Lecture12:

Plasma Physics 1

APPH E6101x

Columbia University

Last Lecture

- Wave energy density and its relationship to the dispersion function, $D(\omega)$
- Measurement of electrostatic plasma waves
- Waves in a (cold) magnetized plasma

This Lecture

- CMA Diagram (Chapter 6)

Waves in Magnetized Plasma

$$\begin{pmatrix} \hat{v}_x \\ \hat{v}_y \\ \hat{v}_z \end{pmatrix} = \mathrm{i}\frac{q}{\omega m} \begin{pmatrix} \dfrac{\omega^2}{\omega^2 - \omega_\mathrm{c}^2} & \mathrm{i}\dfrac{s\omega\omega_\mathrm{c}}{\omega^2 - \omega_\mathrm{c}^2} & 0 \\ -\mathrm{i}\dfrac{s\omega\omega_\mathrm{c}}{\omega^2 - \omega_\mathrm{c}^2} & \dfrac{\omega^2}{\omega^2 - \omega_\mathrm{c}^2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \hat{E}_x \\ \hat{E}_y \\ \hat{E}_z \end{pmatrix}$$

$$\boldsymbol{\sigma}(\omega) = \mathrm{i}\omega\varepsilon_0 \begin{pmatrix} \sum_\alpha \dfrac{\omega_{\mathrm{p}\alpha}^2}{\omega^2 - \omega_{\mathrm{c}\alpha}^2} & \mathrm{i}\sum_\alpha s_\alpha \dfrac{\omega_{\mathrm{p}\alpha}^2}{\omega^2 - \omega_{\mathrm{c}\alpha}^2}\dfrac{\omega_{\mathrm{c}\alpha}}{\omega} & 0 \\ -\mathrm{i}\sum_\alpha s_\alpha \dfrac{\omega_{\mathrm{p}\alpha}^2}{\omega^2 - \omega_{\mathrm{c}\alpha}^2}\dfrac{\omega_{\mathrm{c}\alpha}}{\omega} & \sum_\alpha \dfrac{\omega_{\mathrm{p}\alpha}^2}{\omega^2 - \omega_{\mathrm{c}\alpha}^2} & 0 \\ 0 & 0 & \sum_\alpha \dfrac{\omega_{\mathrm{p}\alpha}^2}{\omega^2} \end{pmatrix}$$

$$\boxed{\boldsymbol{\varepsilon}_\omega = \mathbf{I} + \frac{\mathrm{i}}{\omega\varepsilon_0}\boldsymbol{\sigma}_\omega}$$

Waves in Magnetized Plasma

$$\boxed{\boldsymbol{\varepsilon}_\omega = \mathbf{I} + \frac{\mathrm{i}}{\omega\varepsilon_0}\boldsymbol{\sigma}_\omega}$$

$$S = 1 - \sum_\alpha \frac{\omega_{\mathrm{p}\alpha}^2}{\omega^2 - \omega_{\mathrm{c}\alpha}^2}$$

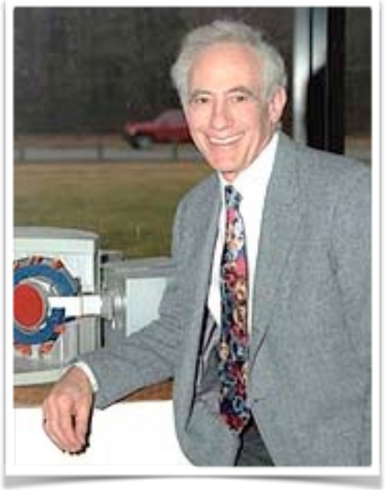
$$D = \sum_\alpha s_\alpha \frac{\omega_{\mathrm{p}\alpha}^2}{\omega^2 - \omega_{\mathrm{c}\alpha}^2}\frac{\omega_{c\alpha}}{\omega}$$

$$\boldsymbol{\varepsilon}(\omega) = \begin{pmatrix} S & -\mathrm{i}D & 0 \\ \mathrm{i}D & S & 0 \\ 0 & 0 & P \end{pmatrix}$$

$$P = 1 - \sum_\alpha \frac{\omega_{\mathrm{p}\alpha}^2}{\omega^2}.$$

http://www.nytimes.com/2001/04/18/nyregion/thomas-h-stix-plasma-physicist-dies-at-76.html

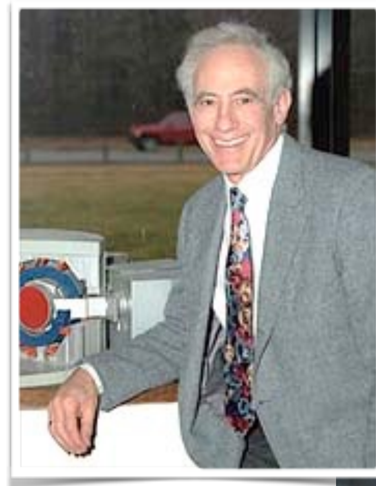
Working both in the laboratory and with theoretical calculations, he found many ways to put waves to work in fusion research in succeeding decades, and his 1962 book, "The Theory of Plasma Waves," codified the subject in mathematical form for the first time.

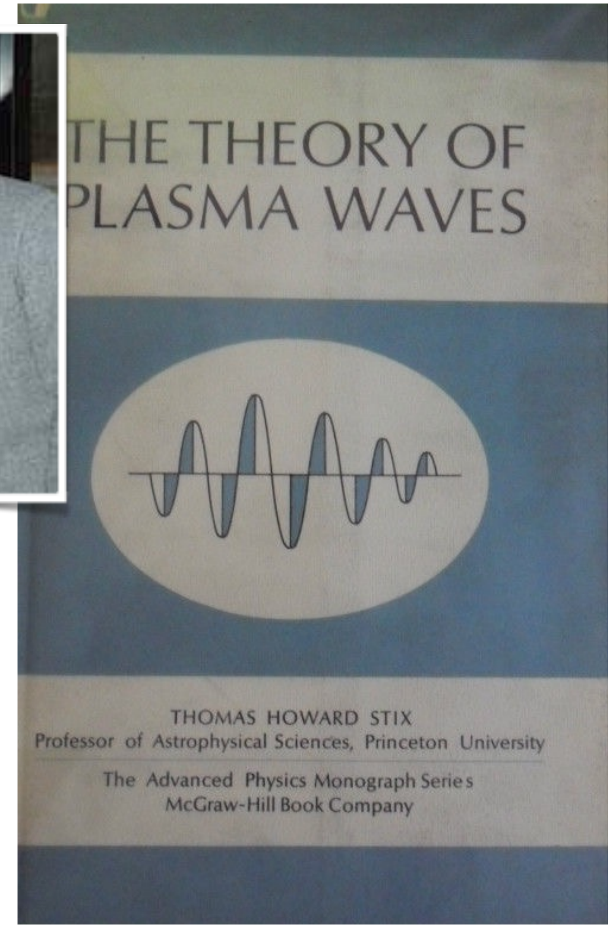


$$S = 1 - \sum_\alpha \frac{\omega_{\mathrm{p}\alpha}^2}{\omega^2 - \omega_{\mathrm{c}\alpha}^2}$$

$$D = \sum_\alpha s_\alpha \frac{\omega_{\mathrm{p}\alpha}^2}{\omega^2 - \omega_{\mathrm{c}\alpha}^2} \frac{\omega_{c\alpha}}{\omega}$$

$$P = 1 - \sum_\alpha \frac{\omega_{\mathrm{p}\alpha}^2}{\omega^2}\,.$$





$$R \equiv 1 - \sum_s \frac{\omega_{ps}^2}{\omega^2}\left(\frac{\omega}{\omega \pm \Omega_s}\right)$$

$$L \equiv 1 - \sum_s \frac{\omega_{ps}^2}{\omega^2}\left(\frac{\omega}{\omega \mp \Omega_s}\right)$$

$$S \equiv \frac{1}{2}(R + L) \qquad D \equiv \frac{1}{2}(R - L)$$

$$P \equiv 1 - \sum_s \frac{\omega_{ps}^2}{\omega^2}.$$

$$\boldsymbol{\varepsilon}(\omega) = \begin{pmatrix} S & -\mathrm{i}D & 0 \\ \mathrm{i}D & S & 0 \\ 0 & 0 & P \end{pmatrix}$$

Waves in Magnetized Plasma

$$\begin{pmatrix} S-\mathcal{N}^2\cos^2\psi & -\mathrm{i}D & \mathcal{N}^2\cos\psi\sin\psi \\ \mathrm{i}D & S-\mathcal{N}^2 & 0 \\ \mathcal{N}^2\cos\psi\sin\psi & 0 & P-\mathcal{N}^2\sin^2\psi \end{pmatrix}\cdot\begin{pmatrix} \hat{E}_x \\ \hat{E}_y \\ \hat{E}_z \end{pmatrix}=0$$

$$\mathbf{k} \;=\; (k\sin\psi, 0, k\cos\psi)$$

Good places to start:
Propagation along B (ψ = 0)
Propagation ⊥ to B (ψ = π/2)

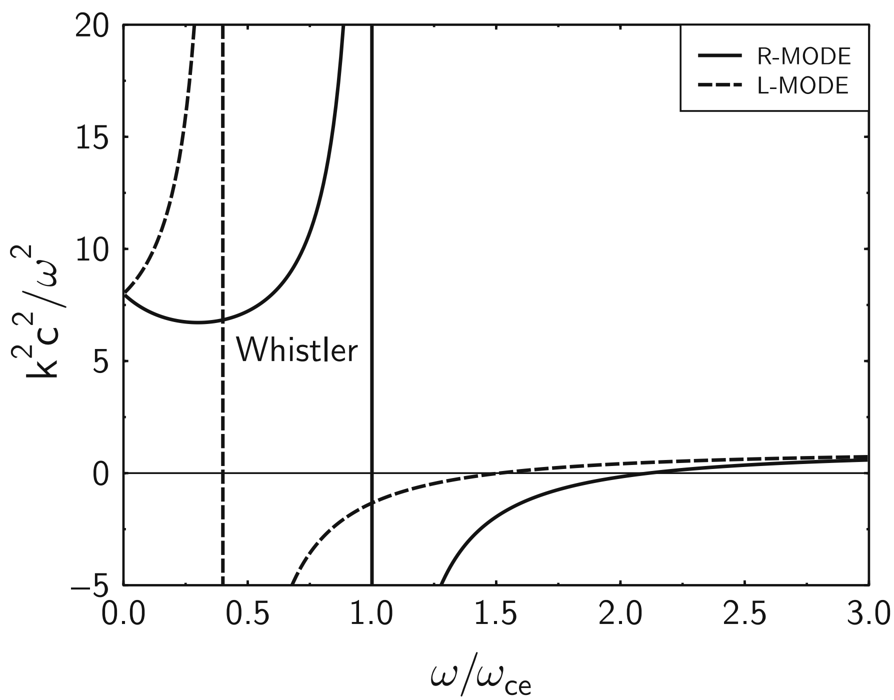
$$S = 1 - \sum_\alpha \frac{\omega_{\mathrm{p}\alpha}^2}{\omega^2-\omega_{\mathrm{c}\alpha}^2}$$

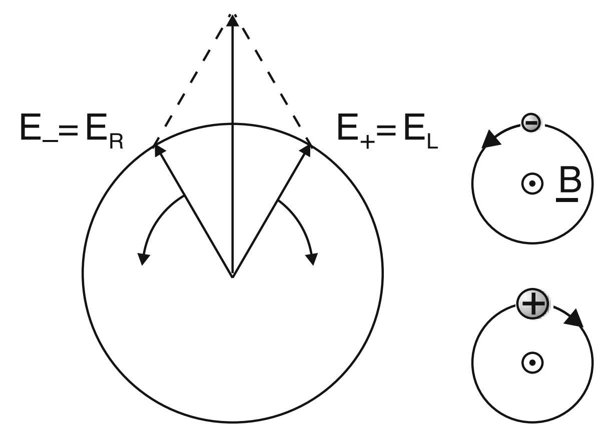
$$D = \sum_\alpha s_\alpha \frac{\omega_{\mathrm{p}\alpha}^2}{\omega^2-\omega_{\mathrm{c}\alpha}^2}\frac{\omega_{c\alpha}}{\omega}$$

$$P = 1 - \sum_\alpha \frac{\omega_{\mathrm{p}\alpha}^2}{\omega^2}\,.$$

Waves k || B

$$\begin{pmatrix} S-\mathcal{N}^2 & -\mathrm{i}D & 0 \\ \mathrm{i}D & S-\mathcal{N}^2 & 0 \\ 0 & 0 & P \end{pmatrix} \cdot \begin{pmatrix} \hat{E}_x \\ \hat{E}_y \\ \hat{E}_z \end{pmatrix} = 0$$





Waves k ⊥ B

Extra-Ordinary Mode

$$\begin{pmatrix} S & -\mathrm{i}D \\ \mathrm{i}D & (S-\mathscr{N}^2) \end{pmatrix} \cdot \begin{pmatrix} \hat{E}_x \\ \hat{E}_y \end{pmatrix} = 0$$

$$\mathscr{N}_{\mathrm{X}} = \left(\frac{S^2 - D^2}{S} \right)^{1/2}$$

$$\omega_{\mathrm{uh}} = (\omega_{\mathrm{ce}}^2 + \omega_{\mathrm{pe}}^2)^{1/2}$$

$$\omega_{\mathrm{lh}} = \left(\omega_{\mathrm{ci}}^2 + \frac{\omega_{\mathrm{pi}}^2 \omega_{\mathrm{ce}}^2}{\omega_{\mathrm{pe}}^2 + \omega_{\mathrm{ce}}^2} \right)^{1/2}$$

Plus: Ordinary Mode

$$\omega^2 = \omega_{\mathrm{pe}}^2 + k^2 c^2$$

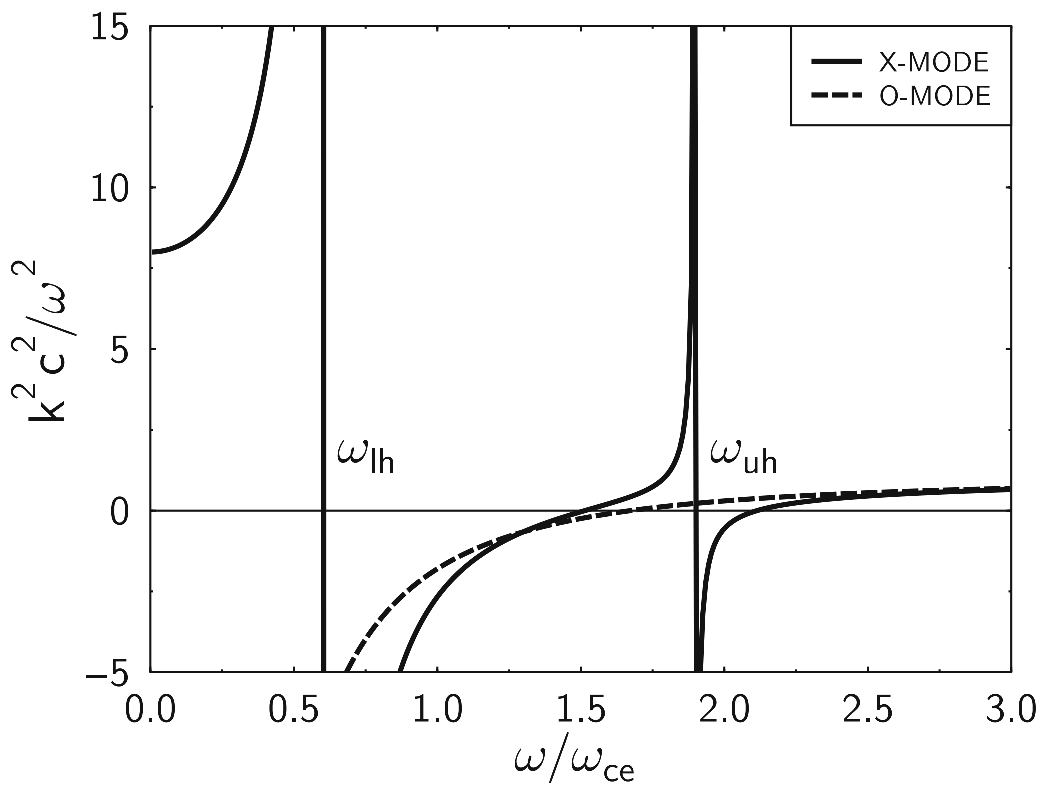
Waves k ⊥ B

Extra-Ordinary Mode

$$\begin{pmatrix} S & -\mathrm{i}D \\ \mathrm{i}D & (S-\mathcal{N}^2) \end{pmatrix} \cdot \begin{pmatrix} \hat{E}_x \\ \hat{E}_y \end{pmatrix} = 0$$

Plus: Ordinary Mode

$$\omega^2 = \omega_{\mathrm{pe}}^2 + k^2 c^2$$



$$S = 1 - \sum_\alpha \frac{\omega_{\mathrm{p}\alpha}^2}{\omega^2 - \omega_{\mathrm{c}\alpha}^2}$$

$$D = \sum_\alpha s_\alpha \frac{\omega_{\mathrm{p}\alpha}^2}{\omega^2 - \omega_{\mathrm{c}\alpha}^2} \frac{\omega_{c\alpha}}{\omega}$$

$$P = 1 - \sum_\alpha \frac{\omega_{\mathrm{p}\alpha}^2}{\omega^2} \, .$$

$$R \equiv 1 - \sum_s \frac{\omega_{ps}^2}{\omega^2} \left(\frac{\omega}{\omega \pm \Omega_s}\right)$$

$$L \equiv 1 - \sum_s \frac{\omega_{ps}^2}{\omega^2} \left(\frac{\omega}{\omega \mp \Omega_s}\right)$$

$$S \equiv \frac{1}{2}(R + L) \qquad D \equiv \frac{1}{2}(R - L)$$

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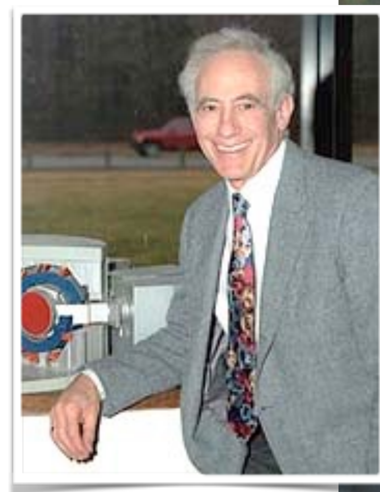
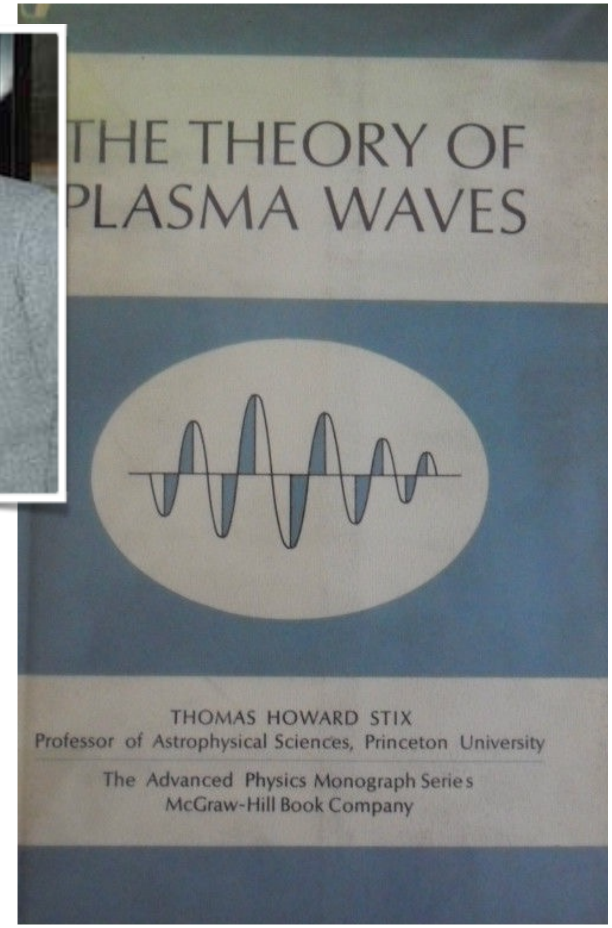
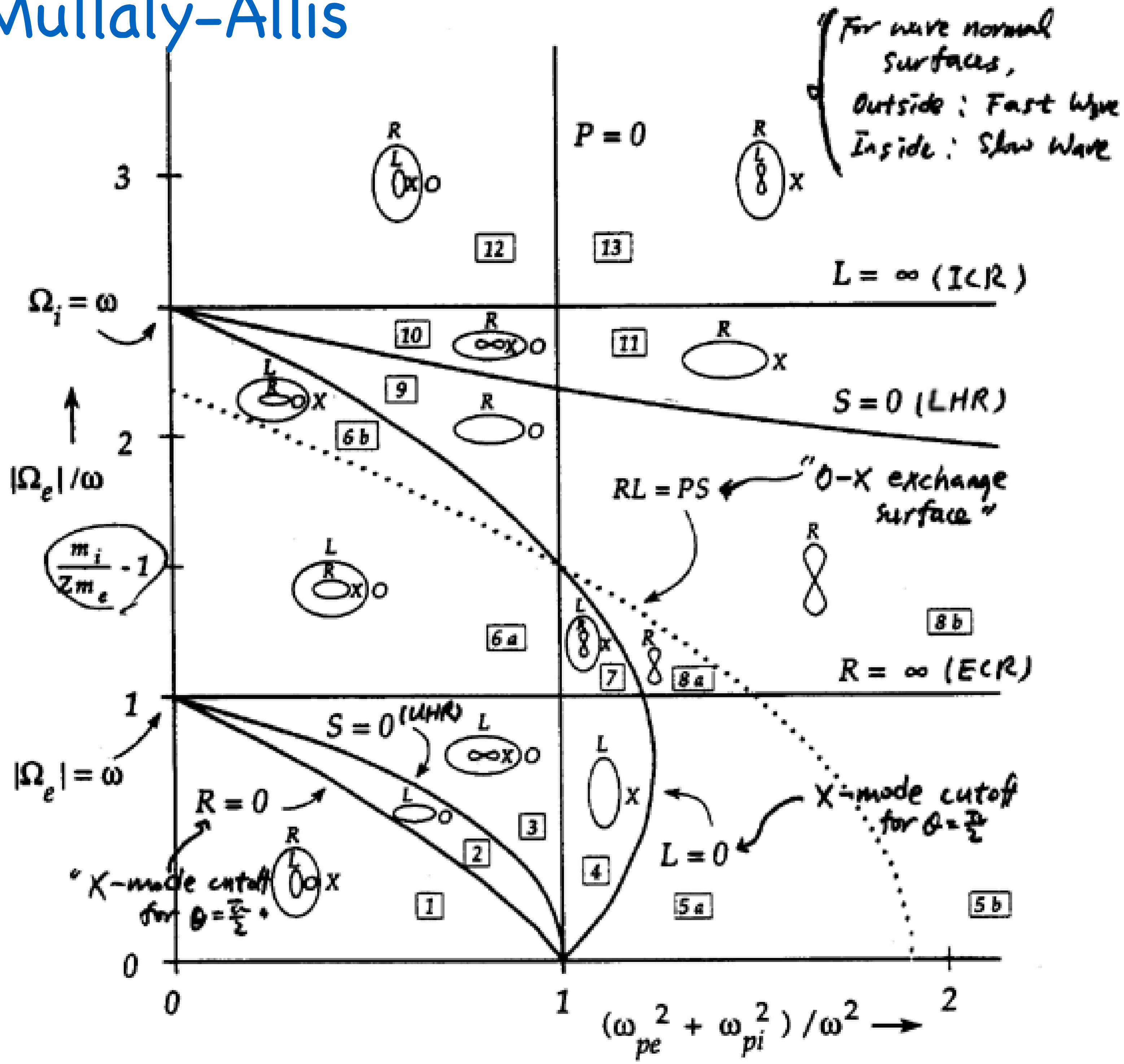




$$\boldsymbol{\varepsilon}(\omega) = \begin{pmatrix} S & -\mathrm{i}D & 0 \\ \mathrm{i}D & S & 0 \\ 0 & 0 & P \end{pmatrix}$$

Clemmow-Mullaly-Allis

CMA Diagram



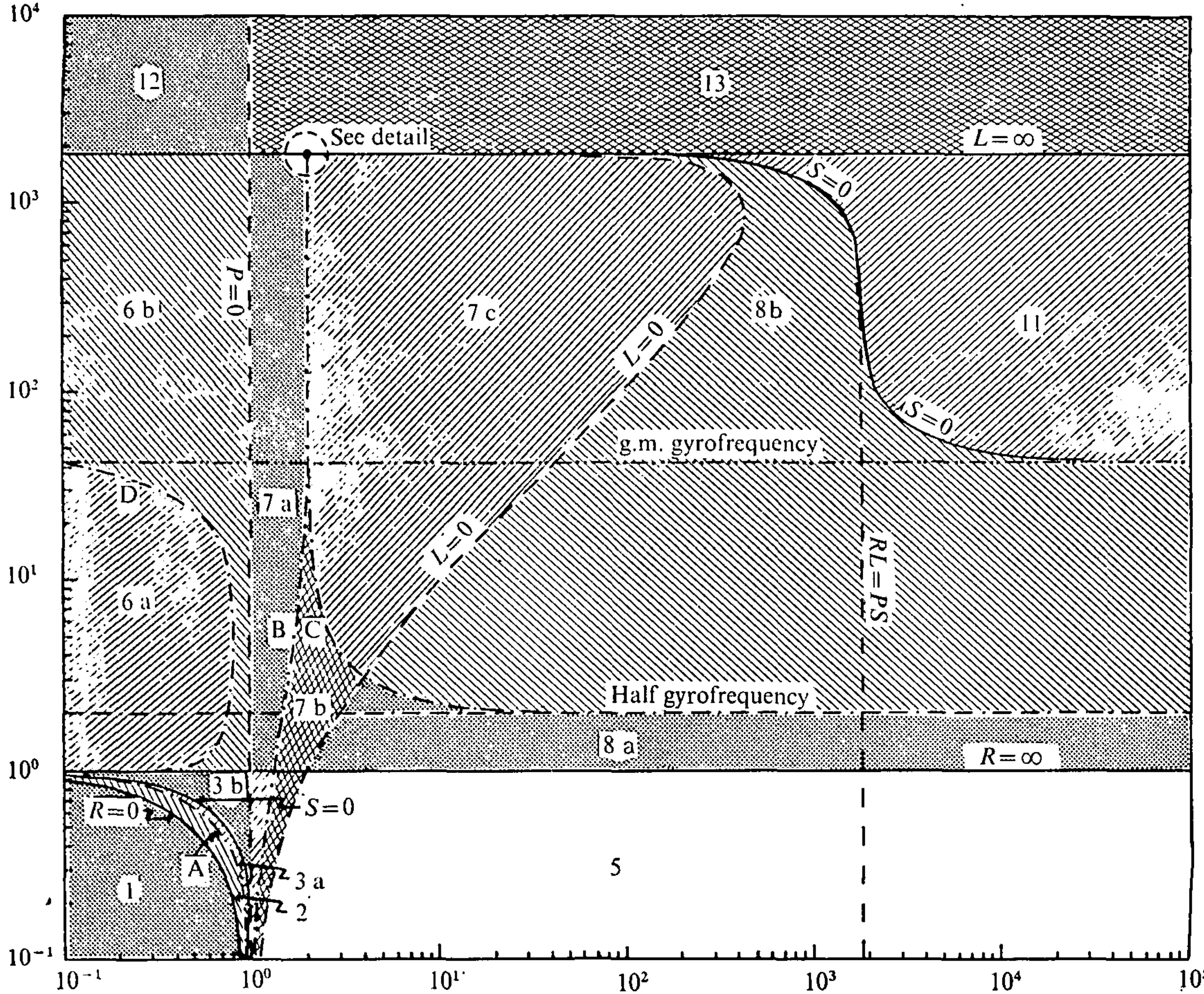


FIGURE 1. Clemmow–Mullaly–Allis diagram for an H^+ plasma ($\eta = 5{\cdot}447.10^{-4}$). The numbering of the regions corresponds to that of Stix (1962), but regions are subdivided according to the positions of the unfoldings. Subdivisions are according to the scheme of

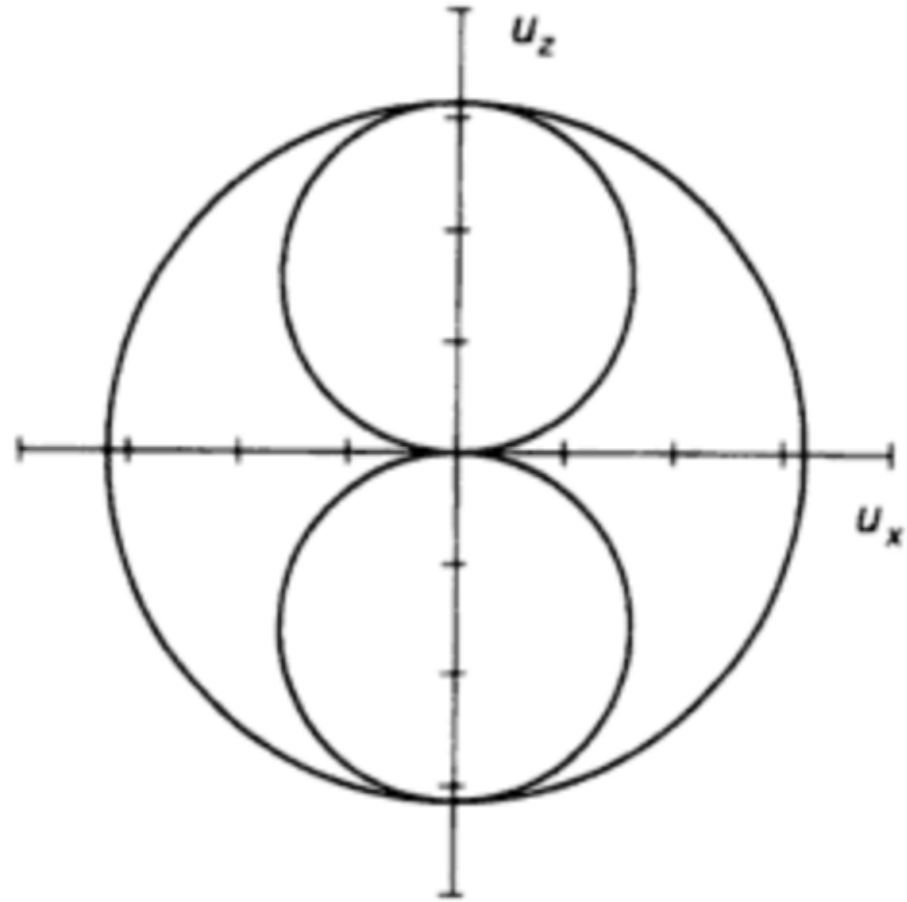


Fig. 1-1. Wave normal surfaces [solutions of Eq. (46) using Eqs. (30)–(32) and set (48)] for $\mu=1836$, $\gamma=1000$, $\beta=1000$ ($R=1000.5$, $L=1002.5$, $P = -1.8 \times 10^{12}$). *The parameters are representative for the shear Alfvén wave (inner figure) and the compressional Alfvén wave (outer figure), Sec. 2-4.* $\mathbf{u}\equiv\omega\mathbf{k}/k^2c$. *The zero-order magnetic field is directed along the z-axis.*

1-6 Wave Normal Surfaces

We now have the pieces of information most needed to discuss normal surfaces for waves propagating through a magnetized uniform cold plasma. As described in the introductory section for this chapter, the wave normal surface is the locus of the phase-velocity vector, $\mathbf{v}_{\text{phase}} = (\omega/k)\hat{\mathbf{k}}$, where $\hat{\mathbf{k}} = \mathbf{k}/k$. The wave normal surfaces are figures of revolution about the $\mathbf{B}_0$ or $\hat{\mathbf{z}}$ axis, and their cross section is a two-dimensional polar plot of ω/k vs θ. With $\mathbf{k}$ in the x,z plane as in Eq. (28), this cross section may be equally well represented as the plot, in Cartesian coordinates, of $\omega k_z/k^2$ vs $\omega k_x/k^2$. In either case, one must keep in mind that ω is the solution of the dispersion relation, Eq. (29), $\omega = \omega(k,\theta)$ or $\omega = \omega(k_x,k_z)$.

The equation for the wave normal surface is easily obtained from Eq. (29), solving for the dimensionless wave phase velocity $u = \omega/kc = 1/n$:

$$Cu^4 - Bu^2 + A = 0\,, \tag{46}$$

with A, B, and C given in Eqs. (30)–(32). The properties of the solutions to Eq. (46) are discussed in detail in the following four sections, but it is immediately obvious that if $u(\theta)$ is a solution, so are $u(-\theta)$, $u(\pi-\theta)$, and $u(\theta-\pi)$. The proof is simply that A, B, and C are functions only of $\sin^2\theta$ and $\cos^2\theta$.

Each of these labeling schemes adds some information to our knowledge of the nature of the wave. We may, therefore, use all of them at once, even though the simple identification of the wave normal surface will be redundant. Repeating at this point the summary of labels given in Sec. 1-1, a wave normal surface for a cold plasma will be identified by the particular bounded volume in parameter space in which it occurs, and then will be labeled:

1. Spheroid, wheel lemniscoid, or dumbbell lemniscoid, according to the shape of the surface.

2. Fast (F) or slow (S), according to the magnitude of the phase velocity at angles between 0 *and* $\pi/2$.

3. Right (R) or left (L), according to the polarization at $\theta=0$.

4. Ordinary (O) or extraordinary (X), according to the dispersion relation at $\theta=\pi/2$. *The ordinary wave obeys* $n^2 = P$; *the extraordinary wave obeys* $n^2=RL/S$.

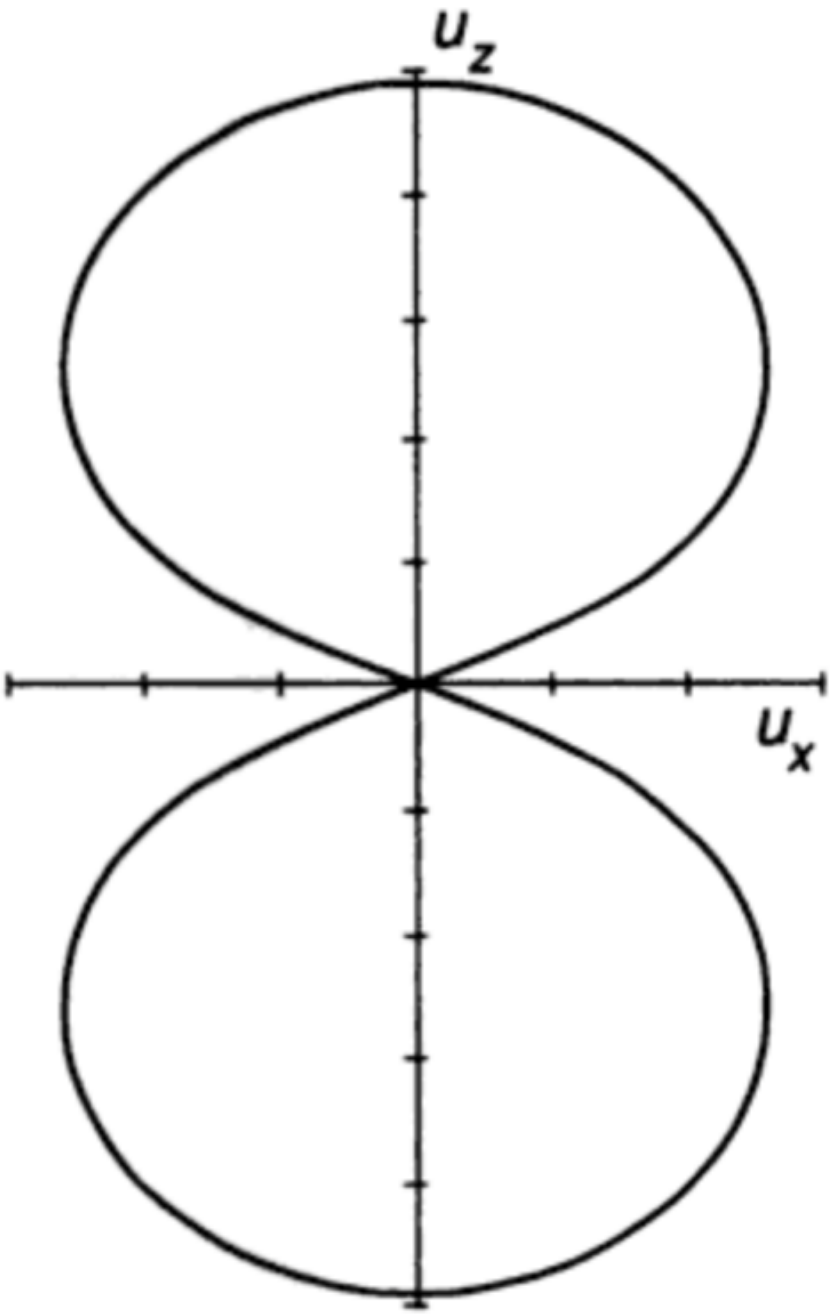


Fig. 1-3. Wave normal surfaces for $\mu = 1836$, $\gamma = 1000$, $\beta = 1/400$ ($R = 4.19$, $L = -1.06$, $P = -10.5$). *The parameters are representative for the whistler mode, Secs. 2-7 and 4-5.* $\mathbf{u} \equiv \omega \mathbf{k}/k^2 c$.

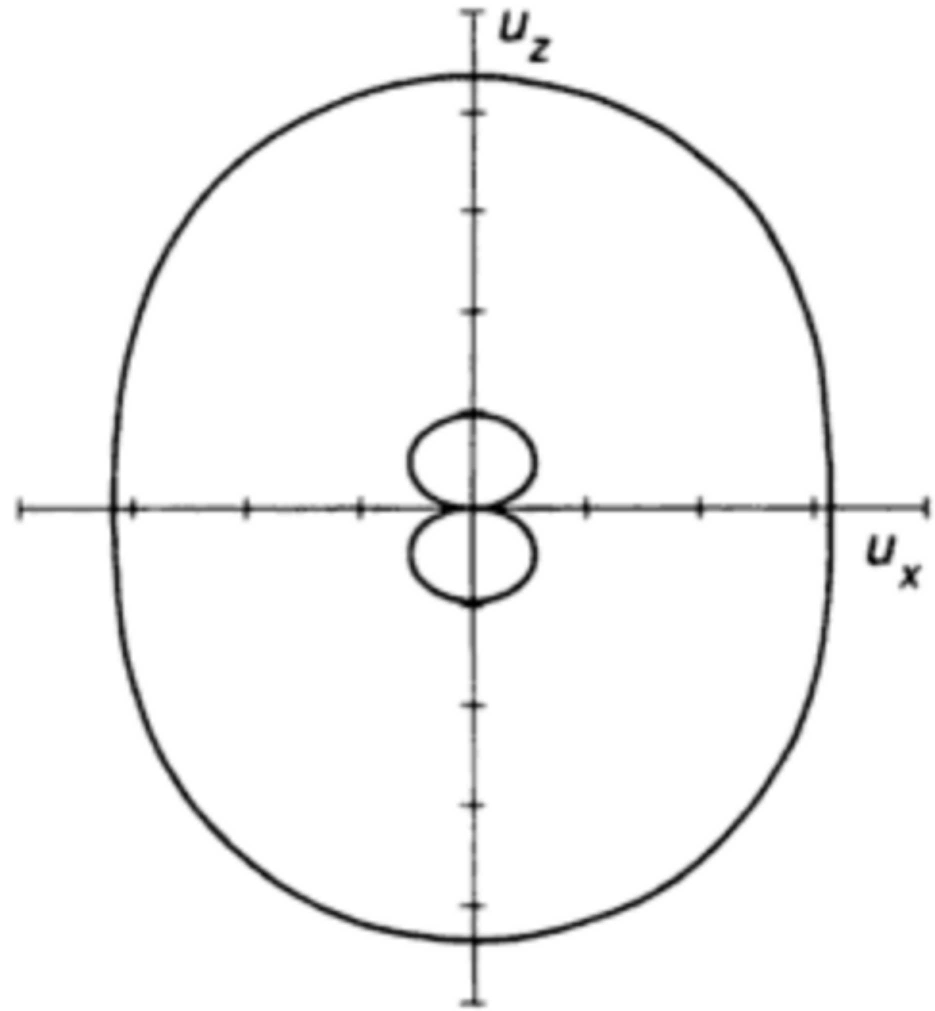


Fig. 1-2. Wave normal surfaces for $\mu = 1836$, $\gamma = 1000$, $\beta = 1.1$ ($R = 525$, $L = 11{,}000$, *and* $P = -2.2 \times 10^6$). *The parameters are representative for ion cyclotron waves (inner figure) and fast waves (outer figure). Sec. 2-5.* $\mathbf{u} \equiv \omega \mathbf{k}/k^2 c$.

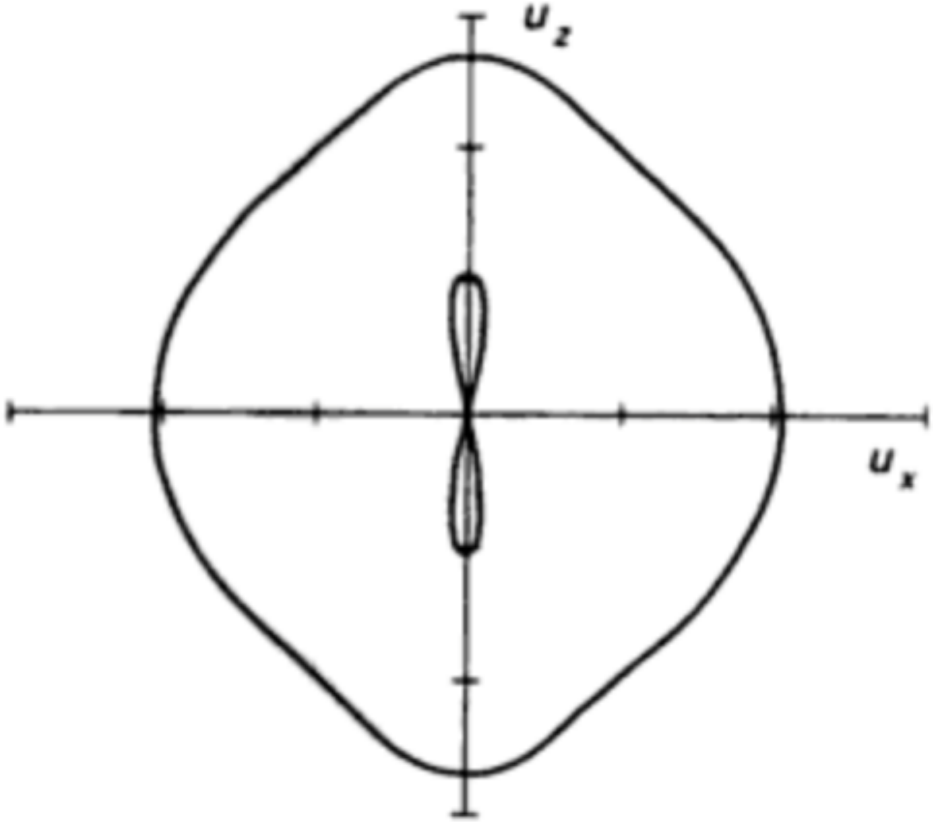


Fig. 1-4. Wave normal surfaces near $P=0$. *Parameters are* $\mu=1836$, $\gamma=1000$, $\beta=1/1300$ ($R=3.63$, $L=0.549$, $P=-0.0870$). $\mathbf{u}\equiv\omega\mathbf{k}/k^2c$.

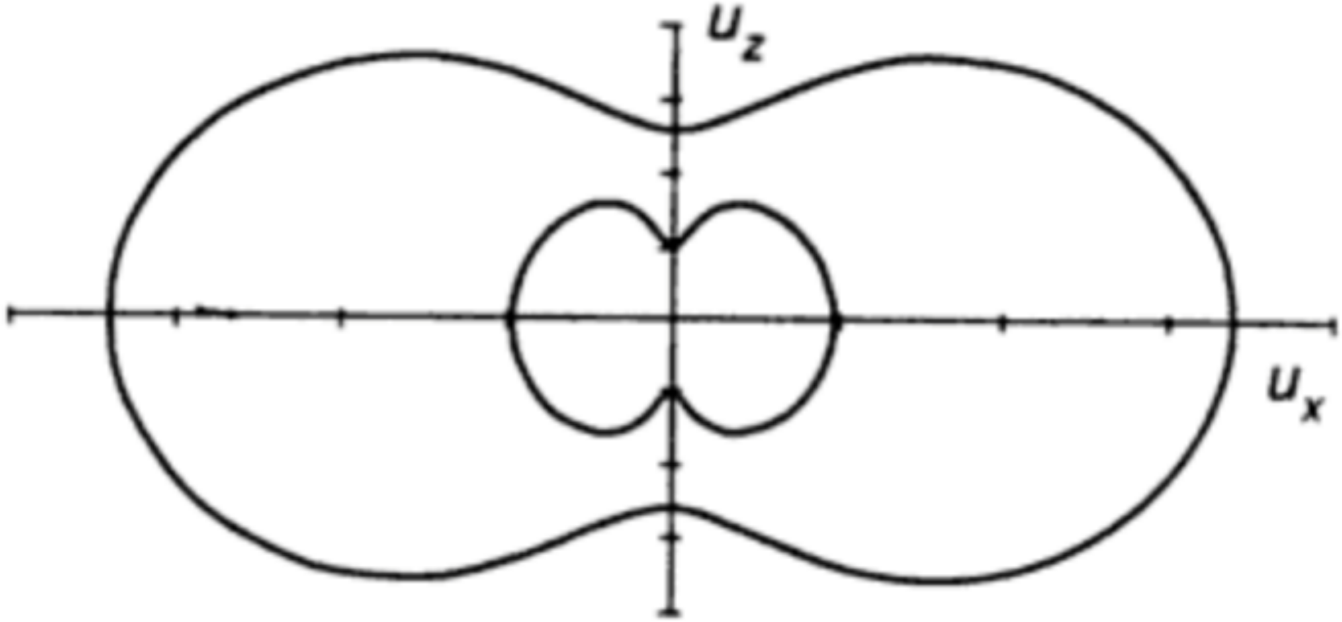


Fig. 1-5. Wave normal surfaces near $P=0$. Parameters are $\mu=1836$, $\gamma=1000$, $\beta=1/1418.5$ ($R=4.10$, $L=0.602$, $P=0.0870$). $\mathbf{u}\equiv\omega\mathbf{k}/k^2c$.

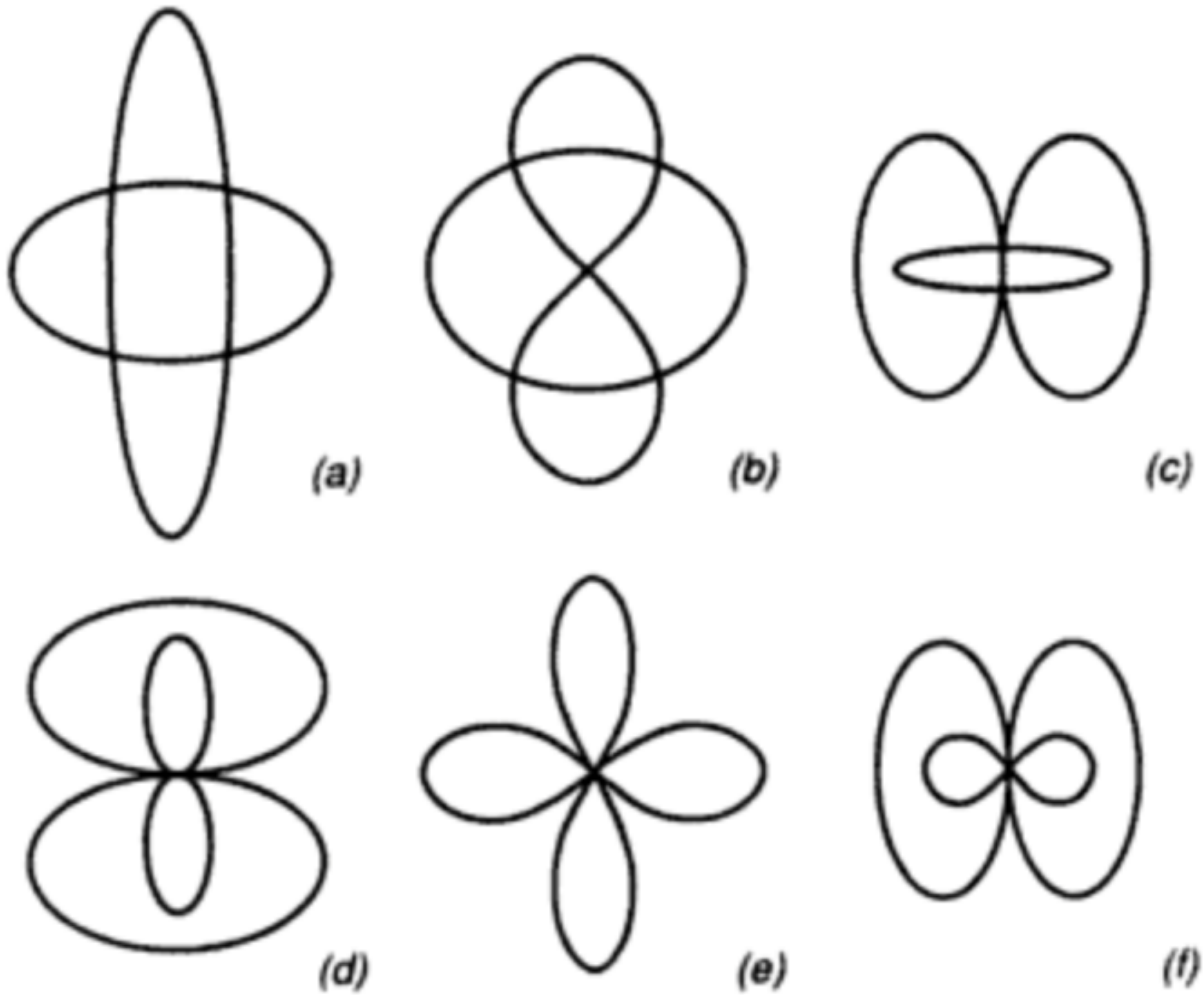


Fig. 1-6. Wave normal surfaces in combinations that never occur as simultaneous solutions of Eq. (46). In (a), (b), and (c) the two surfaces intersect each other; in (d), (e), and (f) both surfaces are lemniscoids (rotated lemniscates).

Next Lecture

- Chapter 7: Langmuir Probes