Lecture 11:

Plasma Physics 1

APPH E6101x
Columbia University
Last Lecture

- Introduction to plasma waves
- Basic review of electromagnetic waves in various media (conducting, dielectric, gyrotropic, ...)
- Basic waves concepts (especially plane waves)
- Electromagnetic waves in unmagnetized plasma
- Electrostatic waves in unmagnetized plasma
This Lecture

- Wave energy density and its relationship to the dispersion function, $D(\omega)$
- Measurement of electrostatic plasma waves
- Waves in a (cold) magnetized plasma
Plasma waves are described by the set of Maxwell’s equations.

6.1.1 Basic Concepts

The propagation speed and polarization of the plasma waves are determined. This model is developed step by step starting from Maxwell’s equations. The wave equation that is expanded by an additional friction force that is described by a collision frequency 

\[ \nu_{\text{coll}} \] \n
determines the relation between the alternating electric current and a proper equation of motion for the plasma species that establishes the relation for the variable \( \nu \).

In warm plasmas, we could include pressure effects by solving the MHD equations. For discussing the propagation properties of the waves, we make the additional simplifying assumption that, at a chosen angular frequency \( \omega \), Landau damping, will be discussed in Chap. 9.

\[
\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}
\]

\[
\nabla \times \mathbf{B} = \mu_0 \left( \mathbf{j} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right)
\]

\[
\nabla \times (\nabla \times \mathbf{E}) = -\nabla \times \frac{\partial \mathbf{B}}{\partial t}
\]

\[
= -\frac{\partial}{\partial t} (\nabla \times \mathbf{B})
\]

\[
= -\mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} - \mu_0 \frac{\partial \mathbf{j}}{\partial t}
\]
\[ \nabla \times (\nabla \times \mathbf{E}) + \frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} = -\mu_0 \frac{\partial \mathbf{j}}{\partial t} \]

\[ i \mathbf{k} \times \hat{\mathbf{E}} = i \omega \hat{\mathbf{B}} \]

\[ i \mathbf{k} \times \hat{\mathbf{B}} = -i \omega \varepsilon_0 \mu_0 \hat{\mathbf{E}} + \mu_0 \mathbf{j}_0 \]

\[ \mathbf{k} \times (\mathbf{k} \times \hat{\mathbf{E}}) = (\mathbf{k} \mathbf{k} - k^2 \hat{\mathbf{I}}) \hat{\mathbf{E}} \]

\[
\left\{ \mathbf{k} \mathbf{k} - k^2 \mathbf{I} + \frac{\omega^2}{c^2} \mathbf{I} + i \omega \mu_0 \sigma(\omega) \right\} \cdot \hat{\mathbf{E}} = 0
\]

\[
\left\{ \mathbf{k} \mathbf{k} - k^2 \mathbf{I} + \frac{\omega^2}{c^2} \mathbf{e}(\omega) \right\} \cdot \hat{\mathbf{E}} = 0
\]
Wave Energy Density

(Poynting’s Theorem)

\[
\frac{2}{\varepsilon_0} \frac{\partial \overline{E}}{\partial t} = -\nabla \times \overline{E} \quad \text{and} \quad \frac{\varepsilon_0}{2} \frac{\partial \overline{H}}{\partial t} = \nabla \times \overline{H} - \overline{J}
\]

\[
\frac{2}{\varepsilon_0} \left( \frac{\varepsilon_0}{2} E^2 \right) + \frac{2}{\mu_0} \left( \frac{B^2}{2} \right) = \overline{E} \cdot \nabla \times \overline{H} - \overline{H} \cdot \nabla \times \overline{E} + \overline{J} \cdot \overline{E}
\]

\[
\overline{B} = \frac{\overline{H} \times \overline{E}}{\omega}
\]

\[
\overline{B}^2 = \overline{E} \cdot \left( \frac{\varepsilon_0}{\omega^2} - \frac{1}{\mu_0} \right) \overline{E}
\]

\[
\frac{2}{2\varepsilon_0} \frac{\partial \omega}{\partial t} + \nabla \cdot \overline{U} \omega = -\overline{J} \cdot \overline{E} - 2\pi \omega
\]
Electrostatic Waves

\[ \nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0} \]

\[ -\nabla \cdot \nabla \Phi = \frac{\rho}{\varepsilon_0} \]

\[ k^2 \Phi = \frac{\rho}{\varepsilon_0} \approx -\frac{en_e}{\varepsilon_0} \]

\[ \frac{\partial n}{\partial t} + \frac{\partial}{\partial x} (nv) = 0 \]

\[ m\ddot{v} = -q \frac{d\phi}{dx} - \gamma \frac{d(nk_B T)}{dx} \]

Electron Pressure Force
Electrostatic Plasma Waves

\[ \nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0} \]

\[ -\nabla \cdot \nabla \Phi = \frac{\rho}{\varepsilon_0} \]

\[ k^2 \Phi = \frac{\rho}{\varepsilon_0} \approx -\frac{en_e}{\varepsilon_0} \]

\[ \frac{\partial n}{\partial t} + \frac{\partial}{\partial x} (nv) = 0 \]

\[ -i\omega \hat{n} + ikn_0 \hat{v} = 0 \]

\[ m \hat{v} = -q \frac{d\phi}{dx} - \frac{\gamma}{n} \frac{d(nk_BT)}{dx} \]

\[ -i\omega m \hat{v} = -ikq \hat{\phi} - ik\gamma k_B T \hat{n} \]
Electrostatic Plasma Waves

\[ \omega = \left( \omega_{pe}^2 + \frac{3}{2}k^2v_{Te}^2 \right)^{1/2} = \omega_{pe} \left( 1 + 3k^2\lambda_D^2 \right)^{1/2} \]
Electrostatic Ion Sound Waves

\[-i\omega m_i \dot{v}_i = e\hat{E} - \frac{ik}{n_{i0}}(\gamma_i k_B T_i)\hat{n}_i\]

\[0 = -e\hat{E} - \frac{ik}{n_{e0}}(k_B T_e)\hat{n}_e\]

\[\hat{n}_i = \frac{ek}{-i\omega^2 m_i + ik^2 \gamma_i k_B T_i} \hat{E}\]

\[\hat{n}_e = \frac{-e}{ik k_B T_e} \hat{E},\]

\[k^2 \Phi = \frac{\rho}{\epsilon_0} \approx \frac{e}{\epsilon_0} (n_i - n_e)\]

\[ik \hat{E} = \left(\frac{n_{i0} e^2}{\epsilon_0 m_i}\right) \frac{k}{-i\omega^2 + ik^2 \gamma_i k_B T_i / m_i} \hat{E} + \left(\frac{n_{e0} e^2}{\epsilon_0 k_B T_e}\right) \frac{1}{ik} \hat{E}\]
Electrostatic Ion Sound Waves

\[ \varepsilon(k, \omega) = 1 - \frac{\omega_{\text{pi}}^2}{\omega^2 - k^2 \gamma_i k_B T_i / m_i} + \frac{1}{k^2 \lambda_{\text{De}}^2} \]

\[ \omega^2 = k^2 \left( \frac{\gamma_i k_B T_i}{m_i} + \frac{\omega_{\text{pi}}^2 \lambda_{\text{De}}^2}{1 + k^2 \lambda_{\text{De}}^2} \right) \]

\[ \omega \approx \frac{k \, C_s}{\sqrt{1 + k^2 \lambda_{\text{De}}^2}} \quad \text{for } T_e \gg T_i \quad C_s = \omega_{\text{pi}} \lambda_{\text{De}} \]
Energy Density for Electrostatic Waves

\[ m \frac{d\vec{v}}{dt} = q \vec{E} e^{-i(\omega t - k \cdot \vec{r})} \]

\[ -i \omega \vec{v} = \frac{q}{m} \vec{E} \quad \Rightarrow \quad \vec{v} = i \frac{q}{\omega m} \vec{E} \]

\[ \left\langle \frac{1}{2} m \vec{v} \cdot \vec{v} \right\rangle = \frac{1}{2} m m_0 \left\langle \frac{1}{2} \vec{v}^* \vec{v} \right\rangle \]

\[ = \frac{1}{2} m m_0 \frac{1}{2} \frac{q^2}{m^2} |E|^2 \]

\[ = \frac{1}{2} \frac{q^2}{\omega^2} E_0^2 \frac{1}{2} |E|^2 \]

For high frequency waves

\[ \omega \gg \omega_p \]

\[ \text{Plasma energy} \sim \frac{\omega_p^2}{\omega^2} \times \text{Electric energy density} \]

For low frequency sound waves

\[ \omega \ll \omega_p \]

\[ \text{Plasma energy} \sim \frac{\omega_p^2}{\omega^2} \times \text{Electric energy} \]

\[ \gg \text{Electric energy} \]

For \( \omega \ll \omega_p \)
Damping and Dispersion

\[ D(\omega, h) = 0 \]

\[ \omega = \omega_r + i \omega_i \]

Waves usually have

\[ |\omega_r| > |\omega_i| \]

\[ D(\omega, h) = 0 \Rightarrow D_r(\omega_r, h) + i \frac{1}{\omega_r} D_i \]

\[ \omega_i = -\frac{D_i}{2D_r/\omega} \left| \omega_r \right| \]

with \[ D_r(\omega_r, h) = 0 \]

\[ D_i \sim \text{energy loss rate} \]

\[ \frac{2D_r}{\omega} \sim \text{energy density of wave} \]
Slowly-Varying Wave

\[ E(\vec{\eta}, t) = R_0 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} E(\vec{\eta}, t) e^{-i(\omega t - \vec{\eta} \cdot \vec{n})} \]

Fast Wave Oscillations

Slowly Varying Amplitude/Phase

\[ \frac{1}{2} \left| E^2(\vec{\eta}, t) \right| = \frac{1}{2} R_0 \int_0^{2\pi} E^* \cdot E \]

But what about plasma current?

\[ \bar{J}(\vec{\eta}, t) = \int S S d\eta' S dt' \bar{g}(\vec{\eta}', t') \cdot E(\vec{\eta}', t') \]

Express this near \( \bar{g}(\vec{\eta}, t) \)

\[ E(\vec{\eta}' - \vec{\eta}, t - t') \approx \left[ \bar{g}(\vec{\eta}, t) - \frac{2E}{\partial t} - \vec{\eta}' \cdot \frac{2E}{\partial \eta} + \ldots \right] e^{-i(\omega t - \vec{\eta} \cdot \vec{n})} \]

\[ \bar{g}(\vec{\eta}', t') = \bar{g}(\vec{\eta}', t') e^{i(\omega t' - \vec{\eta}' \cdot \vec{n})} \]
Slowly-Varying Wave

\[ E(\vec{n}, t) = \left[ E_0(\vec{n}, t) - t \frac{2E}{\omega} \right] e^{\frac{-i(\omega t - \vec{A} \cdot \vec{n})}{\epsilon}} \]

But

\[ \iiint_{dV} \frac{\vec{E}(\vec{n}, t')}{2\epsilon} \cdot \vec{j}(\vec{n} - \vec{n'}, t-t') dV' = \vec{J}(\vec{n}, t) \]

\[ = \vec{6}(\vec{n}, t) \frac{\partial}{\partial t} \]

\[ = \vec{6}(\vec{n}, t) \frac{\partial}{\partial t} \frac{\partial}{\partial t} \]

\[ = -i \frac{2\vec{6}}{2\epsilon} \]

\[ \iiint_{dV} \frac{\vec{E}(\vec{n}, t')}{2\epsilon} \cdot \vec{j}(\vec{n} - \vec{n'}, t-t') dV' = \vec{J}(\vec{n}, t) \]

\[ \iiint_{dV} \frac{\vec{E}(\vec{n}, t')}{2\epsilon} \cdot \vec{j}(\vec{n} - \vec{n'}, t-t') dV' = \vec{J}(\vec{n}, t) \]

\[ \vec{J}(\omega, h) = \vec{6}(h). E(\omega, h) + i \frac{2\vec{6}}{\omega} \frac{2E}{\epsilon} - i \frac{2\vec{6}}{2\epsilon} \frac{2E}{\epsilon} + ... \]

For slowly varying amplitude...
Electrostatic Wave Energy Conservation

\[ \varepsilon_0 \frac{\partial \vec{E}}{\partial t} = -\vec{j} \]

\[ \varepsilon_0 \frac{\partial \vec{E}}{\partial t} - j \omega \varepsilon_0 \vec{E} = -\vec{\nabla} \cdot \hat{\vec{E}} + c \frac{2\vec{E}}{2\varepsilon_0 \frac{\partial}{\partial t}} - c \frac{2\vec{E}}{2k \cdot \vec{E}_0} + \ldots \]

Resistive Part:
\[ G = i G_i + G_R \]

Reactive Part:
\[ \text{For waves } (G_i) \gg |G_R| \]

Imaginary Part:
\[ 0 = -j \omega \varepsilon_0 (1 - \frac{G_i}{\omega \varepsilon_0}) \vec{E} \]

Real Part:
\[ \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \left( 1 + \frac{2G_i}{2\varepsilon_0 \frac{\partial}{\partial t}} \right) = -G_R \vec{E} + \frac{2G_i}{2k} \cdot \frac{\vec{E}}{2} \]
\[ \frac{\partial}{\partial t} \left( \varepsilon_0 \frac{|\vec{E}|^2}{2} \left( 1 - \frac{1}{\varepsilon_0 \frac{\partial}{\partial t}} \right) \right) = -\varepsilon_0 \left( \varepsilon_0 \frac{|\vec{E}|^2}{2} \right) - \nabla \cdot \left( \frac{-2G_i}{2k} \right) \frac{|\vec{E}|^2}{2} \varepsilon_0 \]
Electrostatic Wave Energy Conservation

Define \( W = \text{energy density of wave} \)

\[
W = \frac{\varepsilon_0 (E^2)}{2} \frac{2}{2w} \omega (\omega D) \quad \omega D = W \left(1 - \frac{\sigma}{\varepsilon_0} \right)
\]

\[
\nu_g = -\frac{2E}{2\omega^2} \frac{2w}{2w} \omega (\omega D) = \frac{2w}{2\omega} = \text{group velocity}
\]

\[
\gamma = \text{damping rate}
\]

\[
\gamma = \frac{G_R}{\varepsilon_0}
\]

\[
\frac{2}{2t} (w) + D \cdot \nu_g W = -\gamma W
\]
Electrostatic Wave Energy Conservation

**Plasma Wave**

\[ \omega_D(\omega) = \omega \left( 1 - \frac{\omega_D^2}{\omega^2} \right) \]

\[ \frac{\partial}{\partial \omega} (\omega_D) = 1 + \frac{\omega_D^2}{\omega^2} \]

**Ion Sound Wave**

\[ \omega_D = \omega \left( 1 - \frac{\omega_D^2}{\omega^2} + \frac{1}{k_D^2} \right) \]

\[ \frac{\partial}{\partial \omega} (2\omega_D) = 1 + \frac{1}{k_D^2} + \frac{\omega_D^2}{\omega^2} \]

\[ \approx 1 + \frac{\omega_D^2}{\omega^2} \]
John Malmberg and Chuck Wharton

The first experimental measurement of Landau Damping
Prof. Malmberg joined UCSD from General Atomics in 1969 as a professor of physics. Much of his work revolved around theoretical and experimental investigations of fully ionized gases or plasmas. The field could offer insights into how stars work and how to ignite and control thermonuclear reactions to produce fusion energy--the power that drives the sun.

A plasma is the fourth state of matter, with solids, liquids and gases making up the other three. Most of the matter in the Universe is in the plasma state; for example, the matter of stars is composed of plasmas.

In recent years, Prof. Malmberg had been experimenting with pure electron plasmas that were trapped in a magnetic bottle. By contrast with electrically neutral plasmas that contain an equal number of positive and negative electrons, pure electron plasmas are rare in nature.

Before joining UCSD, Prof. Malmberg was director of the Plasma Turbulence group at General Atomics, where he carried out some of the first and most important experiments to test the basic principals of plasma physics. Perhaps his most important experiment involved the confirmation of the phenomenon called "Landau damping," where electrons surf on a plasma wave, stealing energy from the wave and causing it to damp (decrease in amplitude).

For his pioneering work in testing the basic principals of plasma, and for his more recent work with electron plasmas, Prof. Malmberg was named the recipient of the American Physical Society's James Clerk Maxwell Prize in Plasma Physics in 1985.
Professor Wharton was a staff member at the University of California Lawrence Radiation Laboratory at Livermore and Berkeley, California from 1950-1962. While in this position, he spent a year (1959-60) as engineer-scientist at the Max-Planck Institute for Physics in Munich, Germany, and also as a lecturer at the International Summer Course in Plasma Physics at Riso, Denmark. For the next five years he was a staff member of the Experimental Physics Group at General Atomics in San Diego, California. He joined the EE faculty as a full professor in 1967.

In 1973 he received the Humboldt Prize awarded by the Alexander von Humboldt Foundation. He was elected a fellow of the American Physical Society in 1973. In 1976 he was elected a fellow of the IEEE “in recognition of contributions to the understanding of plasmas and to the development of plasma diagnostic techniques.” In 1979 he was given the award, Socio Onorario, by the International School of Plasma Physics (Milan, Italy).

Charles (Chuck) taught undergraduate courses in electromagnetic theory, plasma physics, and electrical sciences laboratory. His research was primarily in the area of plasma-physics diagnostics, in which he is a recognized world authority, and in plasma interactions and heating with waves and beams with applications to controlled thermonuclear fusion.
COLLISIONLESS DAMPING OF ELECTROSTATIC PLASMA WAVES*

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John Jay Hopkins Laboratory for Pure and Applied Science,
General Atomic Division of General Dynamics Corporation, San Diego, California
(Received 6 July 1964)

DISPERSION OF ELECTRON PLASMA WAVES*

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General Atomic Division of General Dynamics Corporation, San Diego, California
(Received 31 May 1966)
Description of the Experimental Device

The machine which produces the plasma has been described in detail elsewhere. The plasma is produced in a duoplasmatron-type hydrogen arc source and drifts from it into a long uniform magnetic field of a few hundred gauss. The entire machine is steady state. The resulting plasma has, in a typical case, a radius of 7 mm, a length of 230 cm, a density of $5 \times 10^8$ electrons/cm$^3$, and a temperature of $12 \pm 3$ eV as measured by Langmuir probes. The background pressure is $1.7 \times 10^{-5}$ Torr (mostly $H_2$). Hence, the Debye length is about 1 mm, the electron mean free path for electron-ion collisions is of the order of 1000 meters and for electron-neutral collisions is about 40 meters. The plasma is surrounded by a stainless steel tube 3.8 cm in radius which acts as a waveguide beyond cutoff to reduce electromagnetic coupling between probes. The plasma density depends somewhat on distance from the source.

FIGURE 10.8.4
Schematic of experiment used to investigate plasma wave echoes. [After J. H. Malmberg, et al., Proceedings of Conference on Phénomènes d'ionization dans les Gaz, 4; 229 (1963).]
Description of the Experimental Device

If $L$ is large compared with the Landau damping length, and if $\omega_2/(\omega_2 - \omega_1)$ is of order unity, this third electric field, which is the spatial plasma echo, appears at a position well separated from the first two electric field excitation positions. The experiment used by Malmberg et al.\textsuperscript{1} to study the spatial plasma wave echoes is depicted schematically in Fig. 10.8.4. The plasma column is 180 cm long and 5 cm in diameter, with a central density of $1.5 \times 10^8$ cm$^{-3}$. The axial magnetic field is 300 G and can be regarded as infinite for the purposes of the experiment. The plasma has a temperature of 9.4 eV and a Debye length of 2 mm. The electron mean free path is $10^5$ cm for electron-ion collisions and $4 \times 10^4$ cm for electron-neutral collisions. The plasma column is surrounded by a 5.2-cm-radius cylinder that acts as a waveguide beyond cutoff and reduces the stray electromagnetic coupling between the excitation and detection probes.

A plasma wave echo obtained with this experiment is shown in the lower trace of Fig. 10.8.5. The upper trace is the spatial distribution of the 120-MHz signal in the vicinity of the excitation probe at $x = 0$. The middle trace is the spatial distribution of the 130-MHz signal in the vicinity of the second probe at

Space Charge Waves in Cylindrical Plasma Columns

A. W. Trivelpiece† and R. W. Gould
California Institute of Technology, Pasadena, California

Assuming wave solutions \( \exp[j(\omega t - \beta z)] \)

Fig. 10. Schematic of apparatus to measure phase and attenuation characteristics of space charge waves in a plasma.

Table I. Pertinent dimensions of experimental apparatus used in space charge wave propagation experiment.

<table>
<thead>
<tr>
<th>Dimension</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Diameter of plasma</td>
<td>0.328 in.</td>
</tr>
<tr>
<td>Diameter of tube containing plasma</td>
<td>0.410 in.</td>
</tr>
<tr>
<td>Diameter of slotted wave guide</td>
<td>0.410 and 0.750 in.</td>
</tr>
<tr>
<td>Length of plasma column</td>
<td>25 cm</td>
</tr>
<tr>
<td>Signal frequency range</td>
<td>10 to 4000 mc</td>
</tr>
<tr>
<td>Cyclotron frequency range</td>
<td>0 to 5000 mc</td>
</tr>
<tr>
<td>Plasma frequency range</td>
<td>500 to 5000 mc</td>
</tr>
<tr>
<td>Temperature of mercury in tube</td>
<td>300±0.1°K</td>
</tr>
<tr>
<td>Empty wave guide cutoff frequency (approx)</td>
<td>25 000 mc</td>
</tr>
<tr>
<td>Pressure of mercury at 300°K (approx)</td>
<td>2 microns</td>
</tr>
<tr>
<td>Mean free path of plasma electrons (approx)</td>
<td>5 cm</td>
</tr>
</tbody>
</table>

Fig. 13. Measured phase characteristics of plasma space charge waves for no magnetic field for \( a = 0.52 \) cm, \( b = 0.62 \) cm, \( K = 4.6 \).
TG Modes: Low Frequency Surface Waves

Inside Plasma (\( n < a \)):
\[
\begin{align*}
\mathbf{E} &= -\nabla \Phi \\
\nabla \cdot (\varepsilon_0 \mathbf{E}) &= 0 \\
\Rightarrow \nabla^2 \Phi &= 0 \\
k &= 1 - \frac{\omega_p^2}{\omega^2}
\end{align*}
\]

Outside Plasma (\( n > a \)):
\[
\begin{align*}
\mathbf{E} &= -\nabla \Phi \\
\nabla \cdot (\varepsilon_0 \mathbf{E}) &= 0 \\
\nabla^2 \Phi &= 0
\end{align*}
\]

Boundary Conditions (\( n = a \)):
\[
\begin{align*}
\nabla \cdot (\varepsilon_0 \mathbf{E}_n \mathbf{E}_n - \varepsilon_0 \mathbf{E}_o \mathbf{E}_o) &= 0 \\
\Phi_{in} &= \Phi_{out}
\end{align*}
\]

\[
\Phi(n) \sim I_0(hn) \quad \text{inside} \\
K_0(hn) \quad \text{outside}
\]

\[
\begin{align*}
\Phi(n) &= (1 - \frac{\omega_p^2}{\omega^2}) I_0'(h_n) c_1 = K_0'(h_n) c_2 \\
I_0 c_1 &= K_0 c_2 \\
1 - \frac{\omega_p^2}{\omega^2} &= \frac{I_0(h_n) K_0'(h_n)}{I_0'(h_n) K_0(h_n)}
\end{align*}
\]
Raw Data

Two probes, each consisting of a 0.2-mm diameter radial tungsten wire, are placed in the plasma. One probe is connected by coaxial cable to a chopped signal generator. The other probe is connected to a receiver which includes a sharp high-frequency filter, a string of broad-band amplifiers, an rf detector, a video amplifier, and a coherent detector operated at the transmitter chopping frequency. Provision is made to add a reference signal from the transmitter to the receiver rf signal, i.e., we may use the system as an interferometer. The transmitter is set at a series of fixed frequencies, and at each, the receiving probe is moved longitudinally. The position of the receiving probe, which is transduced, is applied to the \(x\) axis of an \(x-y\) recorder, and the interferometer output or the logarithm of the received power is applied to the \(y\) axis.

FIG. 1. Raw data. Upper curve is the logarithm of received power. Lower curve is interferometer output. Abscissa is probe separation.
Waves in Magnetized Plasma

\[
\frac{\partial \mathbf{v}^{(\alpha)}}{\partial t} = \frac{q_{\alpha}}{m_{\alpha}} \left( \mathbf{E}_1 + \mathbf{v}^{(\alpha)} \times \mathbf{B}_0 \right) \quad \alpha = e, i
\]

\[
\hat{v}_x = \frac{i}{\omega m} \left( \hat{E}_x + \hat{v}_y B_0 \right), \quad \hat{v}_y = \frac{i}{\omega m} \left( \hat{E}_y - \hat{v}_x B_0 \right)
\]

\[
\hat{v}^\pm = \hat{v}_x \pm i \hat{v}_y, \quad \hat{E}^\pm = \hat{E}_x \pm i \hat{E}_y
\]

\[
\hat{v}^\pm = \frac{i}{\omega m} \left( \hat{E}^\pm \mp i \hat{v}^\pm B_0 \right)
\]

\[
\hat{v}^\pm = \frac{i}{m} \hat{E}^\pm \frac{1}{\omega \mp s\omega_c}
\]
Waves in Magnetized Plasma

\[
\begin{align*}
(\hat{v}_x, \hat{v}_y, \hat{v}_z) &= i \frac{q}{\omega m} \begin{pmatrix} \frac{\omega^2}{\omega^2 - \omega_c^2} & i \frac{s \omega \omega_c}{\omega^2 - \omega_c^2} & 0 \\ -i \frac{s \omega \omega_c}{\omega^2 - \omega_c^2} & \frac{\omega^2}{\omega^2 - \omega_c^2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot (\hat{E}_x, \hat{E}_y, \hat{E}_z)
\end{align*}
\]

\[
\sigma(\omega) = i \omega \varepsilon_0 \begin{pmatrix}
\sum_{\alpha} \frac{\omega_{p\alpha}^2}{\omega^2 - \omega_{c\alpha}^2} & i \sum_{\alpha} s_{\alpha} \frac{\omega_{p\alpha}^2}{\omega^2 - \omega_{c\alpha}^2} & \frac{\omega_{c\alpha}}{\omega} & 0 \\
-\sum_{\alpha} s_{\alpha} \frac{\omega_{p\alpha}^2}{\omega^2 - \omega_{c\alpha}^2} & \frac{\omega_{c\alpha}}{\omega} & \sum_{\alpha} \frac{\omega_{p\alpha}^2}{\omega^2 - \omega_{c\alpha}^2} & 0 \\
0 & 0 & \sum_{\alpha} \frac{\omega_{p\alpha}^2}{\omega^2} & 0
\end{pmatrix}
\]

\[
\varepsilon_\omega = \mathbf{I} + \frac{i}{\omega \varepsilon_0} \sigma_\omega
\]
Waves in Magnetized Plasma

\[ \varepsilon_\omega = \mathbf{I} + \frac{i}{\omega \varepsilon_0} \sigma_\omega \]

\[ \varepsilon(\omega) = \begin{pmatrix} S & -iD & 0 \\ iD & S & 0 \\ 0 & 0 & P \end{pmatrix} \]

\[ S = 1 - \sum_\alpha \frac{\omega_{p\alpha}^2}{\omega^2 - \omega_{c\alpha}^2} \]

\[ D = \sum_\alpha s_\alpha \frac{\omega_{p\alpha}^2}{\omega^2 - \omega_{c\alpha}^2} \frac{\omega_{c\alpha}}{\omega} \]

\[ P = 1 - \sum_\alpha \frac{\omega_{p\alpha}^2}{\omega^2} . \]


Working both in the laboratory and with theoretical calculations, he found many ways to put waves to work in fusion research in succeeding decades, and his 1962 book, "The Theory of Plasma Waves," codified the subject in mathematical form for the first time.
Waves in Magnetized Plasma

\[
\begin{pmatrix}
S - N^2 \cos^2 \psi & -iD & N^2 \cos \psi \sin \psi \\
 iD & S - N^2 & 0 \\
N^2 \cos \psi \sin \psi & 0 & P - N^2 \sin^2 \psi
\end{pmatrix}
\begin{pmatrix}
\hat{E}_x \\
\hat{E}_y \\
\hat{E}_z
\end{pmatrix} = 0
\]

\[k = (k \sin \psi, 0, k \cos \psi)\]

Good places to start:
Propagation along B (\(\psi = 0\))
Propagation \(\perp\) to B (\(\psi = \pi/2\))

\[
S = 1 - \sum_{\alpha} \frac{\omega_{p\alpha}^2}{\omega^2 - \omega_{c\alpha}^2}
\]

\[
D = \sum_{\alpha} s_{\alpha} \frac{\omega_{p\alpha}^2}{\omega^2 - \omega_{c\alpha}^2} \frac{\omega_{c\alpha}}{\omega}
\]

\[
P = 1 - \sum_{\alpha} \frac{\omega_{p\alpha}^2}{\omega^2}.
\]
Waves $k \parallel B$

\[
\begin{pmatrix}
S - N^2 & -iD & 0 \\
iD & S - N^2 & 0 \\
0 & 0 & P
\end{pmatrix}
\cdot
\begin{pmatrix}
\hat{E}_x \\
\hat{E}_y \\
\hat{E}_z
\end{pmatrix} = 0
\]
Waves $k \perp B$

Extra-Ordinary Mode

\[
\begin{pmatrix}
S & -iD \\
iD (S - N^2)
\end{pmatrix}
\cdot
\begin{pmatrix}
\hat{E}_x \\
\hat{E}_y
\end{pmatrix} = 0
\]

\[N_X = \left(\frac{S^2 - D^2}{S}\right)^{1/2}\]

\[\omega_{uh} = (\omega_{ce}^2 + \omega_{pe}^2)^{1/2}\]

\[\omega_{lh} = \left(\omega_{ci}^2 + \frac{\omega_{pi}^2 \omega_{ce}^2}{\omega_{pe}^2 + \omega_{ce}^2}\right)^{1/2}\]

Plus: Ordinary Mode

\[\omega^2 = \omega_{pe}^2 + k^2 c^2\]
Waves $k \perp B$

Extra-Ordinary Mode

\[
\begin{pmatrix} S & -iD \\ iD (S - N^2) \end{pmatrix} \cdot \begin{pmatrix} \hat{E}_x \\ \hat{E}_y \end{pmatrix} = 0
\]

Plus: Ordinary Mode

\[
\omega^2 = \omega_{pe}^2 + k^2 c^2
\]
Next Lecture

- Chapter 7: “Plasma Boundaries”
- Probes (!)