. 8.16c and explains the name flute instability.
Stability of Plasmas Confined by Magnetic Fields

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In this paper, we examine the question of the stability of plasmas confined by magnetic fields. Whereas previous studies of this problem have started from the magnetohydrodynamic equations, we pay closer attention to the motions of individual particles. Our results are similar to, but more general than, those which follow from the magnetohydrodynamic equations.

\[ E_P = \frac{pv}{\gamma - 1} \]
PLASMA STABILITY

(a) end view unperturbed
(b) end view perturbed
(c) side view

Fig. 6. Illustration of flute-type instability.

The volume $V$ of a flux tube is given by

$$V = \int dl \, A = \phi \int \frac{dl}{B}.$$  \hspace{1cm} (28)

Hence, the change in material energy is given by

$$\Delta E_p = \frac{1}{\gamma - 1} \left( \frac{p_1}{V_1^\gamma} V_{II} + \frac{p_{II}}{V_{II}^\gamma} V_I - p_1 V_I - p_{II} V_{II} \right).$$ \hspace{1cm} (29)

We have used the scalar pressure result that $p$ is constant along a line. If the flux tubes are nearby, we may expand

$$p_{II} = p_1 + \delta p \right)$$

$$V_{II} = V_I + \delta V \right)$$

and find

$$\Delta E_p = \delta p \delta V + \gamma p \frac{(\delta V)^2}{V} = V^{-\gamma} \delta (p V^\gamma) \delta V.$$  \hspace{1cm} (31)
V. **CORRECT TREATMENT OF PLASMA ENERGY**

In this section, we shall discuss the flute-type instability of Section IV, trying, however, to calculate the change in internal energy of the plasma correctly rather than by the magnetohydrodynamic approximation. To do this, we first look at individual particles, calculating the change in energy of a particle as the line to which it is tied moves in a flute-type instability. First we note that there are two adiabatic invariants of the motion

\[ \frac{w \perp}{B} = \mu \quad (37) \]

\[ \int v \parallel \, dl = J. \quad (38) \]

Here $w \perp$ is the particle energy in the plane transverse to the field, and $v \parallel$ is its velocity along the line, the integral being taken between turning points or over a period of the motion.
Two Laboratory Magnetospheres

Levitated Dipole Experiment (LDX)
(1.2 MA \cdot 0.41 \text{ MA m}^2 \cdot 550 \text{ kJ} \cdot 565 \text{ kg})
\text{Nb}_3\text{Sn} \cdot 3 \text{ Hours Float Time}
24 \text{ kW ECRH}

Ring Trap 1 (RT-1)
(0.25 \text{ MA} \cdot 0.17 \text{ MA m}^2 \cdot 22 \text{ kJ} \cdot 112 \text{ kg})
\text{Bi-2223} \cdot 6 \text{ Hours Float Time}
50 \text{ kW ECRH}
Laboratory Magnetospheres: Designed for Maximum Flux Tube Expansion

Levitated Dipole Experiment (LDX)

Flux Tube Expansion:
\[ \frac{\delta V_{\text{out}}}{\delta V_{\text{in}}} = 100 \]

Ring Trap 1 (RT-1)

Flux Tube Expansion:
\[ \frac{\delta V_{\text{out}}}{\delta V_{\text{in}}} = 40 \]
Large Flux Tube Expansion Maximizes Plasma’s Stable Pressure Gradient

Ideal MHD interchange instability limits plasma pressure gradient relative to the rate of **flux-tube expansion**...

\[
\Delta W_p = \Delta \left( PV^{5/3} \right) \frac{\Delta V}{V^{5/3}} > 0
\]

and steep pressure gradients are MHD stable, **even as \( \beta \gg 1 \)**.

MHD stability **requires** finite plasma pressure at edge.

\[
\Delta W_p = \Delta P \Delta V + \frac{5}{3} \frac{P}{V} (\Delta V)^2 > 0
\]

**Magnetosphere:** Magnetopause plasma sustained by solar wind

**Laboratory:** Scrape-off-layer (SOL) maintained by escaping plasma

\[\frac{P(\text{core})}{P(\text{edge})} \leq \left( \frac{V(\text{edge})}{V(\text{core})} \right)^{5/3} \sim 2000 \text{ (LDX)}\]

Gold, “Motions in the magnetosphere of the Earth,” *JGR*, 64, 1219 (1959)
8.4 Macroscopic Instabilities

8.4.2 Pinch Instabilities

The pinch effect was already introduced in Sect. 5.3.4. The pinch effect is not necessarily a homogeneous mechanism. When we assume that the plasma cross section is reduced at some point, the magnetic pressure at the plasma surface will increase, because

$$B_\phi = \mu_0 I \left( \frac{2\pi a}{2} \right)^{-1},$$

as shown in Fig. 8.13a. This increased magnetic pressure further reduces the plasma radius at this point, and the plasma column develops a sausage instability.

The magnetic pressure can also deviate from its equilibrium value, when the plasma column is curved, see Fig. 8.13b. Because the magnetic field lines are perpendicular to the local current direction, the field line density, and the associated magnetic pressure, is higher on the inner side and lower on the outer side of the curved plasma column. Hence, the imbalance of magnetic pressure will further displace the column forming a kink.

The sausage and the kink instability can be stabilized by a superimposed longitudinal magnetic field, which is frozen in the plasma. The magnetic field lines have tension

$$T_{mag} = \frac{B^2}{\mu_0}$$

that tends to straighten the field lines, see Sect. 5.2.2. This gives a net restoring force that counteracts the instability from the magnetic pressure imbalance of the azimuthal magnetic field component, as shown in Fig. 8.13c.

(a) (b) (c)

Fig. 8.13 (a) Sausage instability, (b) kink instability of a pinch plasma. The magnetic pressure increases when the cross-section shrinks or becomes asymmetric when the plasma column is curved. (c) The magnetic tension of a superimposed longitudinal magnetic field counteracts the instability.
Reduced MHD in a Large Aspect Ratio Tokamak

1970's

Hanke Strauss (NYU), Marshall Rosenbluth (Princeton)

The basic analytic tool for understanding tokamaks
until the modern age of computer codes.

Basic Assumptions:

- Low beta \( \langle \beta \rangle \approx (a/R)^2 \ll 1 \quad \beta_n \approx (a/R) \)

- Large aspect ratio \( \varepsilon \approx a/R \ll 1 \)

- With \( a \gtrsim 1 \), then \( B_p/B_T \sim \varepsilon/a \ll 1 \)

- With \( \varepsilon \ll 1 \), then \( B_T \approx B_0 \left( \frac{R_0}{R} \right) \approx B_0 (1 - \varepsilon \cos \Theta + ...) \approx \text{constant} \)

  But 1st order in \( \varepsilon \) is important

- \( \nabla \cdot \Phi = 0 \) (or \( \nabla \cdot \tilde{b} = 0 \) \), eliminating acoustic modes

  (no "G-shirts")

- Basically, a "cylindrical" tokamak (1D equilibrium)
DERIVING THE REDUCED MHD EQUATIONS

REF: KADOMTSEV, "TOKAMAK PLASMA: A COMPLEX PHYSICAL SYSTEM" TOP (1992)

\[ \mathbf{B} = \mathbf{B}_{\perp} + \frac{1}{2} \mathbf{B}_{2} \quad \text{with} \quad \mathbf{B}_{2} = \text{constant} \]

\[ \rho \frac{d \mathbf{u}_{\perp}}{dt} = - \nabla \rho + \mathbf{J} \times \mathbf{B} \]

\[ \mathbf{E} = - \nabla \times \mathbf{B} \quad (\text{ideal}) \]

\[ \nabla \cdot \mathbf{E} = 0 \quad \text{incompressible} \]

\[ \nabla \cdot \mathbf{u}_{\perp} = 0 \]

MAXWELL'S EM

\[ \mathbf{J} = \frac{1}{\mu_0} \nabla \times \mathbf{B} \quad \text{(no displacement current)} \]

\[ \nabla \cdot \mathbf{B} = 0 = \nabla \cdot \mathbf{B}_{\perp} \quad \nabla \cdot \mathbf{J} = 0 \]

\[ \frac{2 \mathbf{B}}{\mu_0} = - \nabla \times \mathbf{E} \]

IDEAL MHD DESCRIBE PLASMA DYNAMICS AT ALFVEN TIME SCALE: \( \sqrt{\frac{B_{\perp}^2}{\mu_0 \rho}} \) (FAST!)
STREAM FUNCTION AND POLAR ONE FLUX

with \( B_2 = \text{constant} \), the reduced RHD dynamics is 
\( \nu_2 = 0 \)

described by four unknown functions of \((\tau, \theta, t)\):
\[ \mathbf{B}_1(\tau, \theta, t) \text{ and } \mathbf{U}_1(\tau, \theta, t) \]

Two potentials instead of two vector fields

We "greatly simplify" the math by introducing
the stream function \( \chi \) and the polar one flux function, \( \psi \)

\[ \mathbf{B}_1(\tau, \theta, t) = \nabla \times \nabla \psi \]
\[ \nabla \cdot \mathbf{B} = 0 \]

Ampere's law

\[ \mathbf{J} = -\mu_0 \nabla \times (B \times \psi) \]
\[ \mu_0 \mathbf{J} = \nabla^2 \psi - (\nabla \cdot \mathbf{U}) \cdot \nabla \psi \]

Axial vorticity

\[ \nabla_2 = \nabla \times \mathbf{U} = \nabla^2 \psi \]

Two unknown potentials:
\[ \psi(\tau, \theta, \tau, t) \text{ and } \chi(\tau, \theta, \tau, t) \]
Simplifying the MHD Equations

\[ \rho \, \frac{d \mathbf{v}_+}{d t} = -\nabla \rho + \frac{1}{\mu_0} (\nabla \times \mathbf{B}) \times \mathbf{B} \]

\[ = -\nabla \left( \rho + \frac{B^2}{2 \mu_0} \right) + \frac{1}{\mu_0} (\mathbf{B} \cdot \nabla) \mathbf{B} \]

\[ \hat{z} \cdot \nabla \left( \mathbf{v} \times \mathbf{v}_+ \right) = \frac{1}{\mu_0} (\mathbf{B} \cdot \nabla) (\hat{z} \cdot \nabla \times \mathbf{B}) \]

\[ \rho \, \frac{d}{dt} \left( \nabla^2 \psi \right) = \frac{1}{\mu_0} (\mathbf{B} \cdot \nabla) \nabla^2 \psi = (\mathbf{B} \cdot \nabla) J_z \]

Axial vorticity changes according to field-aligned variation of axial current.
Simplifying the induction equation

\[
\frac{2 \mathbf{B}}{\Delta t} = \nabla \times (\mathbf{v}_t \times \mathbf{B}) = \nabla \times (\mathbf{v}_t \times \mathbf{B}) \quad \Rightarrow \
\frac{\partial \mathbf{B}_1}{\partial t} + (\mathbf{v}_t \cdot \nabla) \mathbf{B}_1 = \frac{d}{dt} \mathbf{B}_1 = (\mathbf{v}_t \cdot \nabla) \mathbf{B}_1
\]

Substituting flux functions

Polaroid flux evolves dynamically due to field-aligned changes in the stream function.
THE IMPORTANCE OF $\overline{B} \cdot \overline{D}$

\[
P \frac{d}{dt} \nabla^2 \chi = \nabla \left( \overline{B} \cdot \overline{D} \right) \nabla^2 \psi \quad (\text{MHD})
\]

\[
\frac{d}{dt} \psi = \left( \overline{B} \cdot \overline{D} \right) \chi \quad (\text{Induction})
\]

**Linear**

\[
\overline{B} \cdot \overline{D} = B_z \frac{2}{R^2} + B_r \overline{D}_r = -i \frac{m}{R} B_z + i \frac{m}{n} B_\rho = i \frac{B_\rho(n)}{n} \left( m - m_\rho(n) \right)
\]

with \( B(n) = \frac{1}{R} \frac{B_\rho}{B_\rho(n)} \) = safety factor

\[ \overline{B} \cdot \overline{D} \rightarrow 0 \text{ when } m/n = 8(n) \] resonance

when \( \overline{B} \cdot \overline{D} \neq 0 \), then ideal reduced mode make sense

\( \overline{B} \cdot \overline{D} = 0 \), then reduced mode does not describe dynamics

(Side Bar! \( \overline{B} \cdot \overline{D} = 0 \) defines "interchange" modes. These are the dominant modes in magnetospheres and dipoles, etc.)
\[ V_4 = 0, \quad \frac{2}{2c} = 0, \quad 0 = -\nabla P + \mathbf{J} \times \mathbf{B} \]
\[ = -\nabla P + \mathbf{J} \times (\mathbf{\nabla} \times \psi) \]
\[ = -\nabla P - J_z \nabla \psi \]

All equilibrium variation is radial, in \( \nabla \psi \) direction.

So
\[ \frac{\nabla \psi \cdot \nabla P}{(\nabla \psi)^2} = -J_z \Rightarrow \frac{2P}{2\psi} = -J_z \quad (=\text{constant}) \]

When \( J_z = \text{constant} \), Saha-Frank's

\[ B_0 = \frac{n B_z}{R B_p(R)} \]

\[ \nabla \times \mathbf{B}_p = \left\{ \begin{array}{l}
\nabla^2 \psi = -\kappa_0 \frac{2P}{2\psi} \\
\end{array} \right. \]

Equilibrium equations for

\[ \kappa_0 J_z = \nabla \times \mathbf{B}_p = \left\{ \begin{array}{l}
P(\psi), \quad J_z(\psi), \quad B(\psi) \\
\end{array} \right. \]

\[ = \frac{1}{\gamma} \frac{2}{\gamma - 1} \left( \frac{\mathbf{B}_p}{R^2} \right) \]

\[ = \frac{B_z}{R^2} \left( \gamma - 1 \right) \]

\[ P(\psi), \quad J_z(\psi), \quad B(\psi) \]
Shafranov's Simple Constant-J Equilibrium

\[ J_\perp(q) \]

\[ B_\rho(a) = \frac{a \mu_0 J_\perp}{2} \quad \psi(a) = \frac{1}{2} \frac{a \mu_0}{a} \]

\[ \varphi(q) = \varphi(a) = \text{constant} \]

\[ \frac{2 \mu_0}{a} \frac{B_\rho}{R J_\perp} \]

\[ P(q) = (\psi(q) - \psi(a)) \frac{2 B_\rho(a)}{a \mu_0} \]

\[ R = \frac{2 \pi R_s \int_{0}^{\infty} \rho \psi(\rho) d\rho}{2 \pi \mu_0 R^2} = \frac{2 \mu_0}{a} \frac{B_\rho}{R J_\perp} \]

\[ \rho_\rho = \langle B \rangle \frac{B_\rho^2}{B_\perp^2} = \left( \frac{\rho}{B} \right) \frac{B_\rho^2}{B_\perp^2} \]

\[ P_{\rho} = \langle B \rangle \frac{B_\rho^2}{B_\perp^2} = \frac{1}{2} \frac{B_\rho^2}{B_\perp^2} \]

Plasma is not diamagnetic, not paramagnetic, not tiny (or order 1)

Poloidal Field Energy

\[ \psi = \int_{0}^{\infty} 2 \pi \rho \psi(\rho) d\rho = \frac{1}{2} \int_{0}^{\infty} 2 \pi \rho \psi(\rho) d\rho \]

\[ \psi = \int_{0}^{\infty} 2 \pi \rho \psi(\rho) d\rho = \frac{1}{2} \int_{0}^{\infty} 2 \pi \rho \psi(\rho) d\rho \]
Wesson's Cylindrical Pinch Equilibrium

Ref: Wesson, Nucl Fusion 18 (1978) p. 87

\[ J_2(\gamma) = J_0 (1 - \frac{\gamma^2}{\alpha^2})^v \]

\[ \rho_i \neq \frac{1}{2} \]

\[ J_0 = \text{central current density} \]

\[ q(\alpha) = \frac{2B_0}{\mu_0 R R J_0} \]

\[ \mu_0 I_P = \frac{2e^2 q^5 B_0}{\mu_0 B_0 R (1 + \gamma_i)} \]

\[ v = \frac{q(a)}{q(0)} - 1 \]

\[ \langle B_0 \rangle \sim e^2 q^2 \]

\[ \beta_P \sim 1 \]

\[ \langle B_0 \rangle \sim \frac{2}{q_0} \]

Equilibrium set by \( q(0), q(a) \)

Two parameters

\[ \frac{d}{d\alpha} q = S = \text{magnetic shear} \neq 0 \]

Pressure profile can be

\[ \text{more peaked} \]

B.V. \( \rightarrow 0 \) at

Innermost Resonant Layer

\( (m - \frac{1}{2} q(\alpha)) \rightarrow 0 \)
LINEARIZED REDUCED MHD

\[
\frac{\partial}{\partial t} \nabla^2 \psi = \frac{1}{\mu_0} \frac{\partial}{\partial t} \nabla^2 \Psi
\]

\[
\frac{\partial}{\partial t} = -i \omega
\]

\[
B_0 \cdot \nabla \rightarrow i \frac{\mu_0}{\gamma} (m - m_0 \gamma)
\]

\[
(\mathbf{\nabla} \cdot \mathbf{\nabla}) \nabla^2 \psi = (\mathbf{\nabla} \cdot \mathbf{\nabla}) \nabla^2 \Psi_0 + (B_0 \cdot \mathbf{\nabla}) \nabla^2 \tilde{\Psi} + \text{nonlinear terms}
\]

\[
\text{Mo\:}\frac{\mathbf{J}}{\gamma_{\text{ek}}} \left\{ \begin{array}{l}
\text{when equilibrium current density varies within plasma} \\
\text{current density varies within plasma}
\end{array} \right. 
\]

\[
\tilde{\mathbf{B}} \cdot \nabla \tilde{J}_2 (\gamma) = (\mathbf{\nabla} \times \mathbf{\nabla} \Psi), \quad \frac{2 \mathbf{J} \times \mathbf{B}}{2\gamma} = -\mathbf{\nabla} \times \mathbf{\nabla} \frac{2 \mathbf{J} \times \mathbf{B}}{2\gamma}
\]

\[
\text{LINEAR MHD:} \quad -\mathbf{\nabla} \times \mathbf{\nabla} \tilde{\Psi} = \frac{m}{\gamma} \frac{\partial}{\partial t} \tilde{\Psi} + \frac{\mu_0}{\gamma m_0} (m - m_0 \gamma) \nabla^2 \tilde{\Psi}
\]

\[
\text{PLASMA INERTIA:} \quad \frac{\gamma}{3}
\]

\[
\text{LINEAR INDUCTION:} \quad -i \omega \tilde{\Psi} = \frac{\mu_0}{\gamma} (m - m_0 \gamma) \tilde{\Psi}
\]
SHAFRANOV CONSTANT $J_z$ EQUATION

\[ \frac{2J_z}{\delta q} = 0 \]

**Within Plasma**

\[ -\rho \omega \nabla^2 \chi = \frac{B_0}{\kappa_0} (m - m_q) \nabla^2 \psi \]

\[ -\omega \psi = \frac{B_0}{\kappa_0} (m - m_q) \chi \]

**Normal Modes (ALLEN WAVES)**

\[ \begin{pmatrix} \frac{\rho \omega}{\kappa_0} (m - m_q) & \chi \nabla \chi \nabla \psi \\ \frac{B_0}{\kappa_0} (m - m_q) \omega & \psi \nabla \psi \nabla \chi \end{pmatrix} \]

\[ \tilde{\chi}, \tilde{\psi} \sim e^{-j \omega t} \]  

\[ \omega = \omega_A (m^q - m_q) \]

**Note:**

\[ \frac{B_0}{\kappa_0} = \text{constant} = \frac{B_0(\omega)}{e} \]

\[ \omega_A = \frac{B_0}{\kappa_0} = \frac{B_0^2}{\kappa_0^2} \left( \frac{e R}{\delta q} \right)^2 \]

\[ = \frac{V_A}{(e R)^2} \text{ ALLEN TRANSIT Time} \]

**Radial Structure**

Not specified

\[ B_0 = 2.5 \]

\[ m = 3, n = 1 \]

\[ m = 1, n = 1 \]

\[ m = 1, n = 1 \]
GLOBAL EXTERNAL KINK MODES

\[ \nabla \cdot \mathbf{A} = \frac{1}{a} \sum \left( \mathbf{A}_n \right) + \cdots \]

STEP DISCONTINUITY AT PLASMA EDGE

\[ \frac{d}{dt} \mathbf{\nabla} \cdot \left( \mathbf{B} \mathbf{\nabla} \psi \right) = (\mathbf{\nabla} \times \mathbf{J}) \cdot \mathbf{\nabla} + \frac{2J_0}{A_0 a} + \frac{i B_0}{A_0 a} \left( m - m_B \right) \psi^2 \]

\[ \text{VERY BIG AT EDGE} \]

\[ \int_{a_-}^{a_+} \int_{2\pi} \int_0^{\pi} \text{ "mho" } \Rightarrow \]

\[ \omega \rho \frac{\partial \psi}{\partial n} \bigg|_c = \frac{2m B_0(a)}{\mu_0 a^2} \psi^2 + \frac{B_0(a)}{\mu_0 a} (m - m_B) \psi \left[ \frac{1}{\psi} \frac{1}{\Delta^0} \right]^{-1} \]

IMPORTANT:

\[ \Delta'(a) = \frac{1}{\psi} \left( \frac{2 \psi}{\partial n} \right)_{a} - \frac{2 \psi}{\partial n}_{a} \]

THIS MEASURES PERTURBED SURFACE CURRENT ON PLASMA

\[ \Delta'(a) \psi = \mu_0 R^2 (\theta, \phi) / a \]
WHAT ARE \((\psi(\xi), \chi(\eta))\) inside and outside plasma?

\[\nabla^2 \psi = 0\]
\[\nabla^2 \chi = 0\]

(no current) outside plasma

(no current, no flow, no vortex, no current)

(no currents inside plasma)

(no vortexity within plasma)

PERTURBED FIELDS + PLASMA MOTION

BUT no currents or vortexity

\[\psi(\eta) \sim (\frac{\eta}{a})^m \quad \eta < a\]

\[\sim (\frac{\eta}{a})^m \quad \eta > a\]

\[\sim \frac{(\frac{\eta}{a})^m - (\frac{\lambda}{b})^m}{\frac{b}{a} - \frac{a}{b}}\] for \(a < \lambda < b\)

NO WALL

with wall

\[\nabla \psi = \frac{\partial}{\partial \xi} \psi - \frac{\partial}{\partial \eta} \chi = \frac{\partial}{\partial \xi} \psi - \frac{i\partial}{\partial \eta} \chi\]

inside

outside
**Kink Mode**

\[- \omega \, \bar{\psi}_a = \frac{B_0}{a} \left( m - m_b \right) \bar{X}_a\]

\[\omega \, \frac{2 \Omega}{2 \pi} \bar{X}_a = \frac{2mB_0}{\mu_0 a^2} \bar{\psi}_a \left[ \frac{m}{2} \left( \frac{\lambda + 1}{\lambda} \right) - \frac{m^2}{2} \right] - \left( \frac{\omega}{\lambda} \right) \bar{X}_a\]

\[\Delta'(a) = - \frac{2m}{a} \left( \frac{b/a}{(b/a)^m - (\frac{a}{b})^m} \right)\]

\[\frac{2 \Omega}{2 \pi} \bar{X}_a = - \frac{m}{a} \bar{X}_a\]

**Global Kink modes**

\[
\begin{pmatrix}
\omega \\
\frac{2mB_0}{\mu_0 a^2} \left[ \frac{m}{2} \left( \frac{\lambda + 1}{\lambda} \right) - 1 \right]
\end{pmatrix}
\cdot
\begin{pmatrix}
\bar{\psi}_a \\
\omega \\
\frac{2mB_0}{\mu_0 a^2} \left[ \frac{m}{2} \left( \frac{\lambda + 1}{\lambda} \right) - 1 \right] \bar{X}_a
\end{pmatrix} = 0
\]

\[\Delta'(a) = - \frac{m}{a} (\lambda + 1)\]

*Shafrawi's Formula*