Lecture10: Plasma Physics 1

APPH E6101x
Columbia University

Last Lecture

- Conservation principles in magnetized plasma (“frozen-in” and conservation of particles/flux tubes)
- Alfvén waves (without plasma pressure)

This Lecture

- Quiz 1: Wednesday
- Introduction to plasma waves
  - Basic review of electromagnetic waves in various media (conducting, dielectric, gyrotropic, …)
  - Basic waves concepts (especially plane waves)
- Electromagnetic waves in *unmagnetized* plasma
- Electrostatic waves in *unmagnetized* plasma

More Practice Problems

AP 6101
Practice for Quiz #1
From Fitzpatrick, and Gurnett and Bhattacharjee

Dielectric of a Magnetized Plasma

2. A quasi-neutral slab of cold (i.e., $\lambda_D \to 0$) plasma whose bounding surfaces are normal to the x-axis consists of electrons of mass m_e, charge $-e$, and mean number density n_e, as well as ions of mass m_i, charge e, and mean number density n_e. The slab is fully magnetized by a uniform y-directed magnetic field of magnitude B. The slab is then subject to an externally generated, uniform, x-directed electric field that is gradually ramped up to a final magnitude E_0. Show that, as a consequence of ion polarization drift, the final magnitude of the electric field inside the plasma is

$$E_1 \simeq \frac{E_0}{\epsilon},$$

where

$$\epsilon = 1 + \frac{c^2}{V_A^2},$$

and $V_A = B/\sqrt{\mu_0\, n_e\, m_i}$ is the so-called Alfvén velocity.

Thermal Equilibrium

5. A uniform isothermal quasi-neutral plasma with singly-charged ions is placed in a relatively weak gravitational field of acceleration $\mathbf{g} = -g\,\mathbf{e}_z$. Assuming, first, that both species are distributed according to the Maxwell-Boltzmann statistics; second, that the perturbed electrostatic potential is a function of z only; and, third, that the electric field is zero at $z = 0$ (and well behaved as $z \to \infty$), demonstrate that the electric field in the region $z > 0$ takes the form $\mathbf{E} = E_z\,\mathbf{e}_z$, where

$$E_z(z) = E_0\left[1 - \exp\left(\frac{\sqrt{2}\,z}{\lambda_D}\right)\right],$$

and

$$E_0 = \frac{m_i\,g}{2\,e}.$$

Here, λ_D is the Debye length, e the magnitude of the electron charge, and m_i the ion mass.

Adiabatic Invariants

6. A particle of charge e, mass m, and energy $\mathcal{E}$, is trapped in a one-dimensional magnetic well of the form

$$B(x,t) = B_0\,(1 + k^2\,x^2),$$

where B_0 is constant, and $k(t)$ is a very slowly increasing function of time. Suppose that the particle's mirror points lie at $x = \pm x_m(t)$, and that its bounce time is $\tau_b(t)$. Demonstrate that, as a consequence of the conservation of the first and second adiabatic invariants,

$$x_m(t) = x_m(0)\left[\frac{k(0)}{k(t)}\right]^{1/2},$$

$$\tau_b(t) = \tau_b(0)\left[\frac{k(0)}{k(t)}\right],$$

$$\mathcal{E}(t) = \mathcal{E}_{0\perp} + \left[\frac{k(t)}{k(0)}\right]\mathcal{E}_{0\parallel}.$$

Here, $\mathcal{E}_{0\perp}$ is the perpendicular energy [i.e., $(1/2)\,m\,v_\perp^2$], and $\mathcal{E}_{0\parallel}$ is the parallel energy [i.e., $(1/2)\,m\,v_\parallel^2$], both evaluated at $x = 0$ and $t = 0$. Assume that the particle's gyroradius is relatively small, and that the electric field-strength is negligible.

Drift Velocity w Collisions

8. Consider a spatially uniform, unmagnetized plasma in which both species have zero mean flow velocity. Let n_e and T_e be the electron number density and temperature, respectively. Let $\mathbf{E}$ be the ambient electric field. The electron distribution function f_e satisfies the simplified kinetic equation

$$-\frac{e}{m_e}\,\mathbf{E}\cdot\nabla_v f_e = C_e.$$

We can crudely approximate the electron collision operator as

$$C_e = -\nu_e\,(f_e - f_0)$$

where ν_e is the effective electron-ion collision frequency, and

$$f_0 = \frac{n_e}{\pi^{3/2}\,v_{te}^3}\exp\left(-\frac{v^2}{v_{te}^2}\right).$$

Here, $v_{te} = \sqrt{2\,T_e/m_e}$. Suppose that $E \ll m_e\,\nu_e\,v_{te}/e$. Demonstrate that it is a good approximation to write

$$f_e = f_0 + \frac{e}{m_e\,\nu_e}\,\mathbf{E}\cdot\nabla_v f_0.$$

Hence, show that

$$\mathbf{j} = \sigma\,\mathbf{E},$$

where

$$\sigma = \frac{e^2\,n_e}{m_e\,\nu_e}.$$

Static MHD Equilibrium

7.6. For a force-balanced MHD equilibrium in a cylindrical geometry with $\mathbf{B} = [0, B_\phi(\rho), B_z(\rho)]$ the radial component of the pressure balance condition $\mathbf{J} \times \mathbf{B} = \nabla P$ can be written

$$\frac{\partial}{\partial \rho}\left(P + \frac{B_\phi^2}{2\mu_0} + \frac{B_z^2}{2\mu_0}\right) = [(\mathbf{B} \cdot \nabla)\mathbf{B}]_\rho.$$

Show that $[(\mathbf{B} \cdot \nabla)\mathbf{B}]_\rho = -B_\phi^2/\rho$.
Hint: Use the identity $\nabla(\mathbf{F} \cdot \mathbf{G}) = (\mathbf{F} \cdot \nabla)\mathbf{G} + (\mathbf{G} \cdot \nabla)\mathbf{F} + \mathbf{F} \times (\nabla \times \mathbf{G}) + \mathbf{G} \times (\nabla \times \mathbf{G})$.

Alfvén Waves with Collisions/Viscosity

1. We can add viscous effects to the MHD momentum equation by including a term $\mu\,\nabla^2\mathbf{V}$, where μ is the dynamic viscosity, so that

$$\rho\,\frac{d\mathbf{V}}{dt} = \mathbf{j}\times\mathbf{b} - \nabla p + \mu\,\nabla^2\mathbf{V}.$$

Likewise, we can add finite conductivity effects to the Ohm's law by including the term $(1/\mu_0\,\sigma)\,\nabla^2\mathbf{B}$, to give

$$\frac{\partial\mathbf{B}}{\partial t} = \nabla\times(\mathbf{V}\times\mathbf{B}) + \frac{1}{\mu_0\,\sigma}\,\nabla^2\mathbf{B},$$

Show that the modified dispersion relation for Alfvén waves can be obtained from the standard one by multiplying both ω^2 and V_S^2 by a factor

$$[1 + \mathrm{i}\,k^2/(\mu_0\,\sigma\,\omega)],$$

and ω^2 by an additional factor

$$[1 + \mathrm{i}\,\mu\,k^2/(\rho_0\,\omega)].$$

If the finite conductivity and viscous corrections are small (i.e., $\sigma\to\infty$ and $\mu\to 0$), show that, for parallel ($\theta = 0$) propagation, the dispersion relation for the shear-Alfvén wave reduces to

$$k \simeq \frac{\omega}{V_A} + \mathrm{i}\,\frac{\omega^2}{2\,V_A^3}\left(\frac{1}{\mu_0\,\sigma} + \frac{\mu}{\rho_0}\right).$$

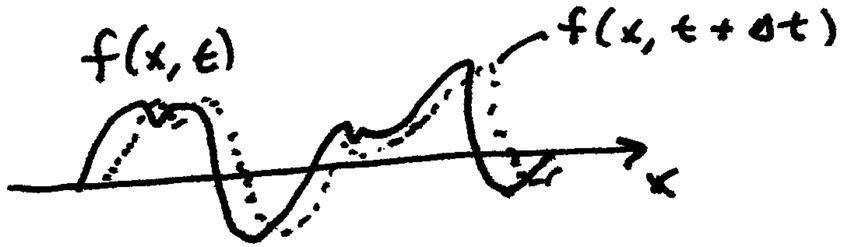
Wave (Helmholtz) Equation

$$\frac{\partial^2 f}{\partial t^2} = c^2 \frac{\partial^2 f}{\partial x^2}$$ (with c ~ constant)

SOLUTION: $f(x,t) = f(x \pm ct)$

$$\frac{\partial f}{\partial t} = \pm c f' \qquad \frac{\partial f}{\partial x} = f'$$

$$\frac{\partial^2 f}{\partial t^2} = c^2 f'' \qquad \frac{\partial^2 f}{\partial x^2} = f''$$



FOR (SIMPLE) HELMHOLTZ EQ, WAVES MOVE AT CONSTANT SPEED c WITHOUT DISTORTION!

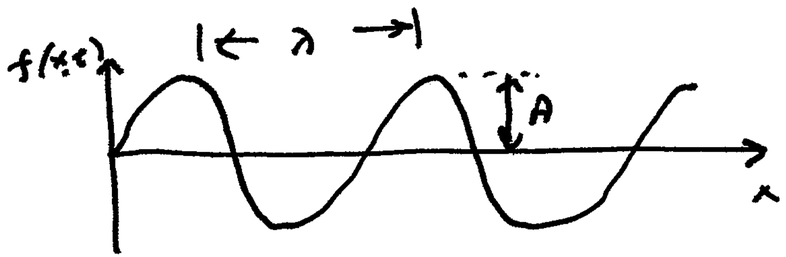
Plane Waves

$$f(x,t) = A \sin\left(\frac{2\pi}{\lambda}(x \pm c t)\right)$$

AMPLITUDE

WAVELENGTH

SPEED



FREQUENCY $= \omega = 2\pi \frac{c}{\lambda}$ RAD/SEC

$\left(\frac{c}{\lambda}\right)$ = FREQUENCY IN HERTZ

$\varphi(x,t) \equiv \frac{2\pi}{\lambda}(x \pm c t)$ = PHASE OF WAVE

"PHASE FRONTS" ARE SURFACES OF CONSTANT PHASE

"PHASE FRONTS" MOVE AT SPEED c

c = PHASE VELOCITY

$$\varphi = \mathbf{k} \cdot \mathbf{r} - \omega t$$

Wave Eq in Multiple Dimensions

$$\frac{\partial^2 f}{\partial t^2} = c^2 \nabla^2 f$$

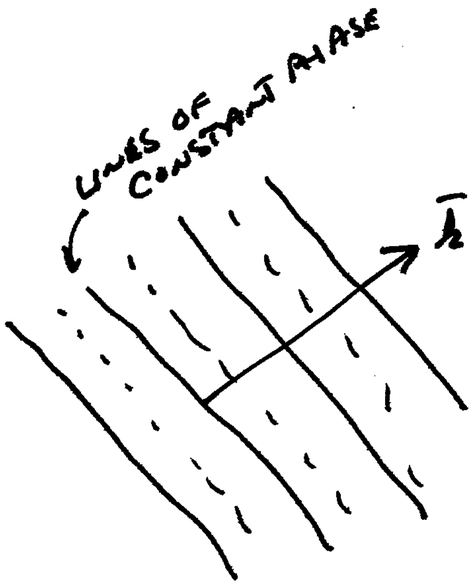
PLANE WAVES:

$$f(x,y,z,t) = A \sin\Big(\underbrace{\frac{2\pi}{\lambda_x}x + \frac{2\pi}{\lambda_y}y + \frac{2\pi}{\lambda_z}z - \omega t}_{\varphi(x,y,z,t) = \text{PHASE FRONT}}\Big)$$

$$\bar{k} = \left(\frac{2\pi}{\lambda_x}, \frac{2\pi}{\lambda_y}, \frac{2\pi}{\lambda_z}\right)$$

= WAVE VECTOR

$\bar{k}$ POINTS IN THE DIRECTION OF PHASE VELOCITY



$$f(x,y,z,t) = A \sin(\bar{k}\cdot\bar{x} - \omega t)$$

$$|c| = \frac{\omega}{|\bar{k}|} \qquad \bar{c} = \bar{k}\left(\frac{\omega}{k^2}\right) = \text{PHASE VELOCITY}$$

$$\varphi = \mathbf{k}\cdot\mathbf{r} - \omega t$$

$$\mathbf{E} = \hat{\mathbf{E}}\,\exp[\mathrm{i}(\mathbf{k}\cdot\mathbf{r} - \omega t)]$$

$$\mathbf{B} = \hat{\mathbf{B}}\,\exp[\mathrm{i}(\mathbf{k}\cdot\mathbf{r} - \omega t)]$$

$$\mathbf{j} = \hat{\mathbf{j}}\,\exp[\mathrm{i}(\mathbf{k}\cdot\mathbf{r} - \omega t)]\,.$$

OTHER SOLUTIONS:

– SUM OF PLANE WAVES

Wave Eq in Cylindrical Coordinates

$$\frac{\partial^2 f}{\partial t^2} = c^2 \nabla^2 f$$

CYLINDRICAL COORDINATES:

$$\frac{\partial^2 f}{\partial t^2} = c^2 \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \varphi^2} + \frac{\partial^2 f}{\partial z^2} \right]$$

CHOOSE SEPARATION OF VARIABLES

$$f(r, \varphi, z, t) \sim g(r)\, e^{\pm i m \theta}\, e^{jkz}\, e^{-j\omega t}$$

$$\frac{\partial^2 f}{\partial \varphi^2} = -m^2 f \qquad \frac{\partial^2 f}{\partial z^2} \sim -k^2 f$$

$$\frac{\partial^2 f}{\partial t^2} = -\omega^2 f$$

SO WAVE EQUATION IS:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial g}{\partial r}\right) + \left(\frac{\omega^2}{c^2} - \frac{m^2}{r^2} - k^2 \right) g(r) = 0$$

CAN BE PUT INTO BESSEL'S EQUATION

$$g(r) = J_m\left(\sqrt{\left(\frac{\omega}{c}\right)^2 - k^2}\; r \right) \; !$$

Wave Packets and Group Velocity

IF $f_1(x,t)$ AND $f_2(x,t)$ ARE SOLUTIONS TO HELMHOLTZ EQUATION, THEN $f_1 + f_2$ IS ALSO A SOLUTION. (THE EQUATION IS <u>LINEAR</u>!)

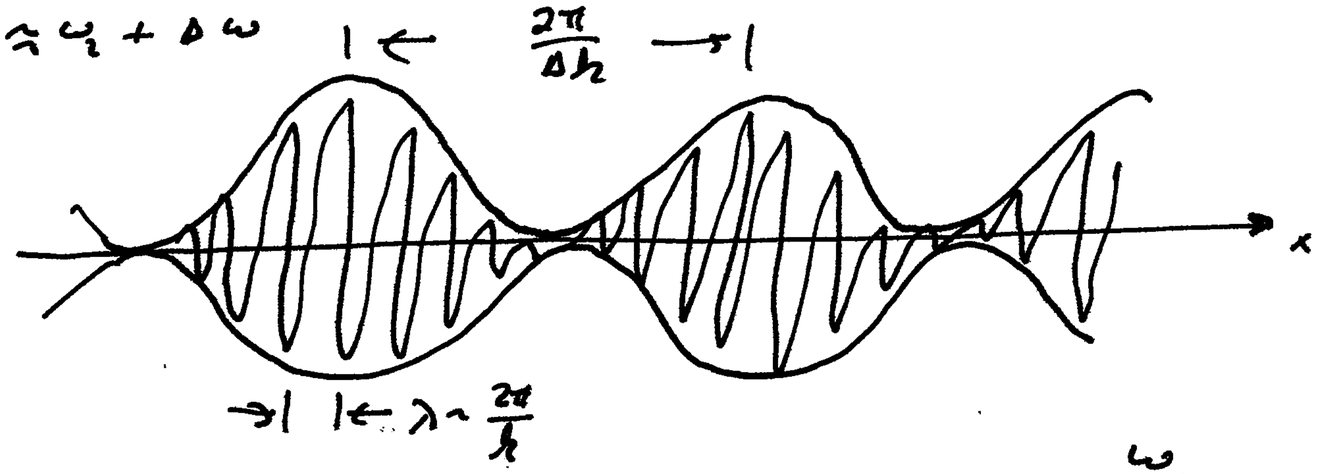
$$f(x,t) = f_1(k_1 x - \omega_1 t) + f_2(k_2 x - \omega_2 t)$$

$$= \sin(k_1 x - \omega_1 t) + \sin(k_2 x - \omega_2 t) \qquad \text{(FOR EXAMPLE)}$$

$$= 2\cos\left(\frac{k_1 - k_2}{2} x - \frac{\omega_1 - \omega_2}{2} t\right) \sin\left(\frac{k_1 + k_2}{2} x - \frac{\omega_1 + \omega_2}{2} t\right)$$

IF $k_1 \approx k_2 + \Delta k$

$\omega_1 \approx \omega_2 + \Delta\omega$

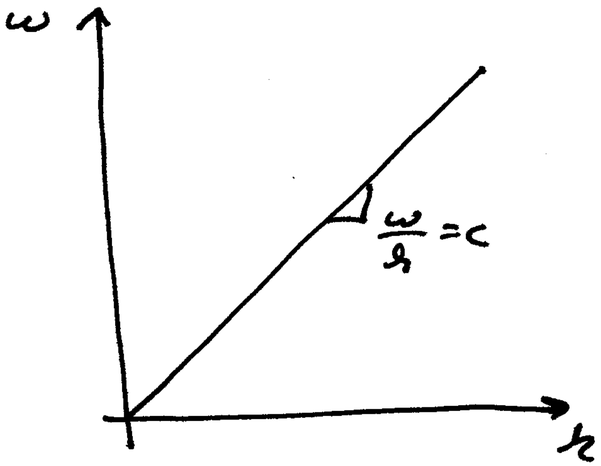


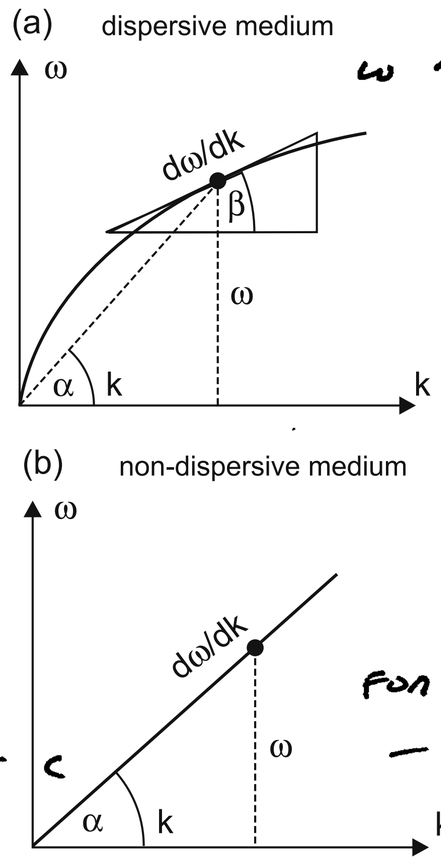
WAVE <u>PHASE FRONTS</u> MOVE AT SPEED $c = \frac{\omega}{|k|}$ = PHASE VELOCITY

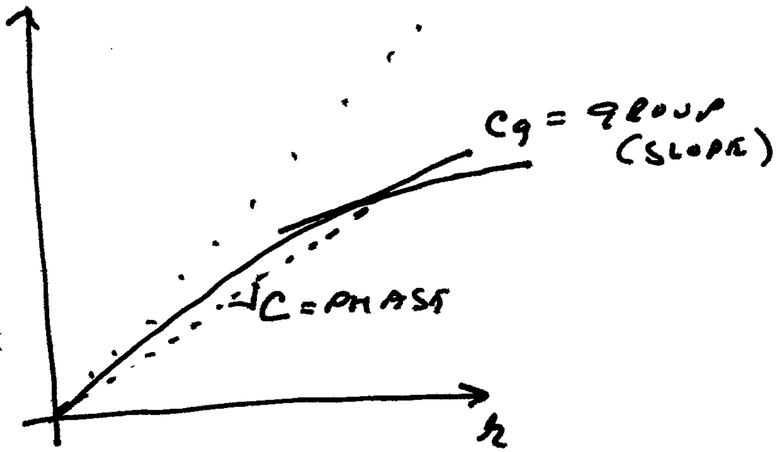
WAVE <u>PACKETS</u> MOVE AT SPEED $c_g = \frac{\Delta\omega}{\Delta k}$ = GROUP VELOCITY

Wave Dispersion

DISPERSIVE WAVES ARE WAVES WHOSE PHASE VELOCITY CHANGES WITH WAVELENGTH (OR FREQUENCY..







NON-DISPERSIVE WAVE

- ALL WAVES MOVE AT C
- GROUP VELOCITY EQUALS PHASE VELOCITY

EXAMPLE DISPERSIVE WAVE

FOR THIS WAVE ...

- SHORT WAVELENGTHS MOVE SLOWER THAN LONG WAVELENGTHS
- WAVE PACKETS OF SHORT WAVELENGTH WAVES HAVE VERY SLOW GROUP VELOCITY

Phasors, Sine, Cosine

$$\nabla \times \mathbf{E} \to \mathrm{i}\mathbf{k} \times \hat{\mathbf{E}}, \quad \nabla \cdot \mathbf{E} \to \mathrm{i}\mathbf{k} \cdot \hat{\mathbf{E}}, \quad \frac{\partial}{\partial t}\mathbf{E} \to -\mathrm{i}\omega\hat{\mathbf{E}}$$

$$\sin(kx - \omega t) = \mathrm{Re}\left\{ e^{i\frac{\pi}{2}} e^{-j(\omega t - kx)} \right\} \qquad \text{PHASOR} = e^{i\pi/2}$$

$$\cos(kx - \omega t) = \mathrm{Re}\left\{ e^{-j(\omega t - kx)} \right\} \qquad \text{PHASOR} = 1$$

$$\cos(kx - \omega t + \theta) = \mathrm{Re}\left\{ e^{i\theta} e^{-j(\omega t - kx)} \right\} \qquad \text{PHASOR} = e^{i\theta}$$

$$\int_0^\lambda \frac{dx}{\lambda} \cos(kx - \omega t) = \int_0^\lambda \frac{dx}{\lambda} \sin(kx - \omega t) = 0$$

$$\int_0^\lambda \frac{dx}{\lambda} \cos^2(kx - \omega t) = \int_0^{2\pi} \frac{d\xi}{2\pi} \cos^2(\xi - \omega t) = \frac{1}{2}$$

$$\int_0^\lambda \frac{dx}{\lambda} \cos(kx - \omega t)\cos(kx - \omega t + \theta) = \frac{1}{2}\cos(\theta)$$

$$\int_0^\lambda \frac{dx}{\lambda} \cos(kx - \omega t)\sin(kx - \omega t) = 0$$

IF A = PHASOR, THEN

$\frac{1}{2}|A|^2 = \frac{1}{2}A^*A$ = SQUARE AVERAGE OF $\mathrm{Re}\left\{A e^{-j(\omega t - kx)}\right\}$

$|A|/\sqrt{2} \equiv$ RMS OF A

PDE's become algebraic!

Review of EM Waves

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{B} = \mu_0 \left(\mathbf{j} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right)$$

$$\begin{aligned} \nabla \times (\nabla \times \mathbf{E}) &= -\nabla \times \frac{\partial \mathbf{B}}{\partial t} \\ &= -\frac{\partial}{\partial t}(\nabla \times \mathbf{B}) \\ &= -\mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} - \mu_0 \frac{\partial \mathbf{j}}{\partial t} \end{aligned}$$

Review of EM Waves

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{B} = \mu_0 \left(\mathbf{j} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right)$$

$$\varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} + \mathbf{j} = \varepsilon_0 \bar{\varepsilon}(\omega) \frac{\partial \mathbf{E}}{\partial t}$$

$$\mathbf{j}(\omega) = \bar{\sigma}(\omega) \cdot \mathbf{E}(\omega)$$

$$\bar{\varepsilon}(\omega) = \bar{1} + \frac{\mathrm{i}}{\omega \varepsilon_0} \bar{\sigma}(\omega)$$

Review of EM Waves

$$\nabla \times (\nabla \times \mathbf{E}) + \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = -\mu_0 \frac{\partial \mathbf{j}}{\partial t}$$

$$\mathrm{i}\mathbf{k} \times \hat{\mathbf{E}} = \mathrm{i}\omega \hat{\mathbf{B}}$$

$$\mathrm{i}\mathbf{k} \times \hat{\mathbf{B}} = -\mathrm{i}\omega\varepsilon_0\mu_0\hat{\mathbf{E}} + \mu_0\hat{\mathbf{j}}_0$$

$$\mathbf{k} \times (\mathbf{k} \times \hat{\mathbf{E}}) = (\mathbf{k}\mathbf{k} - k^2\hat{\mathbf{I}})\hat{\mathbf{E}}$$

All of the plasma physics here

$$\left\{\mathbf{k}\mathbf{k} - k^2\boldsymbol{I} + \frac{\omega^2}{c^2}\boldsymbol{I} + \mathrm{i}\omega\mu_0\boldsymbol{\sigma}(\omega)\right\} \cdot \hat{\mathbf{E}} = 0$$

$$\left\{\mathbf{k}\mathbf{k} - k^2\boldsymbol{I} + \frac{\omega^2}{c^2}\boldsymbol{\varepsilon}(\omega)\right\} \cdot \hat{\mathbf{E}} = 0$$

Normal Modes ("Dispersion Relation:)

$$\begin{pmatrix} k_x k_x - k^2 + \frac{\omega^2}{c^2}\varepsilon_{xx} & k_x k_y + \frac{\omega^2}{c^2}\varepsilon_{xy} & k_x k_z + \frac{\omega^2}{c^2}\varepsilon_{xz} \\ k_y k_x + \frac{\omega^2}{c^2}\varepsilon_{yx} & k_y k_y - k^2 + \frac{\omega^2}{c^2}\varepsilon_{yy} & k_y k_z + \frac{\omega^2}{c^2}\varepsilon_{yz} \\ k_z k_x + \frac{\omega^2}{c^2}\varepsilon_{zx} & k_z k_y + \frac{\omega^2}{c^2}\varepsilon_{zy} & k_z k_z - k^2 + \frac{\omega^2}{c^2}\varepsilon_{zz} \end{pmatrix} \cdot \begin{pmatrix} \hat{E}_x \\ \hat{E}_y \\ \hat{E}_z \end{pmatrix} = 0\,. \tag{6.28}$$

$$0 = D(\omega, \mathbf{k}) = \det\left[\mathbf{k}\mathbf{k} - k^2 \boldsymbol{I} + \frac{\omega^2}{c^2}\boldsymbol{\varepsilon}(\omega)\right]. \tag{6.29}$$

Waves in Unmagnetized Plasma

Collisionless
(reactive)

$$m\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = q\hat{\mathbf{E}}\mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\mathbf{r}-\omega t)}$$

90° out of phase

$$\hat{\mathbf{j}} = nq\hat{\mathbf{v}} = \mathrm{i}\frac{ne^2}{\omega m}\hat{\mathbf{E}}$$

$$\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = \mathrm{i}\frac{ne^2}{\omega m}$$

$$\varepsilon_{xx} = \varepsilon_{yy} = \varepsilon_{zz} = 1 + \frac{\mathrm{i}}{\omega\varepsilon_0}\sigma_{yy} = 1 - \frac{\omega_{\mathrm{pe}}^2}{\omega^2}$$

Collisional
(resistive)

$$m\ (-\mathrm{i}\omega + \nu_{\mathrm{m}})\ \hat{v} = q\ \hat{E}$$

$$\hat{v} = \left[\frac{\nu_{\mathrm{m}}}{\omega^2 + \nu_{\mathrm{m}}^2} + \frac{\mathrm{i}\omega}{\omega^2 + \nu_{\mathrm{m}}^2}\right]\frac{q}{m}\hat{E}$$

In phase
(low-frequency)

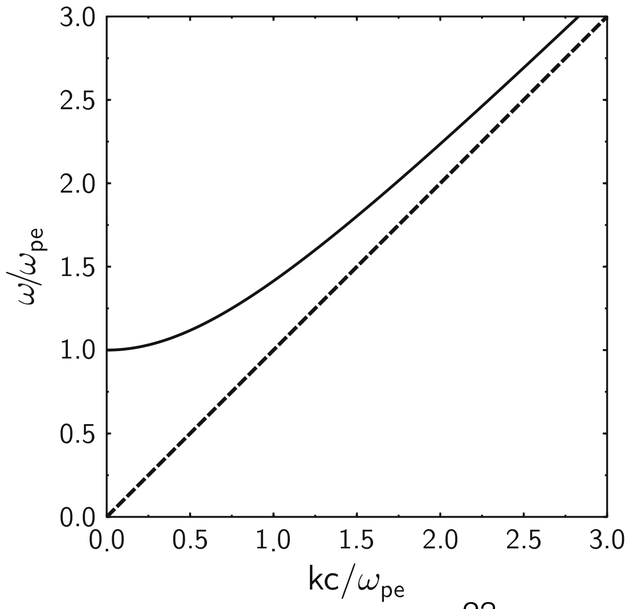
$$\omega_{\mathrm{pe}} = \left(\frac{ne^2}{\varepsilon_0 m_{\mathrm{e}}}\right)^{1/2}$$

Waves in Unmagnetized Plasma

$$0 = D(\omega, \mathbf{k}) = \det\left[\mathbf{kk} - k^2 \boldsymbol{I} + \frac{\omega^2}{c^2}\boldsymbol{\varepsilon}(\omega)\right] \qquad \mathbf{k} = \{k_x, 0, 0\}$$

$$\begin{pmatrix} \frac{\omega^2}{c^2}\left(1 - \frac{\omega_{\rm pe}^2}{\omega^2}\right) & 0 & 0 \\ 0 & -k^2 + \frac{\omega^2}{c^2}\left(1 - \frac{\omega_{\rm pe}^2}{\omega^2}\right) & 0 \\ 0 & 0 & -k^2 + \frac{\omega^2}{c^2}\left(1 - \frac{\omega_{\rm pe}^2}{\omega^2}\right) \end{pmatrix} \cdot \begin{pmatrix} \hat{E}_x \\ \hat{E}_y \\ \hat{E}_z \end{pmatrix} = 0\,. \tag{6.35}$$

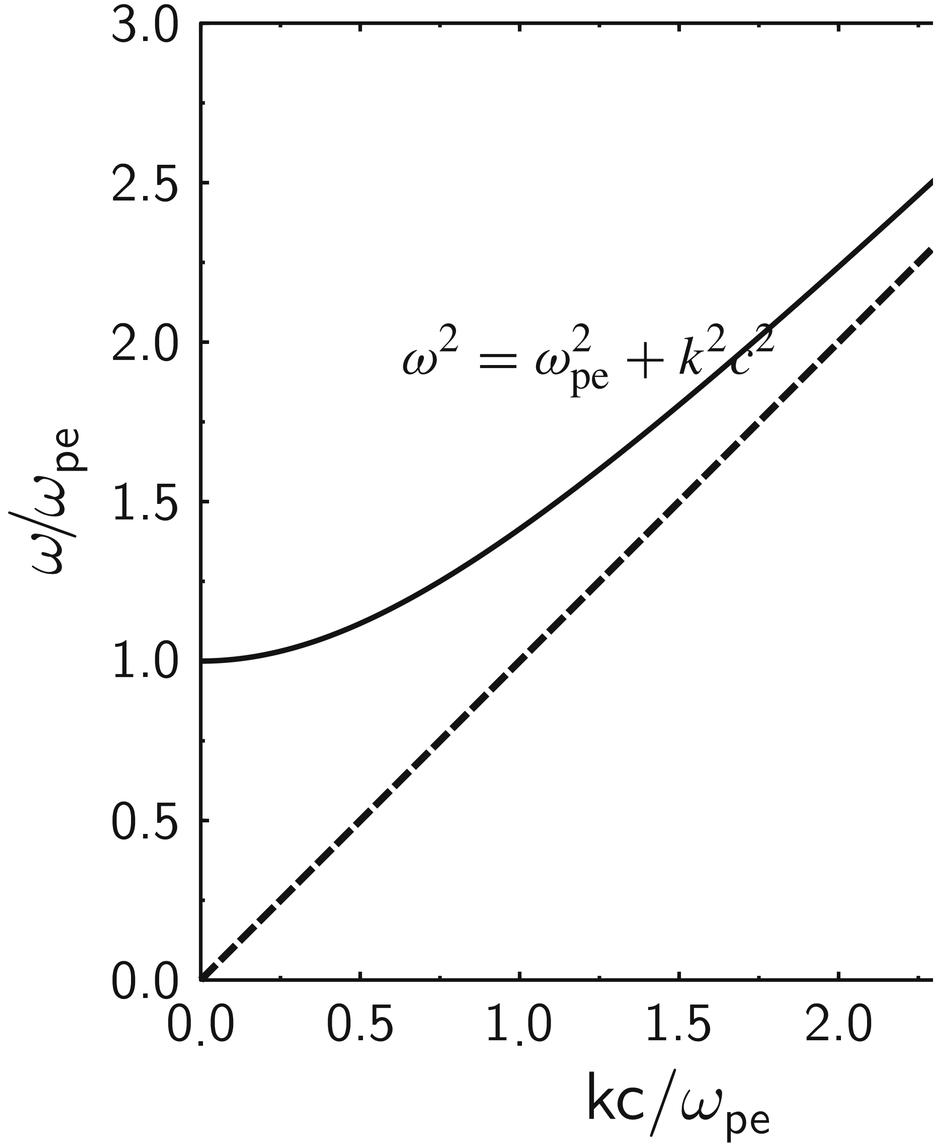
$$\omega^2 = \omega_{\rm pe}^2 + k^2 c^2$$



$$v_\varphi = \left(\frac{\omega_{\rm pe}^2}{k^2} + c^2\right)^{1/2}$$

$$v_{\rm gr} = \frac{kc^2}{\left(\omega_{\rm pe}^2 + k^2 c^2\right)^{1/2}}$$

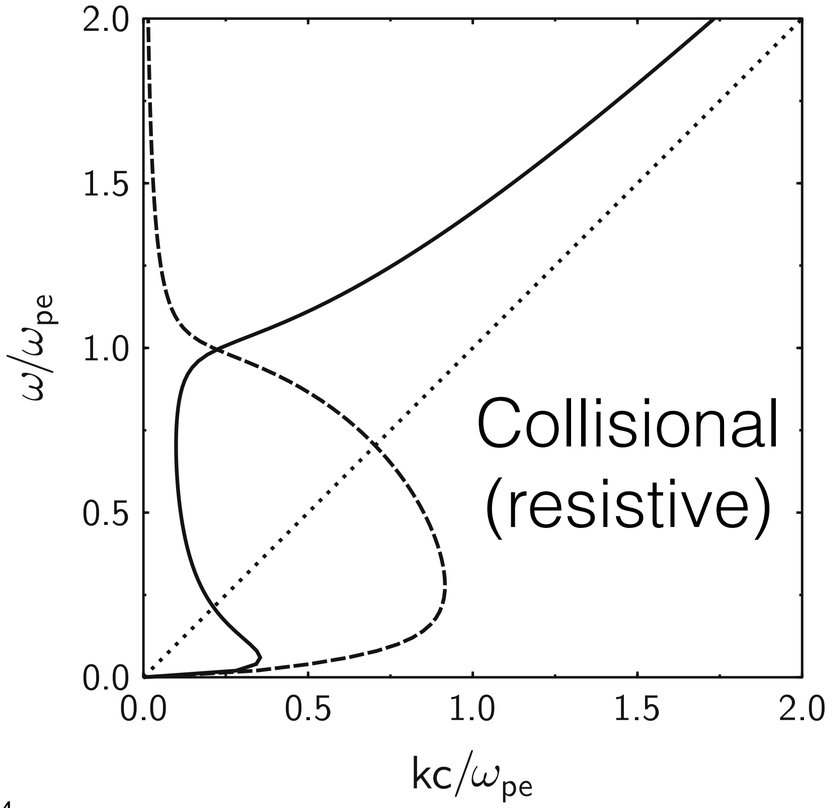
Waves in Unmagnetized Plasma



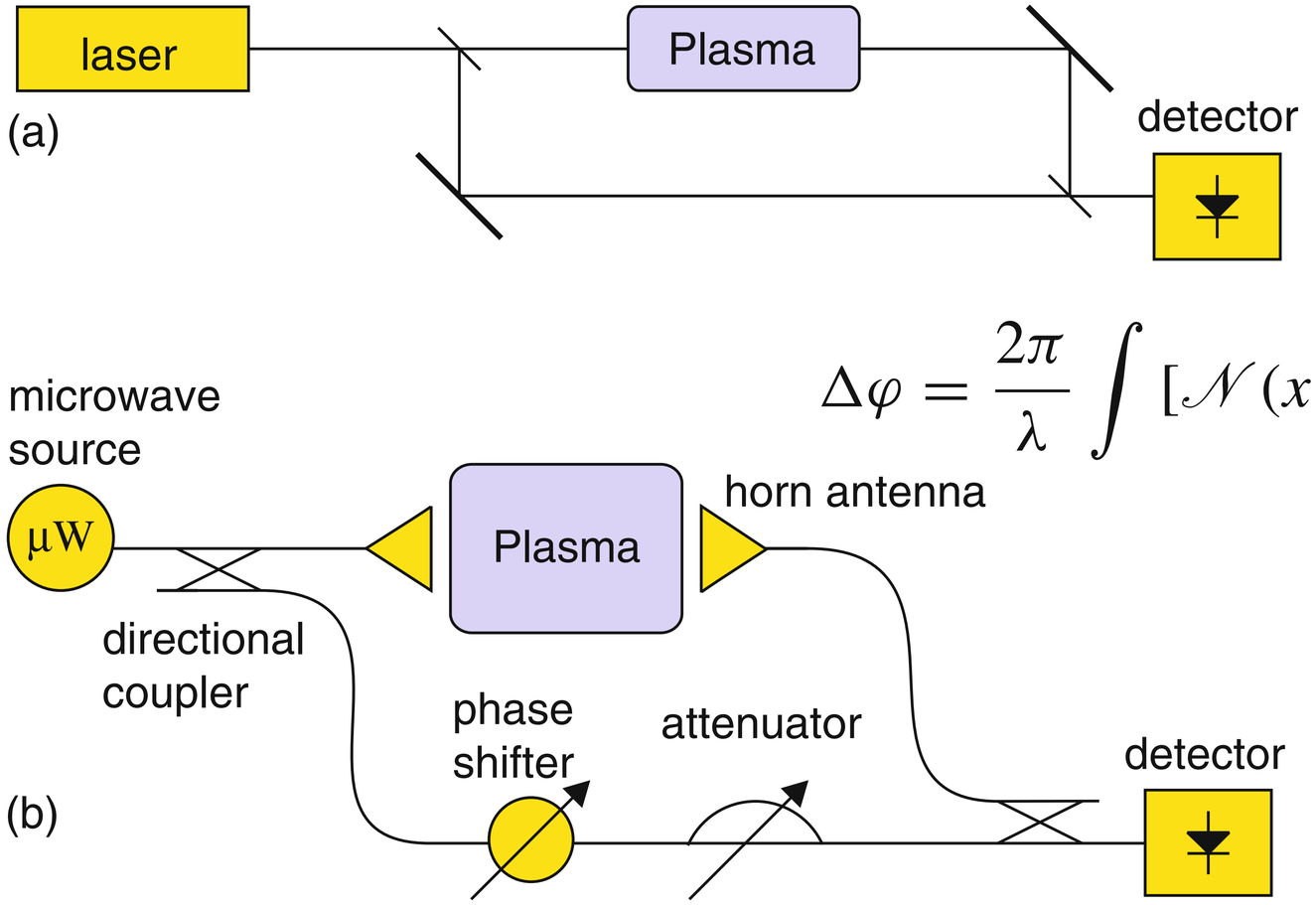
$$m\,(-\mathrm{i}\omega + \nu_\mathrm{m})\,\hat{v} = q\,\hat{E}$$

$$\hat{v} = \left[\frac{\nu_\mathrm{m}}{\omega^2 + \nu_\mathrm{m}^2} + \frac{\mathrm{i}\omega}{\omega^2 + \nu_\mathrm{m}^2}\right]\frac{q}{m}\hat{E}$$

$$k = \frac{1}{c}\left(\omega^2 - \frac{\omega_\mathrm{pe}^2}{1 + \mathrm{i}(\nu_\mathrm{m}/\omega)}\right)^{1/2}$$



Interferometry



$$\Delta\varphi = \frac{2\pi}{\lambda}\int [\mathcal{N}(x) - 1]\,\mathrm{d}x$$

Fig. 6.5 (**a**) Laser interferometer in Mach-Zehnder arrangement, (**b**) microwave interferometer. The optical arrangement uses partially-reflecting and fully-reflecting mirrors. The analog to a partially reflecting mirror is the directional coupler for microwaves

Interferometry

$$\Delta\varphi = \frac{2\pi}{\lambda}\int [\mathcal{N}(x) - 1]\,\mathrm{d}x$$

$$\mathcal{N} = \sqrt{1 - \omega_{\mathrm{pe}}^2/\omega^2} \approx 1 - \frac{1}{2}\frac{\omega_{\mathrm{pe}}^2}{\omega^2} = 1 - \frac{1}{2}\frac{n}{n_{\mathrm{co}}}$$

Table 6.1 Cut-off densities for microwave and laser interferometers

Source	Wavelength λ	Frequency f	Cut-off-density $n_{\mathrm{co}}(\mathrm{m}^{-3})$
Microwave	3 cm	10 GHz	1.2×10^{18}
	8 mm	37 GHz	1.7×10^{19}
	4 mm	75 GHz	7.0×10^{19}
HCN-laser	337 μm	890 GHz	9.8×10^{21}
CO_2 laser	10.6 μm	28 THz	9.9×10^{24}
He-Ne laser	3.39 μm	88 THz	9.7×10^{25}
	0.633 μm	474 THz	2.8×10^{27}

Interferometry

$$\Delta\varphi = \frac{2\pi}{\lambda}\int [\mathcal{N}(x) - 1]\,\mathrm{d}x$$

$$\mathcal{N} = \sqrt{1 - \omega_{\mathrm{pe}}^2/\omega^2} \approx 1 - \frac{1}{2}\frac{\omega_{\mathrm{pe}}^2}{\omega^2} = 1 - \frac{1}{2}\frac{n}{n_{\mathrm{co}}}$$

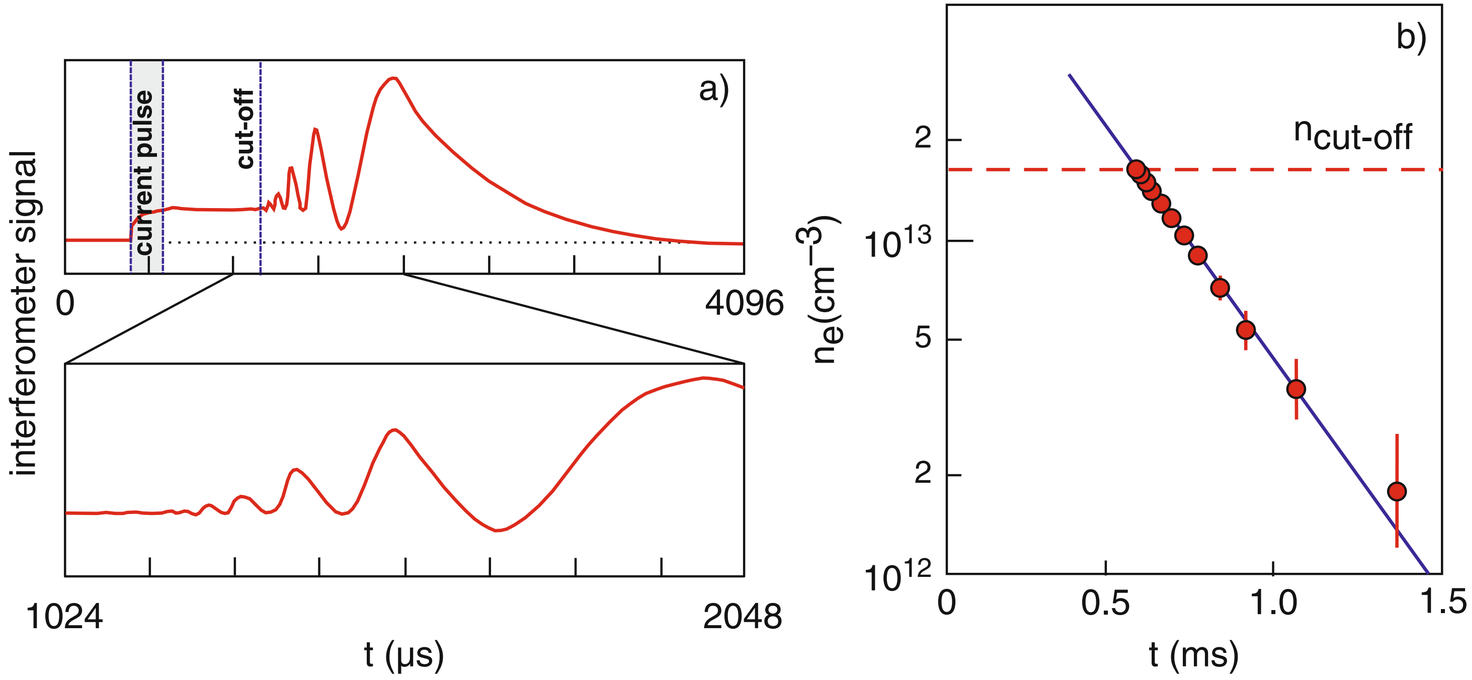


Fig. 6.6 (**a**) Interferogram in a pulsed gas discharge. (**b**) Reconstruction of the decaying electron density by counting interferometer *fringes*

Electrostatic Waves

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$$

$$-\nabla \cdot \nabla \Phi = \frac{\rho}{\epsilon_0}$$

$$k^2 \Phi = \frac{\rho}{\epsilon_0} \approx -\frac{e n_e}{\epsilon_0}$$

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial x}(nv) = 0$$

$$m\dot{v} = -q\frac{\mathrm{d}\phi}{\mathrm{d}x} - \frac{\gamma}{n}\frac{\mathrm{d}(nk_\mathrm{B}T)}{\mathrm{d}x}$$

Electron Pressure Force

$$nm\left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}\right) = nq(\mathbf{E} + \mathbf{u}\times\mathbf{B}) - \nabla p\,. \tag{5.28}$$

Electrostatic Plasma Waves

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$$

$$-\nabla \cdot \nabla \Phi = \frac{\rho}{\epsilon_0}$$

$$k^2 \Phi = \frac{\rho}{\epsilon_0} \approx -\frac{e n_e}{\epsilon_0}$$

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial x}(nv) = 0$$

$$-\mathrm{i}\omega \hat{n} + \mathrm{i}k n_0 \hat{v} = 0$$

$$m\dot{v} = -q\frac{\mathrm{d}\phi}{\mathrm{d}x} - \frac{\gamma}{n}\frac{\mathrm{d}(n k_\mathrm{B} T)}{\mathrm{d}x}$$

$$-\mathrm{i}\omega m \hat{v} = -\mathrm{i}kq\hat{\phi} - \mathrm{i}k\gamma k_\mathrm{B} T \hat{n}$$

Electron Pressure Force

Electrostatic Plasma Waves

$$\omega = \left(\omega_{\mathrm{pe}}^2 + \frac{3}{2}k^2 v_{\mathrm{Te}}^2\right)^{1/2} = \omega_{\mathrm{pe}}\left(1 + 3k^2\lambda_{\mathrm{De}}^2\right)^{1/2}$$

Electron Pressure Force

Electron Pressure Force

Electrostatic Ion Sound Waves

Ion Pressure Force

$$-\mathrm{i}\omega m_\mathrm{i}\hat{v}_\mathrm{i} = e\hat{E} - \frac{\mathrm{i}k}{n_{\mathrm{i}0}}(\gamma_\mathrm{i}k_\mathrm{B}T_\mathrm{i})\hat{n}_\mathrm{i} \qquad \hat{n}_\mathrm{i} = \frac{ek}{-\mathrm{i}\omega^2 m_\mathrm{i} + \mathrm{i}k^2\gamma_\mathrm{i}k_\mathrm{B}T_\mathrm{i}}\hat{E}$$

$$0 = -e\hat{E} - \frac{\mathrm{i}k}{n_{\mathrm{e}0}}(k_\mathrm{B}T_\mathrm{e})\hat{n}_\mathrm{e} \qquad \hat{n}_\mathrm{e} = \frac{-e}{\mathrm{i}kk_\mathrm{B}T_\mathrm{e}}\hat{E}\,,$$

Look!

No electron acceleration

Electron Pressure Force

$$k^2\Phi = \frac{\rho}{\epsilon_0} \approx \frac{e}{\epsilon_0}(n_i - n_e)$$

$$\mathrm{i}k\hat{E} = \left(\frac{n_{\mathrm{i}0}e^2}{\varepsilon_0 m_\mathrm{i}}\right)\frac{k}{-\mathrm{i}\omega^2 + \mathrm{i}k^2\gamma_\mathrm{i}k_\mathrm{B}T_\mathrm{i}/m_\mathrm{i}}\hat{E} + \left(\frac{n_{\mathrm{e}0}e^2}{\varepsilon_0 k_\mathrm{B}T_\mathrm{e}}\right)\frac{1}{\mathrm{i}k}\hat{E}$$

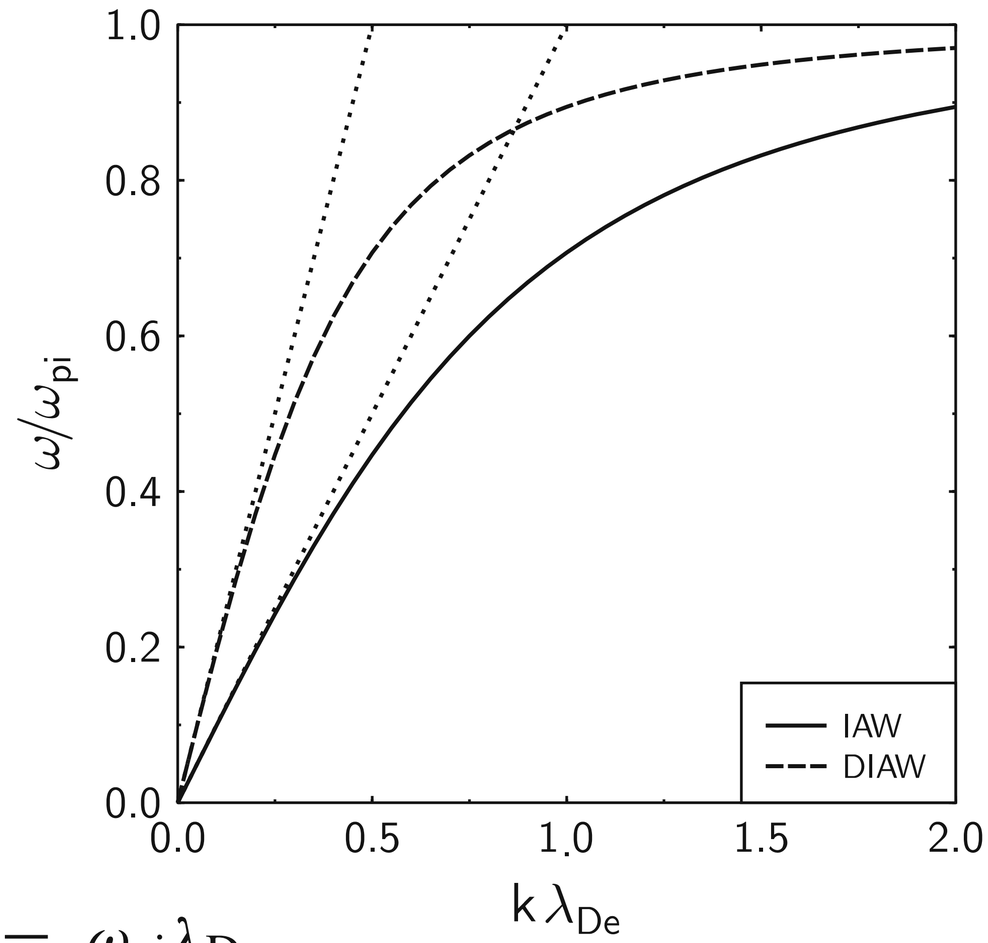
Electrostatic Ion Sound Waves

$$\varepsilon(k,\omega) = 1 - \frac{\omega_{\mathrm{pi}}^2}{\omega^2 - k^2\gamma_{\mathrm{i}}k_{\mathrm{B}}T_{\mathrm{i}}/m_{\mathrm{i}}} + \frac{1}{k^2\lambda_{\mathrm{De}}^2}$$

$$\omega^2 = k^2\left(\frac{\gamma_{\mathrm{i}}k_{\mathrm{B}}T_{\mathrm{i}}}{m_{\mathrm{i}}} + \frac{\omega_{\mathrm{pi}}^2\lambda_{\mathrm{De}}^2}{1+k^2\lambda_{\mathrm{De}}^2}\right)$$

$$\omega \approx \frac{k\,C_{\mathrm{s}}}{\sqrt{1+k^2\lambda_{\mathrm{De}}^2}}$$

$$T_{\mathrm{e}} \gg T_{\mathrm{i}} \qquad C_{\mathrm{s}} = \omega_{\mathrm{pi}}\lambda_{\mathrm{De}}$$



Important Wave Concepts

- Linear vs. nonlinear
- Dispersion
- Phase and group velocity
- “Polarization” and wave structure
- Energy & intensity (Poynting’s Theorem)
- Inhomogeneity

$$W = \frac{1}{2}\int_V dV(\mathbf{H}\cdot\mathbf{B} + \mathbf{E}\cdot\mathbf{D})$$

$$\frac{\partial W}{\partial t} + \int_S \mathbf{N}\cdot d\mathbf{S} = -\int_V dV\,\mathbf{J}\cdot\mathbf{E},$$

Quiz 1 Wednesday

Next Lecture: Video

- Chapter 6: “Plasma Waves”
  - Waves in magnetized plasma