Lecture11: More Plasma Waves
Plasma Physics 1

APPH E6101x
Columbia University

Last Lecture

- Wave energy density and its relationship to the dispersion function, $D(\omega)$
- Measurement of electrostatic plasma waves (briefly)
- Waves in a (cold) magnetized plasma

This Lecture

- Ch. 6 Problems
- Waves in magnetized plasma

Problem 6.1

6.1 In the limit $T_\mathrm{i} \ll T_\mathrm{e}$ the ion-acoustic wave has the dispersion relation

$$\omega(k) = \frac{\omega_\mathrm{pi}\lambda_\mathrm{De}\,k}{(1 + k^2\lambda_\mathrm{De}^2)^{1/2}}$$

(a) Derive an expression for the phase velocity $v_\varphi(k)$ and group velocity $v_\mathrm{g}(k)$ as a function of the wave number k.
(b) Discuss the result with respect to "acoustic behavior" at $k\lambda_\mathrm{De} \ll 1$.

Problem 6.2

6.2 Assume that in a dielectric medium the relation $v_\varphi \cdot v_\mathrm{g} = c^2$ holds. What is the general shape of the dispersion relation $\omega(k)$ for this case?

Problem 6.3

6.3 (a) Show that for $\omega_{\mathrm{pe}}^2 \gg \omega_{\mathrm{ce}}^2 \gg \omega^2$ the refractive index for Whistler waves takes the limiting form

$$\mathscr{N} = \frac{\omega_{\mathrm{pe}}}{(\omega\omega_{\mathrm{ce}})^{1/2}}$$

(b) Calculate phase and group velocity and show that $v_{\mathrm{gr}} = 2v_{\varphi}$.

Problem 6.4

6.4 Determine the minimum plasma density at which a He-Ne Laser at $\lambda = 633\,\text{nm}$ wavelength will be reflected.

Problem 6.5

6.5 Consider an electron-positron plasma with $n_\mathrm{e} = n_\mathrm{p}$. What is the cut-off frequency for electromagnetic waves in this system?

Problem 6.6

6.6 The plasma of the ionospheric F-layer has a density $n_\mathrm{e} \approx 2 \times 10^{12}\,\mathrm{m}^{-3}$. The typical magnetic field at mid-latitude is $B = 50\,\mu\mathrm{T}$. Calculate the electron plasma frequency f_pe, electron cyclotron frequency f_ce and the upper hybrid frequency f_uh.

Problem 6.7

6.7 Prove that $v_\varphi = v_{\text{gr}}$ requires $\omega = v_\varphi\, k$.

T / F The Debye length of a fully-ionized helium plasma is $\sqrt{2/5}$ the Debye length of a fully-ionized hydrogen plasma when, for both plasmas, the ion and electron temperatures are equal.

$$n = n_0 e^{-q\Phi/kT} \approx n_0\left(1 - q\Phi/kT \cdots\right)$$

$$\nabla^2 \Phi = -\frac{1}{\epsilon_0}\left(q_i n_i + q_e n_e\right)$$

Helium: $\quad q_i = -2q_e \qquad n_i = \frac{1}{2} n_e$

So $\quad \nabla^2 \Phi = \frac{q_i^2 n_i(0)}{\epsilon_0 k T_i}\Phi + \frac{q_e^2 n_e(0)}{\epsilon_0 k T_e}\Phi$

$$= \frac{1}{\lambda_D^2}\Phi$$

$$\frac{1}{\lambda_D^2} = \frac{q_i^2 n_i}{\epsilon_0 k T_i} + \frac{q_e^2 n_e}{\epsilon_0 k T_e}$$

with thanks to Andrew Yang

Review of EM Waves

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{B} = \mu_0 \left(\mathbf{j} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right)$$

$$\begin{aligned}
\nabla \times (\nabla \times \mathbf{E}) &= -\nabla \times \frac{\partial \mathbf{B}}{\partial t} \\
&= -\frac{\partial}{\partial t}(\nabla \times \mathbf{B}) \\
&= -\mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} - \mu_0 \frac{\partial \mathbf{j}}{\partial t}
\end{aligned}$$

Review of EM Waves in Medium

$$\varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} + \mathbf{j} = \varepsilon_0 \bar{\varepsilon}(\omega) \frac{\partial \mathbf{E}}{\partial t}$$

$$\mathbf{j}(\omega) = \bar{\sigma}(\omega) \cdot \mathbf{E}(\omega)$$

$$\boxed{\boldsymbol{\varepsilon}_\omega = \mathbf{I} + \frac{\mathrm{i}}{\omega \varepsilon_0} \boldsymbol{\sigma}_\omega}$$

All of the plasma physics here

$$\left\{ \mathbf{kk} - k^2 \boldsymbol{I} + \frac{\omega^2}{c^2} \boldsymbol{I} + \mathrm{i}\omega\mu_0 \boldsymbol{\sigma}(\omega) \right\} \cdot \hat{\mathbf{E}} = 0$$

$$\left\{ \mathbf{kk} - k^2 \boldsymbol{I} + \frac{\omega^2}{c^2} \boldsymbol{\varepsilon}(\omega) \right\} \cdot \hat{\mathbf{E}} = 0$$

EM Waves without Magnetic Field

$$\varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} + \mathbf{j} = \varepsilon_0 \bar{\varepsilon}(\omega) \frac{\partial \mathbf{E}}{\partial t} \qquad \mathbf{j}(\omega) = \bar{\sigma}(\omega) \cdot \mathbf{E}(\omega)$$

$$\boxed{\boldsymbol{\varepsilon}_\omega = \mathbf{I} + \frac{\mathrm{i}}{\omega \varepsilon_0} \boldsymbol{\sigma}_\omega}$$

FOR HIGH FREQUENCY WAVES, $\omega \gg \omega_{pi}$, THEN ONLY ELECTRON CURRENTS

$$\bar{J} = -e n_0 \bar{v}_e$$

$$m_e \frac{dv}{dt} = -eE \Rightarrow -j\omega \bar{v} = -\frac{e}{m_0} \bar{E}$$

$$\bar{v} = j\left(-\frac{e}{m_0}\right) \bar{E}/\omega$$

$$\bar{J} = j \frac{e^2 n}{m_e \omega} \bar{E}$$

$$\bar{J} = j \epsilon_0 (\omega_{pe}^2/\omega) \bar{E}$$

$$\bar{\bar{\sigma}} = j \epsilon_0 \omega \begin{pmatrix} \omega_p^2/\omega^2 & 0 & 0 \\ 0 & \omega_p^2/\omega^2 & 0 \\ 0 & 0 & \omega_p^2/\omega^2 \end{pmatrix} \quad \text{FOR } \bar{E} = (E_x, E_y, E_z)$$

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$$\bar{\bar{\epsilon}} = \bar{\bar{I}} + \frac{i}{\omega \epsilon_0} \bar{\bar{\sigma}} = \bar{\bar{I}}\left(1 - \omega_p^2/\omega^2\right)$$

EM Waves without Magnetic Field

FOR ELECTROMAGNETIC WAVES

$$\mathbf{j}(\omega) = \bar{\sigma}(\omega) \cdot \mathbf{E}(\omega)$$

$$\left\{ \bar{N}\bar{N} - N^2 \bar{\bar{I}} + \bar{\bar{\epsilon}} \right\} \cdot \bar{E} = 0$$

$$\bar{N} = \frac{c\bar{k}}{\omega}$$

INDEX OF REFRACTION

TAKE $\bar{k} = \hat{z} k$ OR $\bar{N} = \hat{z} N$

$$\boxed{\boldsymbol{\varepsilon}_\omega = \mathbf{I} + \frac{\mathrm{i}}{\omega\varepsilon_0}\boldsymbol{\sigma}_\omega}$$

$$\begin{Bmatrix} 1 - N^2 - \frac{\omega_p^2}{\omega^2} & 0 & 0 \\ 0 & 1 - N^2 - \frac{\omega_p^2}{\omega^2} & 0 \\ 0 & 0 & 1 - \frac{\omega_p^2}{\omega^2} \end{Bmatrix} \cdot \begin{Bmatrix} E_x \\ E_y \\ E_z \end{Bmatrix}$$

POLARIZATION:

TRANSVERSE WAVES $E \perp \bar{k}$

LONGITUDINAL WAVES $E \parallel \bar{k}$

FARADAY'S LAW

$$\bar{k} \times E = \omega \bar{B}$$

SO $B = 0$ FOR LONGITUDINAL WAVES

EM Waves without Magnetic Field

WITH ELECTRON PRESSURE...

$$\left.\begin{aligned} \frac{\partial n}{\partial t} + \bar{\nabla}\cdot(n\bar{v}) = 0 \\ m\frac{\partial v}{\partial t} = q\bar{E} - \nabla p \end{aligned}\right\} \Rightarrow \quad \begin{aligned} \tilde{n} = n_0 \frac{\bar{k}\cdot\bar{v}}{\omega} \\ -j\omega\bar{v} = \frac{q}{m}\bar{E} - jk\gamma v_{TH}^2\tilde{n} \end{aligned}$$

WHEN $\bar{k}\cdot\bar{v} = 0 \Rightarrow$ NO CHANGE OF DENSITY

WHEN $\bar{k}\cdot\bar{v} \neq 0 \Rightarrow$ COMPRESSION/DECOMPRESSION

$\therefore$ FOR LONGITUDINAL WAVES

$$-j\omega\bar{v} = \frac{q}{m}\bar{E} - j\gamma\frac{k^2\lambda_D^2}{\omega}\omega_p^2\bar{v}$$

OR

$$-j\omega\bar{v}\left(1 - \gamma k^2\lambda_D^2\frac{\omega_p^2}{\omega^2}\right) = \frac{q}{m}\bar{E}$$

AND

$$\bar{J} = qn\bar{v} \Rightarrow \quad \sigma = j\omega\epsilon_0\frac{\omega_p^2/\omega^2}{1 - \gamma k^2\lambda_D^2(\omega_p^2/\omega^2)}$$

$$\Rightarrow \quad \omega^2 = \omega_p^2(1 + \gamma k^2\lambda_D^2) \qquad \text{ELECTRON PLASMA WAVES}$$

Waves in Magnetized Plasma

$$\frac{\partial \mathbf{v}^{(\alpha)}}{\partial t} = \frac{q_\alpha}{m_\alpha}\left(\mathbf{E}_1 + \mathbf{v}^{(\alpha)} \times \mathbf{B}_0\right) \qquad \alpha = \mathrm{e}, \mathrm{i}$$

$$\hat{v}^{\pm} = \mathrm{i}\frac{q}{\omega m}(\hat{E}^{\pm} \mp \mathrm{i}\hat{v}^{\pm} B_0)$$

$$\hat{v}_x = \mathrm{i}\frac{q}{\omega m}(\hat{E}_x + \hat{v}_y B_0)\,, \qquad \hat{v}_y = \mathrm{i}\frac{q}{\omega m}(\hat{E}_y - \hat{v}_x B_0)$$

$$\boxed{\hat{v}^{\pm} = \mathrm{i}\frac{q}{m}\hat{E}^{\pm}\frac{1}{\omega \mp s\omega_c}}$$

$$\hat{v}^{\pm} = \hat{v}_x \pm \mathrm{i}\hat{v}_y\,, \qquad \hat{E}^{\pm} = \hat{E}_x \pm \mathrm{i}\hat{E}_y$$

Waves in Magnetized Plasma

$$\begin{pmatrix} \hat{v}_x \\ \hat{v}_y \\ \hat{v}_z \end{pmatrix} = \mathrm{i}\frac{q}{\omega m} \begin{pmatrix} \dfrac{\omega^2}{\omega^2 - \omega_c^2} & \mathrm{i}\dfrac{s\omega\omega_c}{\omega^2 - \omega_c^2} & 0 \\ -\mathrm{i}\dfrac{s\omega\omega_c}{\omega^2 - \omega_c^2} & \dfrac{\omega^2}{\omega^2 - \omega_c^2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \hat{E}_x \\ \hat{E}_y \\ \hat{E}_z \end{pmatrix}$$

$$\boldsymbol{\sigma}(\omega) = \mathrm{i}\omega\varepsilon_0 \begin{pmatrix} \sum_\alpha \dfrac{\omega_{p\alpha}^2}{\omega^2 - \omega_{c\alpha}^2} & \mathrm{i}\sum_\alpha s_\alpha \dfrac{\omega_{p\alpha}^2}{\omega^2 - \omega_{c\alpha}^2}\dfrac{\omega_{c\alpha}}{\omega} & 0 \\ -\mathrm{i}\sum_\alpha s_\alpha \dfrac{\omega_{p\alpha}^2}{\omega^2 - \omega_{c\alpha}^2}\dfrac{\omega_{c\alpha}}{\omega} & \sum_\alpha \dfrac{\omega_{p\alpha}^2}{\omega^2 - \omega_{c\alpha}^2} & 0 \\ 0 & 0 & \sum_\alpha \dfrac{\omega_{p\alpha}^2}{\omega^2} \end{pmatrix}$$

$$\boxed{\boldsymbol{\varepsilon}_\omega = \mathbf{I} + \frac{\mathrm{i}}{\omega\varepsilon_0}\boldsymbol{\sigma}_\omega}$$

Waves in Magnetized Plasma

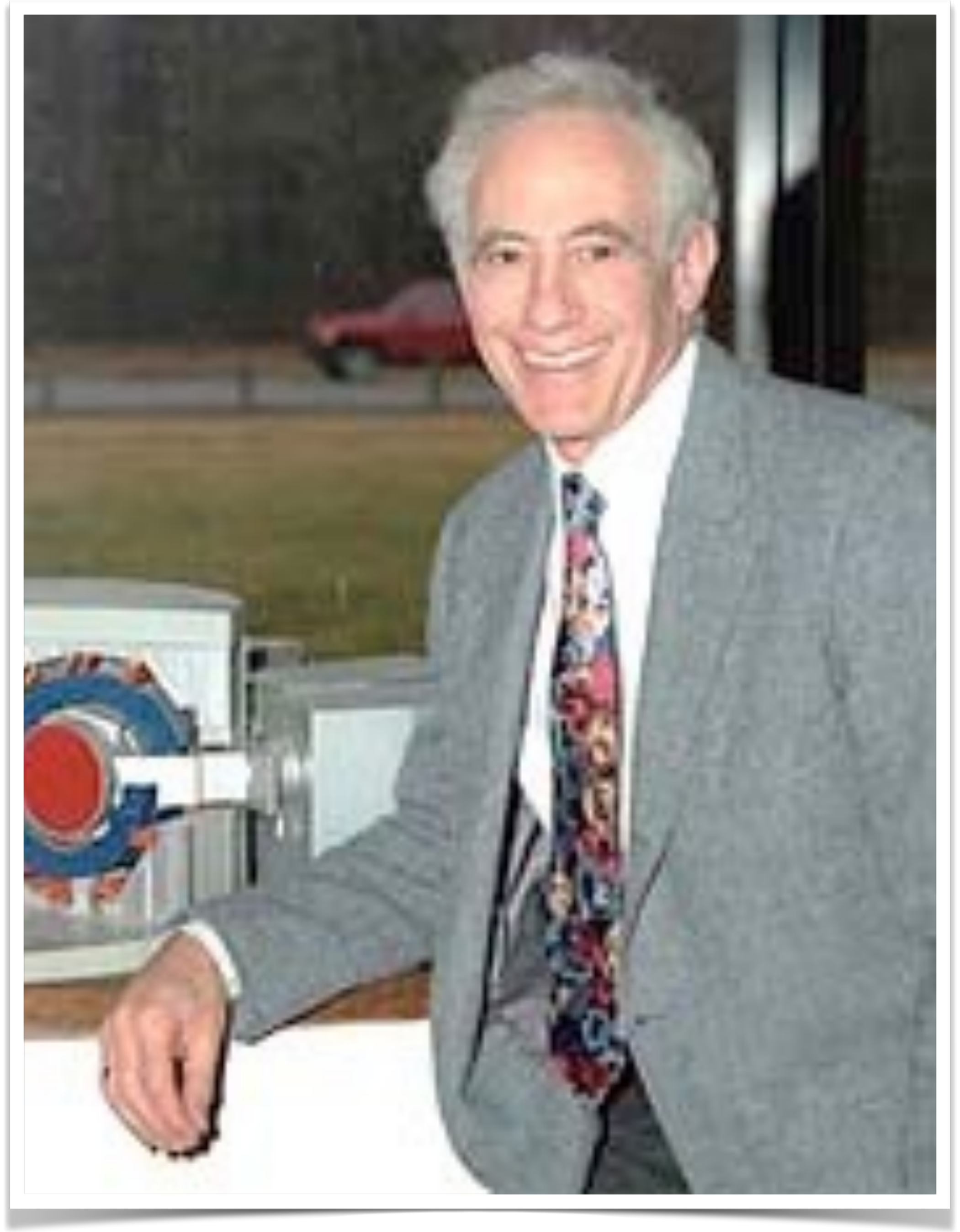
$$\boxed{\boldsymbol{\varepsilon}_\omega = \mathbf{I} + \frac{\mathrm{i}}{\omega\varepsilon_0}\boldsymbol{\sigma}_\omega}$$

$$\boldsymbol{\varepsilon}(\omega) = \begin{pmatrix} S & -\mathrm{i}D & 0 \\ \mathrm{i}D & S & 0 \\ 0 & 0 & P \end{pmatrix}$$

$$S = 1 - \sum_\alpha \frac{\omega_{\mathrm{p}\alpha}^2}{\omega^2 - \omega_{\mathrm{c}\alpha}^2}$$

$$D = \sum_\alpha s_\alpha \frac{\omega_{\mathrm{p}\alpha}^2}{\omega^2 - \omega_{\mathrm{c}\alpha}^2}\frac{\omega_{c\alpha}}{\omega}$$

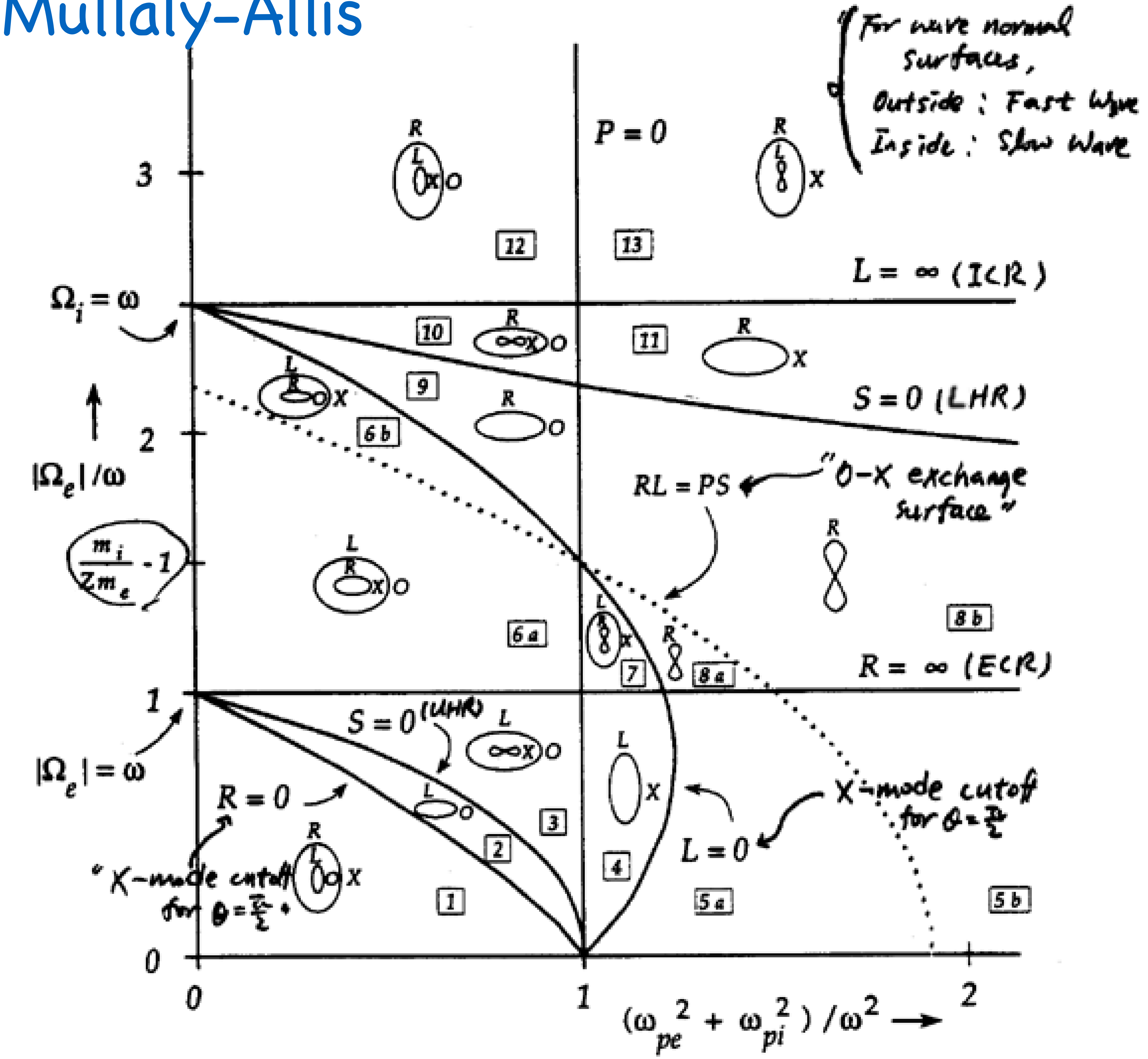
$$P = 1 - \sum_\alpha \frac{\omega_{\mathrm{p}\alpha}^2}{\omega^2} .$$



Working both in the laboratory and with theoretical calculations, he found many ways to put waves to work in fusion research in succeeding decades, and his 1962 book, "The Theory of Plasma Waves," codified the subject in mathematical form for the first time.

http://www.nytimes.com/2001/04/18/nyregion/thomas-h-stix-plasma-physicist-dies-at-76.html

Clemmow-Mullaly-Allis



Lecture 12:
CMA Diagram

$\mathbf{u} \equiv \omega\mathbf{k}/k^2c$

Waves in Magnetized Plasma

$$\begin{pmatrix} S - \mathcal{N}^2 \cos^2\psi & -\mathrm{i}D & \mathcal{N}^2 \cos\psi \sin\psi \\ \mathrm{i}D & S - \mathcal{N}^2 & 0 \\ \mathcal{N}^2 \cos\psi \sin\psi & 0 & P - \mathcal{N}^2 \sin^2\psi \end{pmatrix} \cdot \begin{pmatrix} \hat{E}_x \\ \hat{E}_y \\ \hat{E}_z \end{pmatrix} = 0$$

$$\mathbf{k} = (k \sin\psi, 0, k \cos\psi)$$

$$S = 1 - \sum_\alpha \frac{\omega_{\mathrm{p}\alpha}^2}{\omega^2 - \omega_{\mathrm{c}\alpha}^2}$$

$$D = \sum_\alpha s_\alpha \frac{\omega_{\mathrm{p}\alpha}^2}{\omega^2 - \omega_{\mathrm{c}\alpha}^2} \frac{\omega_{\mathrm{c}\alpha}}{\omega}$$

$$P = 1 - \sum_\alpha \frac{\omega_{\mathrm{p}\alpha}^2}{\omega^2} .$$

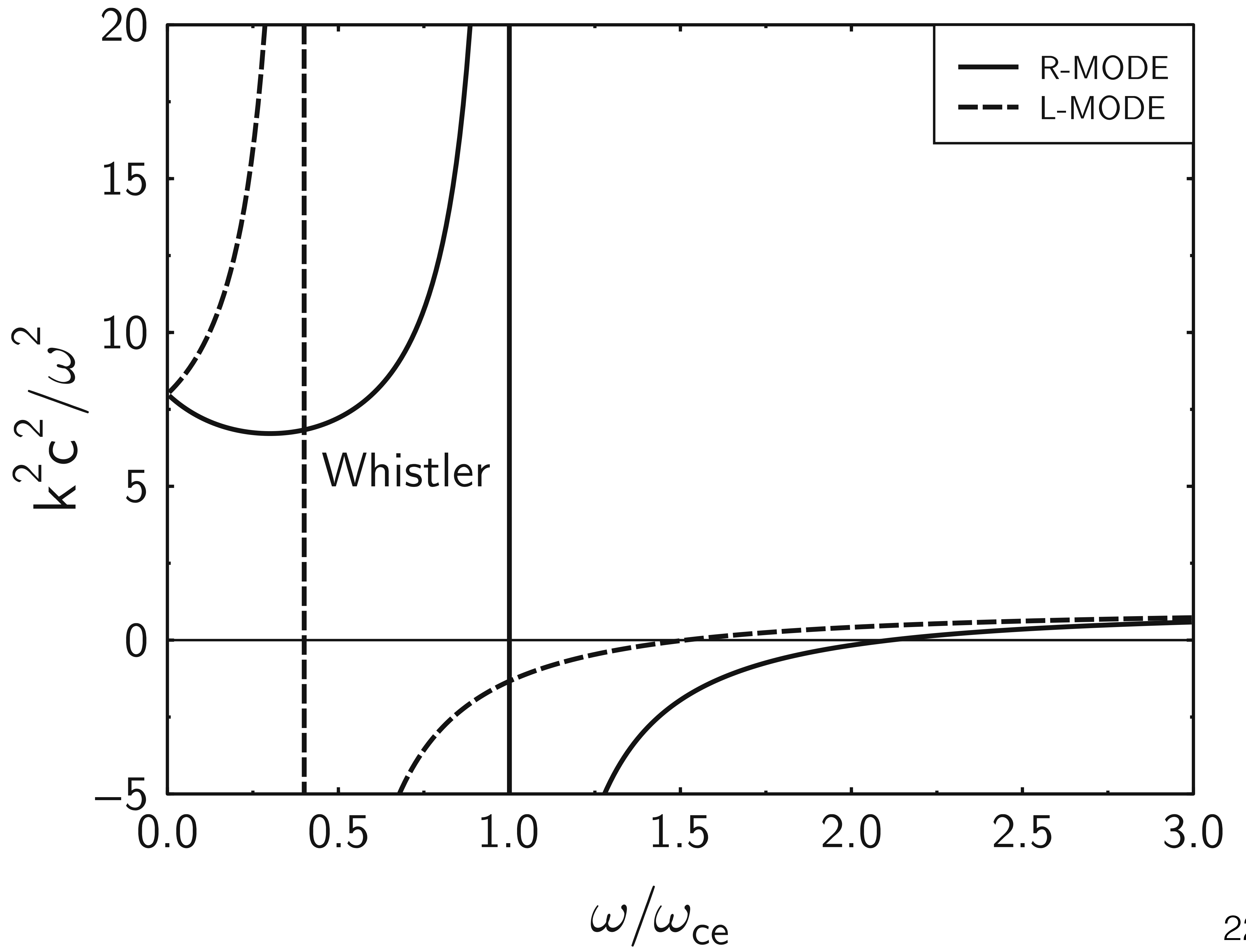
Good places to start:

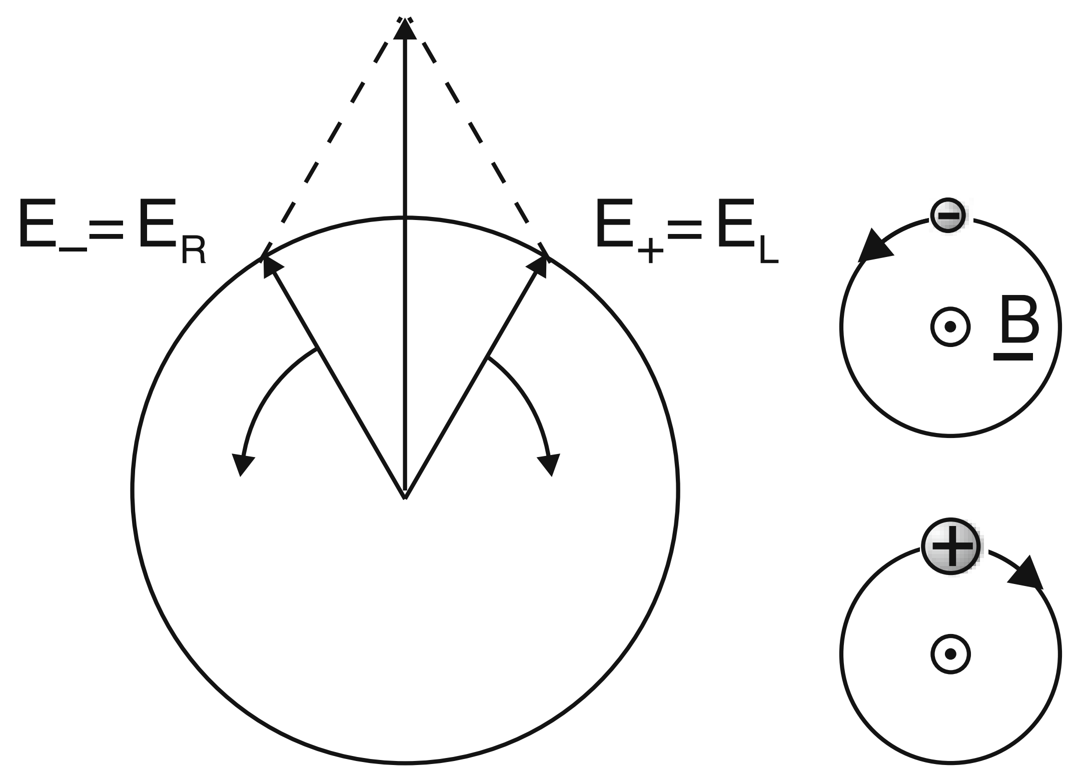
Propagation along B ($\psi = 0$)

Propagation $\perp$ to B ($\psi = \pi/2$)

Waves k || B

$$\begin{pmatrix} S-\mathscr{N}^2 & -\mathrm{i}D & 0 \\ \mathrm{i}D & S-\mathscr{N}^2 & 0 \\ 0 & 0 & P \end{pmatrix} \cdot \begin{pmatrix} \hat{E}_x \\ \hat{E}_y \\ \hat{E}_z \end{pmatrix} = 0$$





Waves k ⊥ B

Extra-Ordinary Mode

$$\begin{pmatrix} S & -\mathrm{i}D \\ \mathrm{i}D & (S-\mathcal{N}^2) \end{pmatrix} \cdot \begin{pmatrix} \hat{E}_x \\ \hat{E}_y \end{pmatrix} = 0$$

$$\mathcal{N}_\mathrm{X} = \left(\frac{S^2 - D^2}{S} \right)^{1/2}$$

$$\omega_\mathrm{uh} = (\omega_\mathrm{ce}^2 + \omega_\mathrm{pe}^2)^{1/2}$$

$$\omega_\mathrm{lh} = \left(\omega_\mathrm{ci}^2 + \frac{\omega_\mathrm{pi}^2 \omega_\mathrm{ce}^2}{\omega_\mathrm{pe}^2 + \omega_\mathrm{ce}^2} \right)^{1/2}$$

Plus: Ordinary Mode

$$\omega^2 = \omega_\mathrm{pe}^2 + k^2 c^2$$

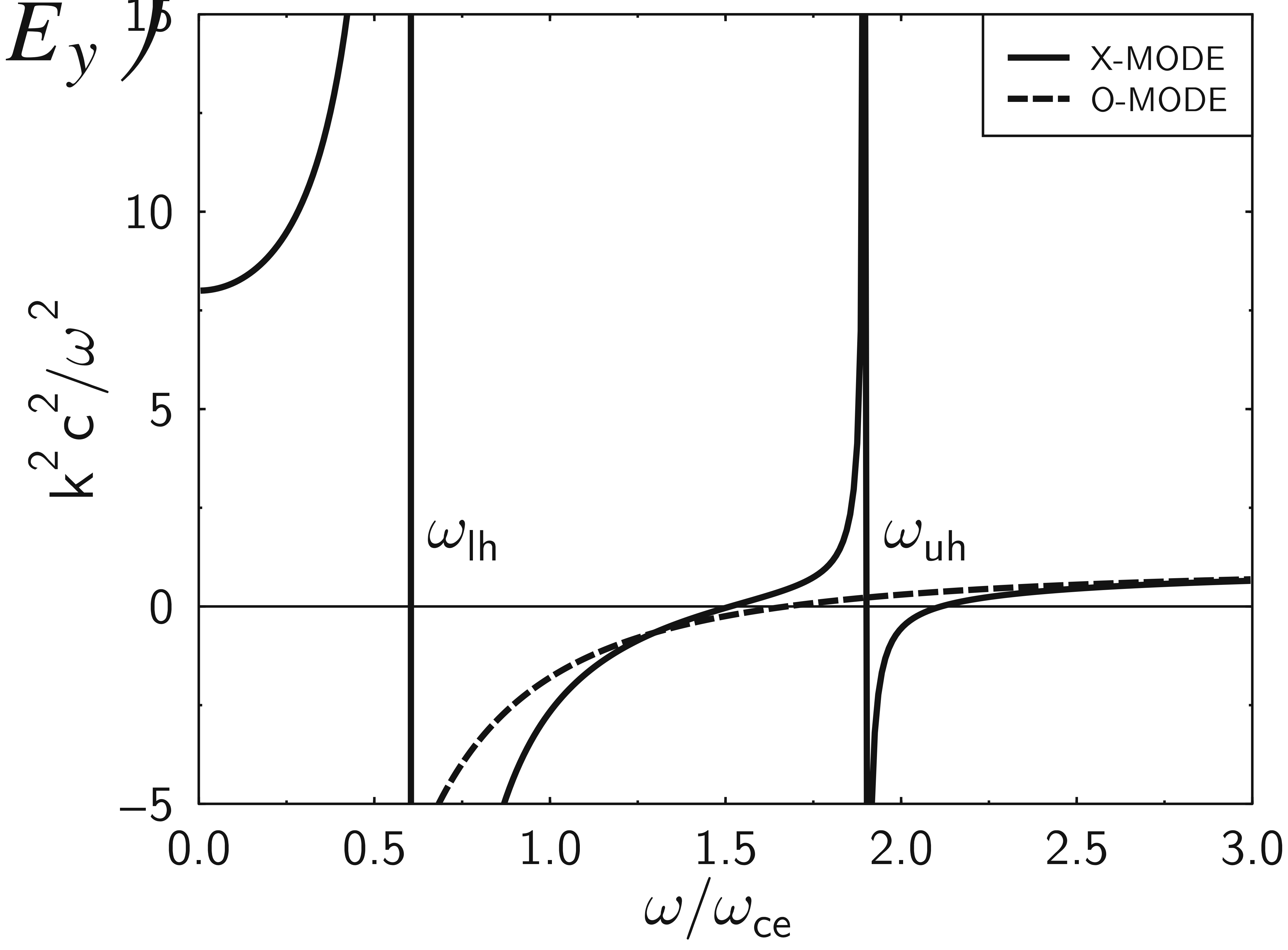
Waves k ⊥ B

Extra-Ordinary Mode

$$\begin{pmatrix} S & -\mathrm{i}D \\ \mathrm{i}D & (S-\mathcal{N}^2) \end{pmatrix} \cdot \begin{pmatrix} \hat{E}_x \\ \hat{E}_y \end{pmatrix} = 0$$

Plus: Ordinary Mode

$$\omega^2 = \omega_{\mathrm{pe}}^2 + k^2 c^2$$



Next …

- APS Division of Plasma Physics (Next Week)
- Chapter 7: “Plasma Boundaries” (Monday, November 6)
    - Probes (!)
- More plasma boundaries, and
- Wednesday, November 8: ***Review***
- ***Quiz #2***: Monday, November 13