Lecture13:

# Plasma Physics 1

APPH E6101x

Columbia University

Fall, 2015

# This Lecture

- 1995 Midterm
- 2005 Midterm

# E6101x Plasma Physics I
# Midterm Examination
# (November 16, 1995)

## Instructions

1. Relax!

2. *Do not guess.* Several of the questions have positive score for correct answers and *negative scores* for incorrect answers.

3. Record all of your answers on this examination sheet in the spaces provided. *You must return all copies of this exam when finished.*

4. The midterm is open book.

6. You have 1 hour and 25 minutes to complete the exam.

## Part 1: .True. or .False.

**(2 points for each correct answer, and -1 point for each incorrect answer.)**

Plasmas are collections of electrons and protons of sufficiently high energy that no states of bound electrons exist.

All plasmas must have a large number of charged particles within a Debye sphere in order for collective effects to dominate over single-particle effects.

In typical plasmas, it is unlikely for ion and electron Debye lengths to be comparable since the ion mass is much larger than the electron mass.

When the plasma parameter is large, the ratio of the energy density of collective plasma oscillations is much, much larger than the energy density of the plasma's thermal motion.

In any magnetized plasma, there are three adiabatic invariants, $\mu$, $J$, $\psi$, corresponding to the periodic cyclotron, bounce, and drift motions.

The polarization drift and the $\mathbf{E} \times \mathbf{B}$ drift can never be co-directional.

# Part 1: .True. or .False.

**(2 points for each correct answer, and**
**-1 point for each incorrect answer.)**

In a steady alternating electric field, the ratio of the magnitude of the polarization drift to the magnitude of the $\mathbf{E} \times \mathbf{B}$ drift is $\omega/\omega_c$.

For a plasma consisting of equal densities of electrons and positrons, no current is induced by mutually perpendicular gravitational and magnetic fields.

For a plasma consisting of equal densities of electrons and positrons, no current is induced by the polarization drift.

In an axisymmetric magnetic mirror, the curvature and gradient-$B$ drifts are in the radial direction.

Since the curvature and gradient drifts depend linearly upon a particle's kinetic energy and not on its mass, a plasma with equal ion and electron temperatures has no current induced by inhomogeneous magnetic fields.

The group velocities of plasma waves can be either faster or slower than their phase velocities.

# Part 1: .True. or .False.

**(2 points for each correct answer, and**
**-1 point for each incorrect answer.)**

Using knowledge of the plasma's dielectric tensor and Maxwell's equations, we can always investigate the types of waves that can propagate in the plasma even if we don't know the plasma's spatial dimensions.

In a tokamak, magnetic fields form nested toroidal surfaces. Thus, in the absence of collisions (or wave-particle interactions), particles circulate either clockwise or counter-clockwise around the torus, but they never reverse directions.

For ~~gives~~ WAVES propagating along a magnetic field, the phase velocity of the right-hand circularly polarized (RHCP) wave is faster than the phase velocity of the left-hand circularly polarized (LHCP) wave. Thus, the electric field vector will rotates in a right-handed sense if the wave energy is equally divided between the RHCP and LHCP waves.

The wavelength becomes small as a wave approaches a resonance.

The wavelength becomes small as the wave approaches a cutoff.

In the absence of collisions, there can be no damping of plasma waves.

# Part 1: .True. or .False.

**(2 points for each correct answer, and -1 point for each incorrect answer.)**

For high-frequency electromagnetic plasma waves, the damping due to collisions is linearly proportional to the collision frequency. This can be understood simply by calculating the work performed by collisions due to the electron's quiver velocity.

The ion's quiver velocity due to an oscillating electric field is related to the electron's quiver velocity by the square-root of the ratio of their temperatures, $T_i/T_e$.

In a plasma with static Maxwellian velocity distributions, the negative energy wave corresponds to waves with negative frequencies.

As a two-stream instability grows, the kinetic energy of plasma also grows.

The Vlasov equation conserves phase-space density.

An exact solution to the Vlasov equation is just as difficult to find than an exact solution to the individual particle trajectories.

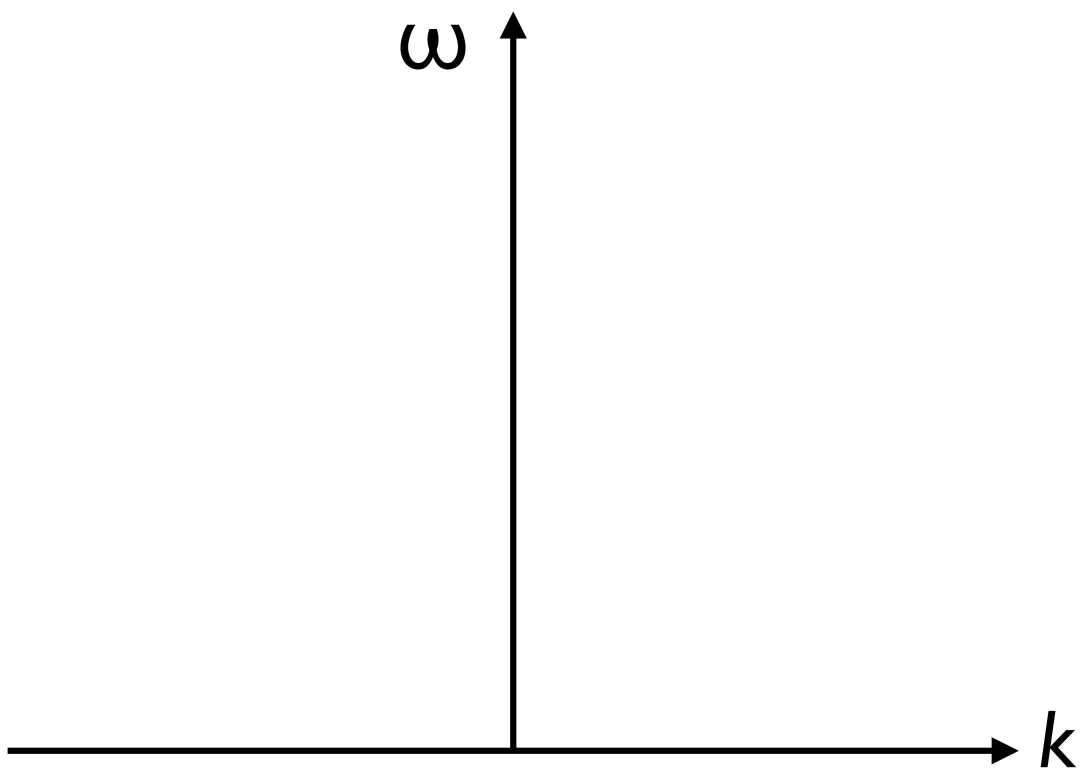
The zeros of the dispersion relation correspond to the poles of the Laplace transform of a linearized plasma wave initial value problem.

# Part 2: Dispersion Plots

## (5 points each)

For each of the graphs below, sketch the solutions of the dispersion relation for homogeneous plasmas by plotting frequency, $\omega$, as a function of wavenumber, $k$. For each graph, assume the plasma consists of electron densities of electrons and protons. Label key resonances and cutoffs. If your scales are mixed or nonlinear, indicate so. Finally, note that just rough sketches are required. You are not given enough information to make exact dispersion plots.

### Field-Free Plasma with Zero Electron Pressure

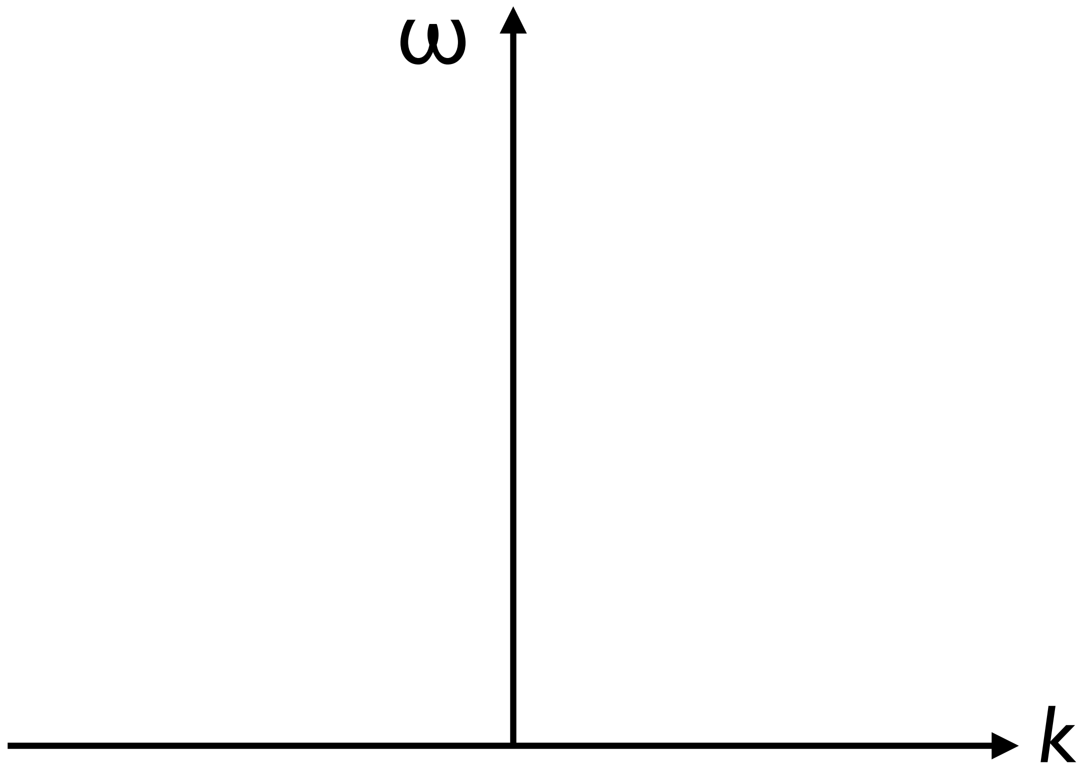


# Part 2: Dispersion Plots

## (5 points each)

For each of the graphs below, sketch the solutions of the dispersion relation for homogeneous plasmas by plotting frequency, $\omega$, as a function of wavenumber, $k$. For each graph, assume the plasma consists of electron densities of electrons and protons. Label key resonances and cutoffs. If your scales are mixed or nonlinear, indicate so. Finally, note that just rough sketches are required. You are not given enough information to make exact dispersion plots.

### Magnetized Plasma with Zero Electron Pressure, Infinitely Massive Ions, and Propagation Along B

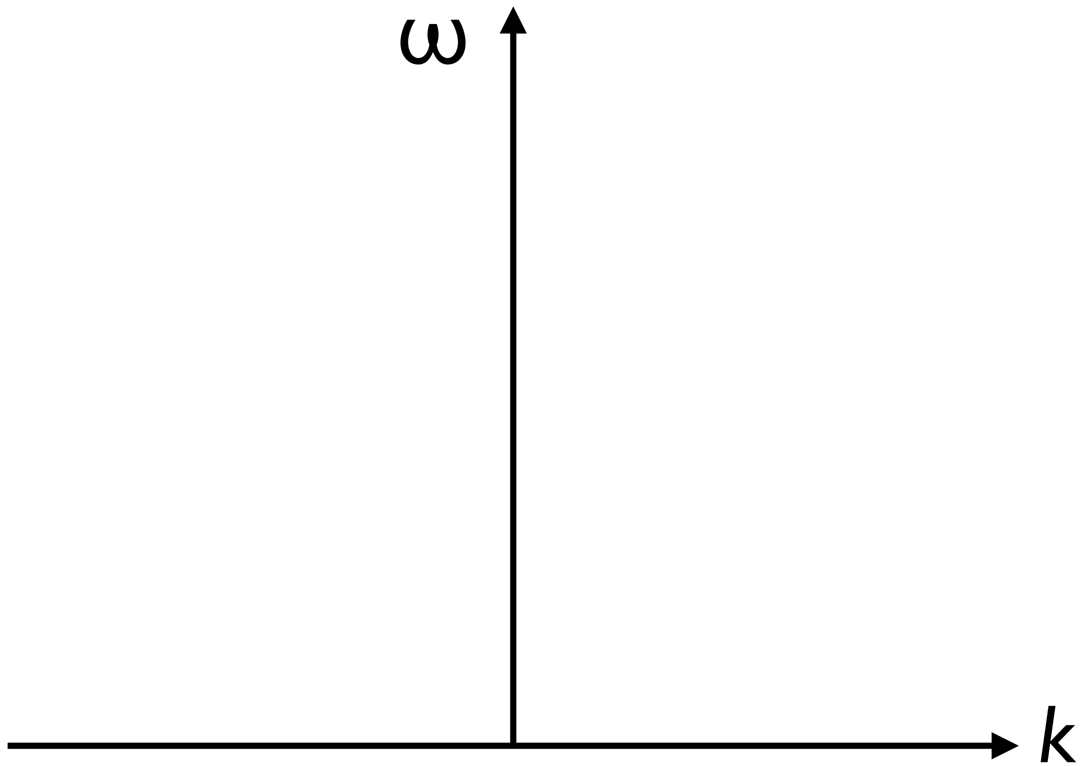


# Part 2: Dispersion Plots

## (5 points each)

For each of the graphs below, sketch the solutions of the dispersion relation for homogeneous plasmas by plotting frequency, $\omega$, as a function of wavenumber, $k$. For each graph, assume the plasma consists of electron densities of electrons and protons. Label key resonances and cutoffs. If your scales are mixed or nonlinear, indicate so. Finally, note that just rough sketches are required. You are not given enough information to make exact dispersion plots.

**Magnetized Plasma with Zero Electron Pressure, Infinitely Massive Ions, and Propagation Across B**

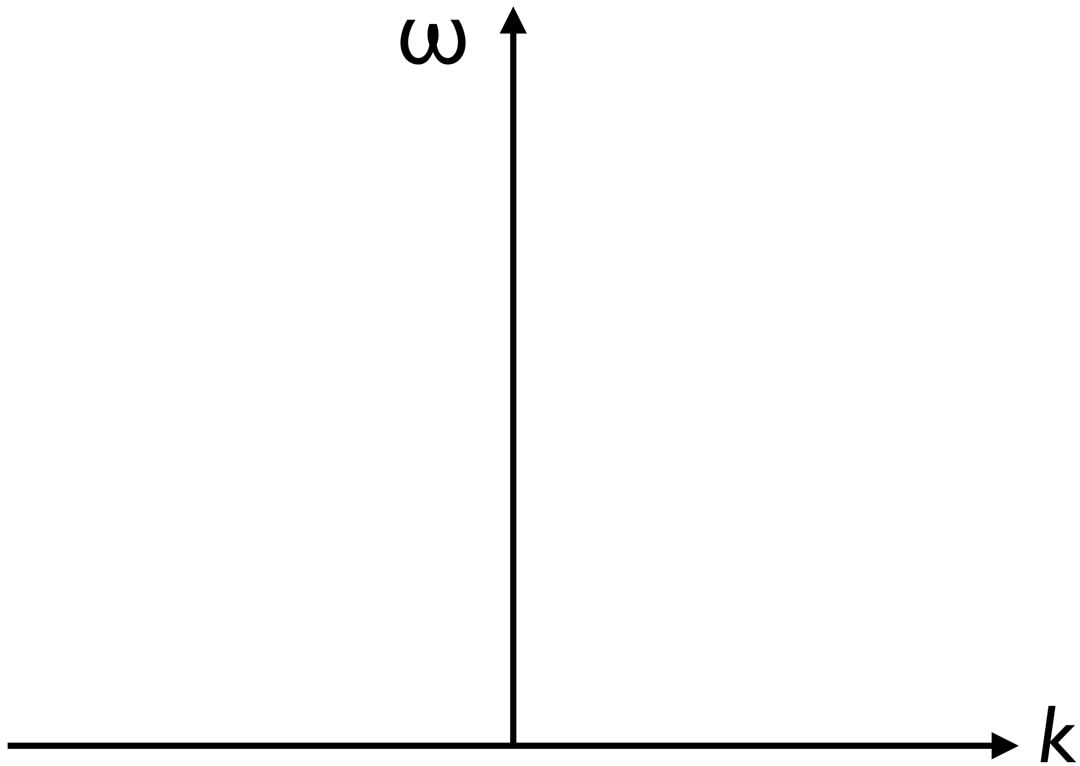


## Part 2: Dispersion Plots

### (5 points each)

For each of the graphs below, sketch the solutions of the dispersion relation for homogeneous plasmas by plotting frequency, $\omega$, as a function of wavenumber, $k$. For each graph, assume the plasma consists of electron densities of electrons and protons. Label key resonances and cutoffs. If your scales are mixed or nonlinear, indicate so. Finally, note that just rough sketches are required. You are not given enough information to make exact dispersion plots.

**Magnetized Plasma with Zero Pressure with Frequency Well-Below the Ion Cyclotron Frequency**

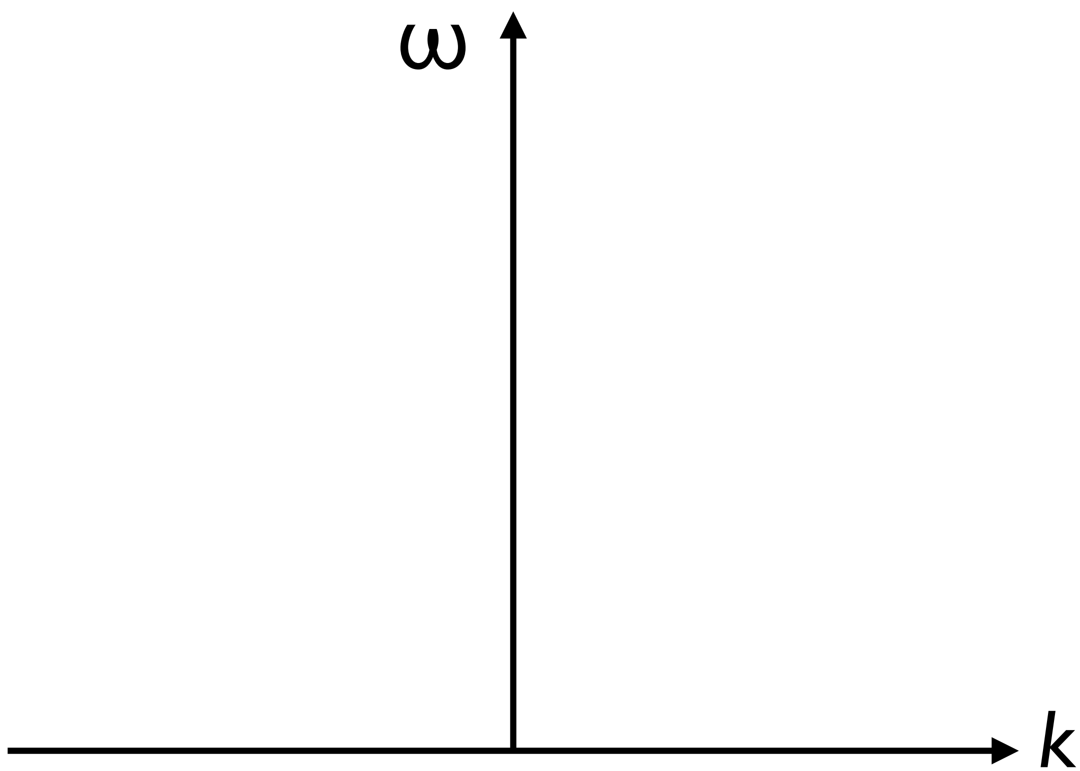


# Part 2: Dispersion Plots

## (5 points each)

For each of the graphs below, sketch the solutions of the dispersion relation for homogeneous plasmas by plotting frequency, $\omega$, as a function of wavenumber, $k$. For each graph, assume the plasma consists of electron densities of electrons and protons. Label key resonances and cutoffs. If your scales are mixed or nonlinear, indicate so. Finally, note that just rough sketches are required. You are not given enough information to make exact dispersion plots.

### Field-Free, Cold Plasma Penetrated by a Cold, Electron Beam



# Part 3: Particle Orbit I

## (25 points total)

Consider an electron trapped within a magnetic trap. The magnetic field is strong and the particle is bouncing along the magnetic field between two regions of very high field strength. The particle's gyroradius is very small compared with the dimensions of the magnetic trap.

The two regions of high field strength begin to move apart. (For example, if the magnetic trap was a flux tube emerging from a coronal hole, this motion could result from plasma convection at the sun's surface.) The actual value of the field strength changes little.

1. Assuming that this geometric motion of the magnetic trap is very slow, describe the particle's motion in phase space. Use the fact that trapped particles can be described with a hierarchy of adiabatic invariants.

2. Form simple expressions for these adiabatic invariants—since the exact expression require detailed models of the magnetic field geometry.

3. Describe approximately what happens to characteristic frequencies of the particle motion.

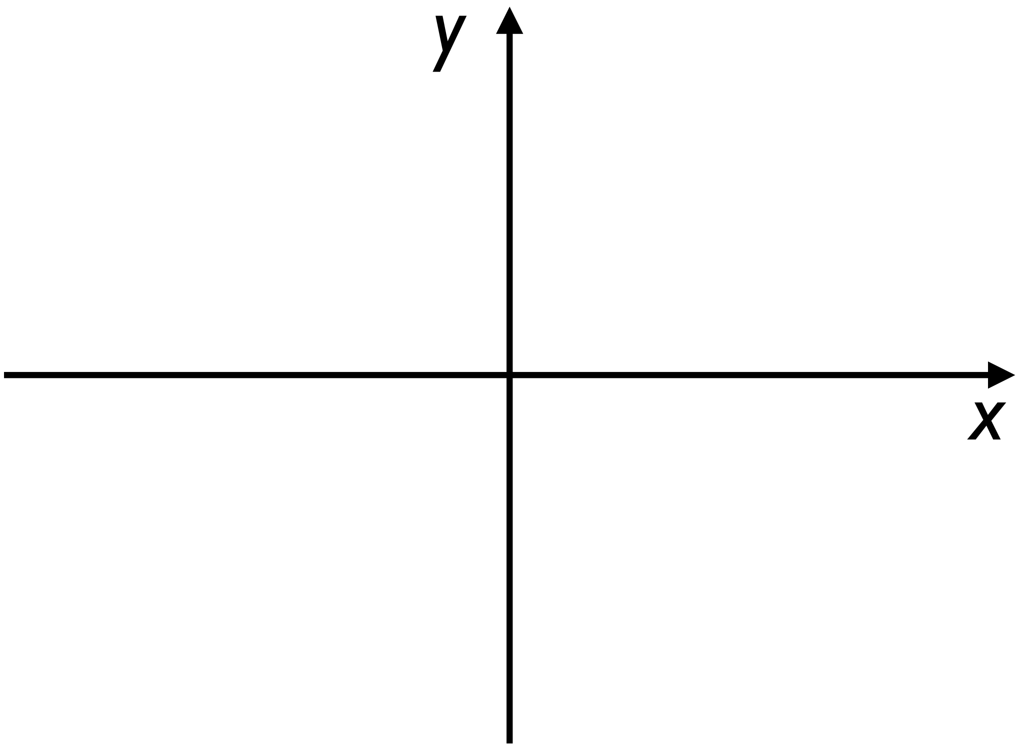
4. Describe approximately what happens to the particle's energy and velocity pitch angle with respect to the magnetic field. Is the particle likely or unlikely to escape from the magnetic trap is the length of the trap increases without changing significantly the strength of the magnetic field?

# Part 4: Particle Orbit II

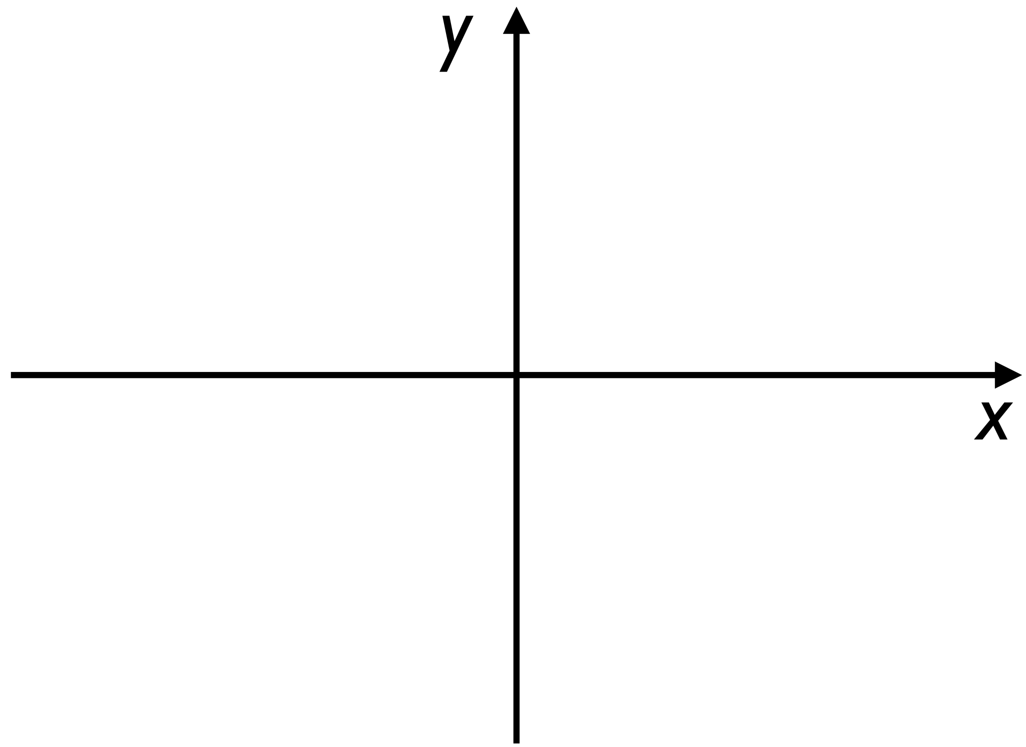
## (25 points total)

Consider an ion initially at rest within a uniform magnetic field, $\mathbf{B} = B\hat{\mathbf{z}}$. At $t > 0$, an electric field is switched on described by $\mathbf{E} = E\cos(\omega t)\,\hat{\mathbf{x}}$. If $\omega_c \gg \omega$, we can consider both the initial motion of the particle and the final motion of the particle. For the initial transient, sketch the motion of the particle for a few cyclotron periods, $2\pi/\omega_c$. For the long-time motion, indicate graphically the gyrocenter motion for long times, $t \gg 1/\omega$.

**Initial Transient**




**Long-Time Motion**



# Part 5: Derive Dispersion Relations

## (50 points total)

In this problem, you are to derive the dispersion relation for electrostatic plasma waves in three ways. These derivations are in the book, but the purpose of this question is for you to show the steps of the derivations. The correct answer is approximately $\omega^2 \approx \omega_p{}^2$ for all three cases. For each case, you must first list a complete set of equations needed to describe the wave propagation. List your unknown perturbed quantities and show that the number of equations equals the number of unknowns.

15 1. Using electron fluid equations in a cold pressureless plasma.

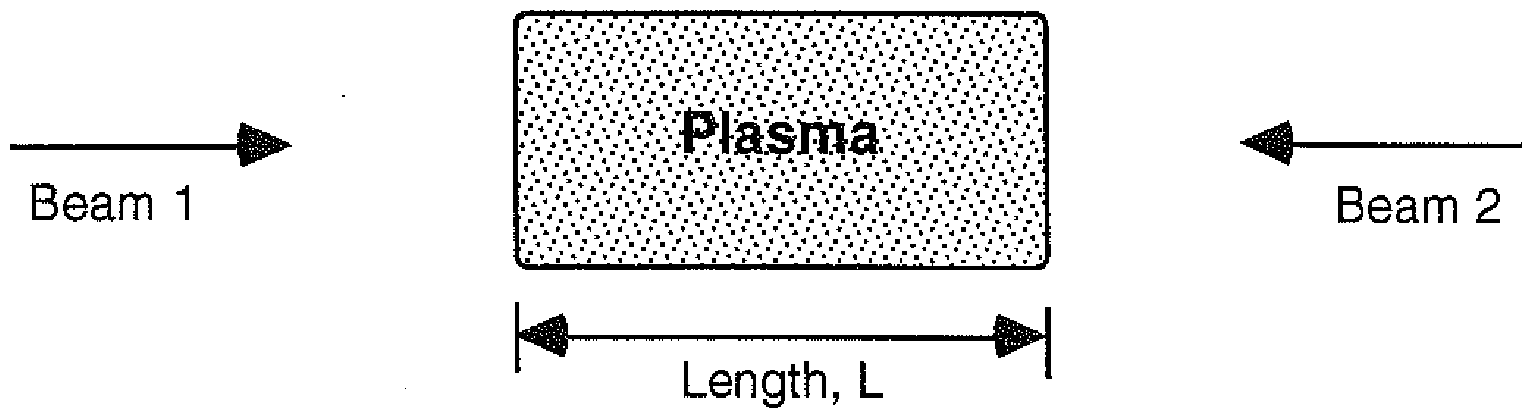
20 2. Using the linearized Vlasov equation retaining the first significant contribution to the dispersion relation due to small but finite electron pressure.

15 3. Using the electron fluid equations with finite pressure. Use the equation of state relating density and pressure perturbations, $(\delta p/p) = \gamma(\delta n/n)$ with $\gamma$ the adiabatic constant.

# Part 6: Two beams and a Plasma

## (25 points total)

Consider a finite-sized plasma into which two counter-streaming beams are injected.




Each beam has the same velocity, $V_0$, and the same plasma density, and the density of the stationary plasma is equal to the densities of each beam.

The dispersion relation for electrostatic plasma waves in this system is given by

$$D(\omega,k) = 1 - \frac{\omega_p^2}{\omega^2} - \frac{\omega_p^2}{(\omega - kV_0)^2} - \frac{\omega_p^2}{(\omega + kV_0)^2}$$

If we assume that the longest wavelength that can exist within the plasma is given by the plasma length, $L$, then prove that two-stream instabilities are stabilized when the density is sufficiently small. Indeed, the stability requirement is $\omega_p/V_0 < K/L$, where $K$ is a constant of the order of 3.

# APPH 6101 Plasma Physics I

## Midterm Exam

## Submit your solutions before noon, Monday, Oct. 31, 2005.

*This is an open book, open note, open anything exam. I intend for the exam to be completed in about 2 hours of work, but I suspect that some of you may take longer. (That's ok, but not required.) Since this is an open-book exam, full credit is given* **only** *if you show your work and explain your reasoning. Please write clearly so that I can understand your solutions.* Each question counts for the same fraction of the total exam score. Please, on your honor, do not speak with any other student, colleague, or scientist, either within or outside Columbia University. You are on your honor to do this exam **entirely** by yourself.

# Question 1

Describe trapped particle motion in large-aspect ratio tokamak geometry and show that the particle motion is described by an orbit shaped like a "banana" when viewed on a poloidal plane. Also give the ratios between the bounce and magnetic drift frequencies as compared to the cyclotron frequency.

*Background:* This is a well-known problem in toroidal magnetic physics described in textbooks about tokamaks. (You do not need to reference these other textbooks, but it is allowed.) Your starting point must be a formulation of the magnetic field. In the "large-aspect-ratio" limit, the magnetic field is nearly toroidal. There is a weak poloidal field such that the magnetic field makes a helical trajectory as it goes around the plasma torus. The geometry that we'll use for the magnetic field is approximately cylindrical, with $(\rho,\ \theta)$ representing the minor coordinates from the major radius of the torus. In this coordinate system, the toroidal field must decrease with radius, and

$$B_t(\rho,\ \theta) = B_0 \frac{R_0}{R_0 + \rho\cos\theta} = \frac{B_0}{1 + \epsilon\cos\theta}$$

where $\epsilon \equiv \rho/R_0 \ll 1$. When $\theta = 0$, the field line is on the "outside" of the torus, and the toroidal field is relatively weak. When $\theta = \pi$, the field line is on the "inside" of the torus, and the toroidal field is stronger. This variation of the strong toroidal field causes (1) particle trapping when $v_\|/v_\perp$ is sufficiently small and (2) magnetic drifts. As $\epsilon \to 0$, the poloidal field depends only upon $\rho$. If the plasma current density is a constant within the plasma, then $B_p(\rho) = \mu_0 \rho J/2\pi$. The "safety factor" is the ratio of the number of times that the magnetic field line goes the "long-way" around the torus to the number of times that the field line goes the "short-way" around the torus. The symbol $q$ is used to describe the safety factor, and it's given by $q(\rho) = \rho B_0/R_0 B_p(\rho)$. For a constant $J$, the safety factor is constant, $q = 2\pi B_0/\mu_0 J$, and $B_p(\rho) = \epsilon B_0/q \ll B_0$. Therefore, the total magnetic field is $\mathbf{B} = \hat{\phi} B_t + \hat{\theta} B_p$, where $\hat{\phi}$ is the unit vector in the toroidal direction and $\hat{\theta}$ is the unit vector in the poloidal direction.

With the magnetic field defined, you should consider the constants of particle motion. Assume that the particle's gyroradius is small compared with the size of of the tokamak (or the drift orbit.) Then, the magnetic moment, $\mu$, and particle energy, $E$, are conserved. Because the tokamak is axisymmetric, the total (canonical) angular momentum is also conserved (as described in Section 3.7 of the textbook.) You need to know the component of the vector potential in the direction of symmetry, *i.e.* the toroidal direction, $A_\phi = \hat{\phi} \cdot \mathbf{A}$. For the approximations used above,

$$A_\phi \approx \frac{\rho^2}{2} \frac{B_0}{R_0 q}.$$

# Question 2

Describe the conditions when light can propagate through *and be guided by* a "channel" through a plasma. In other words, imagine an infinite uniform plasma with a straight channel of dimension $a$ across it's cross-section where the plasma density within the channel is different from the plasma density outside channel. When does the channel act like an "optical fiber" and guide the light for long distances?

# Question 3

Consider a uniformly magnetized and a fully-ionized plasma made from carbon. There would be six times the density of electrons than of ions (but the plasma would still be approximately charge-neutral.)

Describe the Alfvén wave, the electron whistler wave, and the ordinary wave in this plasma. How do these waves (in a fully-ionized carbon plasma) compare to the same waves in a plasma made from singly-ionized carbon having the same mass density (*i.e.* the same density of carbon)?

# Question 4

**Part A.** Write the equations for the first three moments of the particle distribution function given by

$$f(\mathbf{x},\ \mathbf{v},\ t) = n\,\delta(v_x - V_0)\delta(v_y)\delta(v_z)$$

where $(\mathbf{x},\ \mathbf{v})$ are expressed in cartesian coordinates, $V_0$ is a constant, and where you should assume that there is no magnetic force field.

**Part B.** Assume that initially the ions are described by $f_i = n\delta(v_x)\delta(v_y)\delta(v_z)$ and the electrons are described by the distribution function given in Part A. Describe the linear electrostatic waves that exist in this plasma.