Lecture 15:
Plasma Physics 1
APPH E6101x
Columbia University
Last Lecture

- Langmuir probes
- Langmuir probes in magnetized plasma
This Lecture

- Beam-plasma instability
Instabilities

- “Micro-instabilities”
- “Macro-instabilities”
Observations of the Beam-Plasma Instability

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The nonlinear limit of the instability driven by a low density cold electron beam in a collisionless plasma is experimentally found to be determined by the trapping of the beam by the most rapidly growing wave.

The apparatus has previously been described, where it was used for a similar experiment to test quasilinear theory. The apparatus and techniques are also very similar to those used by Malmberg and Wharton for the problem. The present work complements and extends the results of that paper in the nonlinear regime. For these experiments, the two-meter column contained a 3-cm diam plasma with a density near 10^9 and a temperature of 20 eV. The magnetic field of 1 kG was sufficient to render the dynamics one dimensional and the background pressure of 8×10^{-6} Torr precluded collisional effects. The plasma is quiet, with low-frequency density fluctuations of only a few percent. The axial density uniformity is excellent. Although the density drops approximately 20% over the 40 cm near the gun, no measurable gradient exists over the remainder of the column. A gradient of more than a few percent would be readily detectable. The density gradient near the gun is not significant. The waves are still far from saturation when they enter the uniform region, and all the important physics occurs in that region.
Experimental Test of Quasilinear Theory*

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The shape and amplitude of the electron-plasma wave spectrum resulting from a
"gentle bump" on the tail of the electron velocity distribution of a plasma is measured
and found to be in good agreement with quasilinear theory.

The plasma is produced by ionization of hydrogen gas in a coaxial stub microwave cavity, and
it drifts along magnetic field lines down a 250-
cm aluminum tube 10 cm in diameter. The plasma
is terminated by a plate with a \( \frac{3}{4} \text{-cm} \) hole
behind which the electron gun is mounted. The plate is biased to reflect electrons with velocities
less than the slowest of those that come from the
gun. The tube acts as a waveguide beyond cutoff
for electromagnetic propagation at the wave fre-
cequencies used, and has four longitudinal slots
equispaced around the circumference along which
antenna probes may be moved. The complete
assembly is contained in a vacuum chamber and
maintained at a pressure of less than \( 10^{-5} \) Torr
by diffusion and Ti sublimation pumps. Axial
magnetic field coils are mounted around the vacu-
um chamber and provide a magnetic field of
about 1 kG. The electron gun is a simple diode
with a large-aperture (2.5-cm diam) plate
mounted inside a soft-iron cylinder. The result
is a beam distribution with a large spread in
parallel energy, which when injected into the
plasma makes a bump on the tail of the parallel
electron-velocity distribution. The energy dis-
btribution of the beam is measured directly using
a large-aperture gridded energy analyzer.
Typical results for the development of the instability are shown in Fig. 1. The wave power is measured with a loosely coupled, calibrated (−32 dB) coaxial probe connected to a broad band amplifier and rf voltmeter. The probe may be moved the length of the machine, and the figure shows the result for the absolute power in the waves. The qualitative behavior is precisely that predicted: linear growth to a peak, followed by slow oscillation. Although the theory was presented for infinite geometry and an initial value problem, the argument can easily be applied to finite geometry and growth in space. The peak wave power should be

\[ P_w = 2^{2/3} \eta^{1/2} P_b, \quad (1) \]

where \( P_b = I_b V_b \), the input beam power, and

\[ \eta = \int_0^a n_b(r) r \, dr \left( \int_0^a n_p(r) r \, dr \right)^{-1}. \quad (2) \]

\[ \text{Fig. 1. Total wave power as a function of distance from the point of injection of the beam into the plasma column. The background plasma had a density of } 7 \times 10^8 \text{ and a temperature of } 18 \text{ eV.} \]
Fig. 2. Total wave power measured at the spatial maximum as a function of beam current. The wave power is in relative units.

growth in space. The peak wave power should be

\[ P_w = 2^{2/3} \eta^{1/3} P_B, \]

where \( P_b = I_b V_b \), the input beam power, and

sections in the potential well. In space, the oscillations will appear with a wavelength determined by the particle oscillation frequency in the well and the propagation velocity, the beam velocity \( u \). If the potential is sinusoidal and the particles are trapped near the bottom of the well, the oscillation length is given by

\[ \lambda_{osc} = 2\pi u (m/ekE)^{1/2}. \]  

(3)

Making use of Eq. (1), we note that this implies that

\[ \lambda_{osc} \propto I_b^{-1/3}. \]  

(4)

Fig. 3. Oscillation wavelength of the wave energy as a function of beam current.
NONLINEAR EVOLUTION OF A TWO-STREAM INSTABILITY*

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Calculations of a two-stream instability have been made by following the motion of the phase-space boundaries of an incompressible and constant-density phase-space fluid. Because of the condensation of holes, which to a good approximation act as gravitational particles, large-scale nonlinear pulses develop.

\[ \frac{\partial f}{\partial t} + v \frac{\partial f}{\partial x} - \frac{\partial \varphi}{\partial x} \frac{\partial f}{\partial v} = 0, \]

\[ \frac{dx}{dt} = v, \quad \frac{dv}{dt} = -\frac{\partial \varphi}{\partial x}, \]

\[ \frac{\partial^2 \varphi}{\partial x^2} = \omega_p^2 \left[ \int f \frac{dv}{v_0} - 1 \right], \]

The example to be discussed is a two-stream instability, in which the electron plasma is slightly perturbed at \( t=0 \) from an equilibrium characterized by four straight lines in phase space: \( f = 1 \) for \( \frac{1}{2}v_0 < |w| < v_0 \) and \( f = 0 \) elsewhere. Periodic boundary conditions are imposed at \( x = (0, L) \) and the parameters of the problem are \( v_0 \Delta t / \Delta x = 0.25, \omega_p \Delta t = 1/20, \) and \( \Delta x = L/64, \) where \( \Delta x \) is the grid used for evaluating Poisson's equation. The unstable wave numbers are \( k = 2m/L \) with \( n = (1, 2), \) and the linear growth rates are \( \gamma / \omega_p = 0.30, 0.315. \)
The most striking feature of the calculation is the behavior of the \( f = 0 \) "cavity" which initially occupies the strip \( |v| < \frac{1}{2} v_0 \) between the two plasma layers. This must preserve constant area as it deforms, and it is seen in Fig. 1 to coalesce into holes of roughly elliptical shape, so that a large-amplitude electrostatic wave is set up. Superimposed on this wave are coherent oscillations due to rotation of the holes in phase space, and also random fluctuations due to the motion of smaller elements of the hole "fluid." The two outer curves adjust almost adiabatically to the instantaneous potential function.

**FIG. 1.** Evolution in phase space of a two-stream instability from time steps 200 to 600 at intervals of 50 steps. Each step is 1/20 of a plasma period, and the horizontal and vertical coordinates are \( x, v \), respectively. Periodic boundaries have been imposed and three identical periods are shown along each row. The shaded area represents the \( f = 0 \) region enclosed by the plasma fluid.
8.3.2 The Dispersion Relation for a Free Electron Beam

\[ 0 = \varepsilon(\omega, k) = 1 - \frac{\omega_{pe}^2}{(\omega - k v_0)^2}. \]  \hspace{1cm} (8.25)

In Sect. 6.4 we had used first-order perturbation theory to derive the dielectric function of a cold plasma as

\[ \varepsilon(\omega) = 1 - \frac{\omega_{pe}^2}{\omega^2} = 1 + \chi_p. \]  \hspace{1cm} (8.2)
8.1.2 Dispersion of the Beam-Plasma Modes

\[ \varepsilon(\omega, k) = 1 + \chi_p + \chi_b = 1 - \frac{(1 - \alpha_b)\omega_{pe}^2}{\omega^2} - \frac{\alpha_b\omega_{pe}^2}{(\omega - kv_0)^2}. \]
\[ \varepsilon(\omega, k) = 1 + \chi_p + \chi_b = 1 - \frac{(1 - \alpha_b)\omega_p^2}{\omega^2} - \frac{\alpha_b\omega_p^2}{(\omega - k v_0)^2}. \]
Fig. 8.4 The dispersion relation for the beam-plasma modes at $\alpha_b = 0.01$. The dotted lines mark the asymptotes $\omega = \omega_{pe}$ and $\omega = k v_0$. The plasma mode develops into the fast space-charge wave which then approaches the fast beam mode. For $k v_0 / \omega_{pe} < 1$, the beam mode is a complex conjugate slow space-charge wave. At the triple point it splits into stable modes, a slow beam mode and a plasma mode. The second plasma mode with negative $\omega$ remains unaffected by the beam.

For non-vanishing $\alpha_b$, the positive plasma mode connects to the beam mode and becomes the fast space-charge wave, $\omega / k > v_0$. For $k v_0 / \omega_{pe} < 1/3$, the beam modes form a complex conjugate pair. These waves are propagating more slowly than the beam and are called slow space-charge waves. The one with $\omega I > 0$ is exponentially growing in time. The growth rate takes a maximum value near the intersection $\omega_{pe} = k v_0$. At the triple point, the slow space-charge waves become real and form the slow beam mode and the plasma mode. The second plasma mode with negative $\omega$ remains unaffected by the beam.

8.1.3 Growth Rate for a Weak Beam

For small values of $\alpha_b$, the slow space-charge wave that has the maximum growth rate $\gamma = \omega_I$, is not close to $k v_0 / \omega_{pe} = 1$. This means that the phase velocity of the wave is nearly resonant with the electron beam, $v_\phi \approx v_0$. Therefore, it is reasonable to seek an approximate solution for $\varepsilon(\omega, k) = 0$ in the vicinity of the resonance point $\omega_{pe} = k v_0$. Introducing $\omega = \omega_{pe} + \Delta_1 \omega$, we can rewrite the dielectric function in this regime as...
8.1.3 Growth Rate for a Weak Beam

\[ \omega = \omega_{pe} + \Delta \omega \]

\[ 0 = \varepsilon = 1 - \frac{\omega_{pe}^2}{(\omega_{pe} + \Delta \omega)^2} - \frac{\alpha_b \omega_{pe}^2}{(\omega_{pe} + \Delta \omega - kv_0)^2} \]

\[ \approx 1 - \frac{\omega_{pe}^2}{\omega_{pe}^2} + \frac{2 \Delta \omega}{\omega_{pe}^3} - \frac{\alpha_b \omega_{pe}^2}{(\Delta \omega)^2} . \]

\[ \Delta \omega = \left( \frac{\alpha_b}{2} \right)^{1/3} \omega_{pe} e^{n2\pi i/3} \quad \text{with:} \quad n = 0, 1, 2 . \]
\[ \Delta \omega = \left( \frac{\alpha_b}{2} \right)^{1/3} \omega_{pe} e^{n2\pi i/3} \]

\[ \omega = \omega_{pe} + \Delta \omega \]

with: \( n = 0, 1, 2 \).

\[ \omega = \omega_{pe} \left[ 1 + \left( \frac{\alpha_b}{2} \right)^{1/3} \right] \]

\[ \omega_R = \omega_{pe} \left[ 1 - \frac{1}{2} \left( \frac{\alpha_b}{2} \right)^{1/3} \right] \]

\[ \omega_I = \pm \frac{3^{1/2}}{2} \left( \frac{\alpha_b}{2} \right)^{1/3} \omega_{pe}. \]

A fourth root at \( \omega = -\omega_{pe} \) is non-resonant.

The most spectacular result for the unstable mode is the fact, that the growth rate depends on the third root of the beam fraction \( \alpha_b \). Therefore, a beam fraction of \( \alpha = 0.002 \) generates a wave that has an e-folding after only ten wave periods \( (\omega_R/\omega_I \approx 10) \). The real part of the frequency is close to \( \omega_{pe} \) and this wave can therefore be identified as the Langmuir wave. The high growth rate explains why self-excited Langmuir oscillations are so ubiquitous in dc discharges.
8.1.4 Why is the Slow Space-Charge Wave Unstable?

\[
\frac{\partial v_b}{\partial t} + v_0 \frac{\partial v_b}{\partial x} = -\frac{e}{m_e} \hat{E} \exp[i(kx - \omega t)].
\]

\[
\hat{v}_b = \frac{e}{i(\omega - kv_0)m_e} \hat{E}
\]

Again, the electron behaves as an inductor, as long as \( \omega - kv_0 > 0 \), which is realized for the fast space-charge wave. But, for the slow space-charge wave, the opposite case is realized with \( \omega - kv_0 < 0 \). Therefore, beam electrons show a strange behavior when they interact with the slow wave. Their motion in the wave field is such as if they had a “negative mass”. In the language of electronics, the electron now behaves like a capacitor for which the current leads the voltage by a phase shift of 90°.
\[ n_b = n_{b0} + n_{b1}, \quad v_b = v_0 + v_{b1}. \]

\[ \hat{v}_b = \frac{e}{i(\omega - kv_0)m_e} \hat{E} \]

\[ \frac{\partial n_b}{\partial t} + \frac{\partial (n_b v_b)}{\partial x} = 0 \]

\[ (-i\omega + kv_0)\hat{n} + ikn_{b0}\hat{v}_b = 0 \]

\[ \hat{n}_b = \frac{n_{b0}k}{\omega - kv_0} \hat{v}_b \]

\[ \hat{n}_b = \frac{en_{b0}k}{i(\omega - kv_0)^2} \hat{E} \]
\[ \langle W_{\text{kin}} \rangle = \frac{1}{2} m_e \langle (n_{b0} + \hat{n}_b)(v_0 + \hat{v}_b)^2 \rangle \]

\[ = \frac{1}{2} m_e \left( n_{b0} v_0^2 + 2 n_{b0} v_0 \hat{v}_b + n_{b0} \hat{v}_b^2 + \hat{n}_b v_0^2 + 2 \hat{n}_b v_0 \hat{v}_b + \hat{n}_b \hat{v}_b^2 \right) \]

\[ \langle W_{\text{kin}} \rangle = \frac{1}{2} n_{b0} m_e v_0^2 + \frac{1}{2} m_e \langle n_{b0} \hat{v}_b^2 + 2 \hat{n}_b v_0 \hat{v}_b \rangle. \]  

(8.18)

Using (8.15), we find that for nearly resonant particles, \(|\omega/k - v_0| \ll v_0\), the second term in the angle brackets is much larger than the first. Moreover, this contribution of the wave to the beam energy has different signs for the slow wave and the fast wave.
8.1.5 Temporal or Spatial Growth

\[ kv_0 = \omega + i \left( \frac{\alpha_b \omega_{pe}^2 \omega^2}{\omega_{pe}^2 - \omega^2} \right)^{1/2} \]

\[ k_1 = \frac{\omega_{pe}}{v_0} \frac{\alpha_{b1/2} \omega}{(\omega_{pe}^2 - \omega^2)^{1/2}} \]
Next Lecture (Monday)

- Numerical Simulation of two-stream
- Ch. 8: MHD Instabilities (Part 1)