

Handout:

# Plasma Physics 1

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# The Motion of a Charged Particle in a Strong Magnetic Field

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## Abstract

The reduced description in terms of drifts and adiabatic invariants of the motion of a charged particle in a strong magnetic field is derived. The demonstration employs systematically two time scales and an iteration scheme for each quasiperiodicity. This leads to a particularly expeditious derivation, as well as the details of the rapid oscillations at each stage. Moreover the motivation of each part is clear, as is the relation to simple problems in dynamics. The small parameters, the existence of which underlines the method, are displayed explicitly.

# Constructing Gyro/Drift Formulation

IN THIS HANDOUT, WE FOLLOW BERNSTEIN'S DERIVATION OF THE LEADING ORDER gyromotion OF A PARTICLE IN A STRONG MAGNETIC FIELD. THIS DEMONSTRATES THAT THE MAGNETIC MOMENT,  $\mu$ , IS AN APPROXIMATE CONSTANT OF THE MOTION PROVIDED THAT THE MAGNETIC FIELD CHANGES IN TIME AND SPACE MORE SLOWLY THAN  $\Sigma'$  AND  $\rho$ .

BERNSTEIN'S CONTRIBUTION IS TO INTRODUCE THE NOTATION

$$\vec{r} = \bar{\vec{r}}(\epsilon) + \hat{\vec{r}}(\theta, t)$$

WHERE  $\theta$  IS THE FAST GYROMOTION AND  $t$  REPRESENTS THE MORE SLOWLY VARYING PARTS OF MOTION.

# Periodic Gyroangle

To do this, we must require  $\bar{\phi}(\theta)$  to be periodic, and we'll define its time variation to be

$$\frac{d\theta}{dt} = \gamma(t) \Rightarrow \theta = \int_0^t dt' \gamma(t') + \theta_0$$

The motion is actually quite complicated, but we're going to just look at the most significant approximation to the exact motion.

# Spatial expansion...

IN OUR CHOSEN NOTATION,

$$\frac{d\bar{\pi}}{dt} = \frac{d\bar{R}}{dt} + \gamma \frac{d\bar{\phi}}{2\theta} + \frac{d\bar{p}}{2t}$$

                                        
THIS IS      THIS  
FAST          IS SLOW

$$\frac{d^2\bar{\pi}}{dt^2} = \frac{d^2\bar{R}}{dt^2} + \gamma^2 \frac{d^2\bar{\phi}}{2\theta^2} + 2\gamma \frac{d\bar{\phi}}{2\theta dt} + \frac{dv}{dt} \frac{2\bar{\phi}}{2\theta} + \frac{d^2\bar{p}}{2t^2}$$

NEXT, WE CAN EXPAND ABOUT THE GYROCENTER

$$\bar{\alpha}(\bar{\tau}, t) \approx \bar{\alpha}(\bar{R}, t) + (\bar{p} \cdot \bar{\nabla}) \bar{\alpha} + \frac{1}{2} \bar{p} \bar{g} : \bar{\nabla} \bar{\nabla} \bar{\alpha} + \dots$$

$$\bar{\Sigma}(\bar{\tau}, t) \approx \bar{\Sigma}(\bar{R}, t) + (\bar{p} \cdot \bar{\nabla}) \bar{\Sigma} + \frac{1}{2} \bar{g} \bar{g} : \bar{\nabla} \bar{\nabla} \bar{\Sigma} + \dots$$

BUT WE WILL (FOR SIMPLICITY) ASSUME  $\bar{\alpha}$  DOES NOT VARY IN SPACE AND NEGLECT THE 2ND ORDER TERMS IN  $\bar{\Sigma}(\bar{\tau})$ .

# Solution Outline

WE NEED TO DO 3 STEPS:

STEP #1      GYROAVERAGF TO GET DRIFT motion

STEP #2      SEE HOW  $\theta(t)$  EVOLVES

STEP #3      PROVE THAT  $\mu = \text{constant}$

# Step 1

STEP #1 GYRO AVERAGE

WE OBTAIN SIMPLY

$$\left\langle \frac{d^2 \bar{r}}{dt^2} = \bar{a} - \bar{r} \times \frac{d\bar{r}}{dt} \right\rangle_0$$

$$\Rightarrow \frac{d^2 \bar{r}}{dt^2} = \bar{a} - \bar{r} \times \bar{V} + \left\langle \gamma \frac{2\bar{\theta}}{2\theta} \times (\bar{\theta} \cdot \bar{\sigma}) \bar{r} \right\rangle_0$$

BUT, THIS IS EXACTLY THE SAME RESULT THAT WE  
ALREADY DISCUSSED IN CLASS.

# Step 2

STEP #2 SEE HOW  $\theta(t)$  EVOLVES

TAKE THE FULL EQUATION OF MOTION AND  
SUBTRACT OFF THE gyration-center motion  
FOUND IN STEP #1. THIS gives

$$\left( \gamma^2 \frac{\partial^2 \bar{\phi}}{\partial \theta^2} + \bar{r} \times \gamma \frac{\partial \bar{\phi}}{\partial \theta} \right) = -2\sqrt{\frac{\partial^2 \bar{\phi}}{\partial \theta^2 t}} - \frac{dV}{dt} \frac{\partial \bar{\phi}}{\partial \theta} - \cancel{\frac{\partial^2 \bar{\phi}}{\partial t^2}} + (\bar{p} \cdot \bar{v})(\bar{a} + \bar{v} \times \bar{r}) - \cancel{(\bar{g} \cdot \bar{r}) \bar{r} \times \bar{v}} - (\bar{p} \cdot \bar{v}) \bar{r} \times \sqrt{\frac{\partial \bar{\phi}}{\partial \theta}} - \bar{r} \times \frac{\partial \bar{\phi}}{\partial t} - \langle \sqrt{\frac{\partial \bar{\phi}}{\partial \theta}} \times (\bar{p} \cdot \bar{v}) \bar{r} \rangle_\theta + \dots$$



small

\* BY FAR, THE LARGEST TERMS ARE ON THE LHS.  
NEGLECTING THE RHS, THE APPROXIMATE SOLUTION TO  
THE FAST gyromotion IS

$$\text{AND } \gamma = \sqrt{2k}$$

$$\bar{\phi} = \frac{w}{\gamma} [\hat{e}_1 \sin \theta + \hat{e}_2 \cos \theta] \text{ with } \theta(t) = \int_0^t dt' \omega(t') + \phi$$

## Step 3

STEP #3 SHOW THAT  $\mu \approx \text{constant}$

TAKE THE EQUATION in STEP #2 AND FORM THE DOT PRODUCT WITH  $\frac{\partial \vec{r}}{\partial \theta}$ . THEN, AFTER REARRANGING TERMS

$$\begin{aligned} \frac{\partial}{\partial \theta} \left( \frac{1}{2} \gamma^2 \left( \frac{\partial \vec{r}}{\partial \theta} \right)^2 \right) + \frac{\partial}{\partial t} \left( \gamma \left( \frac{\partial \vec{r}}{\partial \theta} \right)^2 \right) &= -\frac{\partial \vec{r}}{\partial \theta} \cdot \left( \vec{r} \times \frac{\partial \vec{r}}{\partial t} \right) \\ &\quad + \frac{\partial \vec{r}}{\partial \theta} \cdot (\vec{r} \cdot \vec{\nabla}) (\vec{a} + \vec{v} \times \vec{r}) \\ &\quad - \frac{\partial \vec{r}}{\partial \theta} \cdot \left\langle \gamma \frac{\partial \vec{r}}{\partial \theta} \times (\vec{r} \cdot \vec{\nabla}) \vec{r} \right\rangle_0 \\ &\quad + \dots \end{aligned}$$

## Step 3

NOW, LET'S GYROAVERAGE AGAIN...

WE GET

$$\frac{d}{dt} \left( \langle \sqrt{\left( \frac{d\bar{\theta}}{d\theta} \right)^2} \rangle_{\theta} \right) = \left\langle \frac{2\bar{\theta}}{2\theta} \cdot (\bar{p} \cdot \bar{\sigma}) (\bar{a} + \bar{v} \times \bar{r}) + \frac{d\bar{\theta}}{d\theta} \times \frac{2\bar{\theta}}{2\theta} \cdot \bar{n} \right\rangle_{\theta}$$

NOW, IN THE SPIRIT OF OUR PERTURBATION APPROACH  
WE USE  $\bar{\rho}(\theta, t)$  FOUND IN STEP #2 AND SUBSTITUTE  
THAT INTO THE EQUATION ABOVE.

THE GYROAVERAGES ARE THEN...

$$\langle \left( \frac{d\theta}{d\theta} \right)^2 \rangle_{\theta} = \rho^2$$

$$\langle \frac{2\bar{\theta}}{2\theta} \cdot \bar{\rho} \rangle_{\theta} = \frac{\rho^2}{2} (\hat{l}_1 \hat{l}_2 - \hat{l}_2 \hat{l}_1)$$

## Step 3

And

$$\left\langle \frac{\partial \bar{g}}{\partial \theta} \times \frac{\partial \bar{g}}{\partial t} \cdot \bar{n} \right\rangle_0 = \left\langle \bar{n} \cdot \hat{g} \times \frac{\partial \bar{g}}{\partial \theta} \cdot \frac{\partial \bar{g}}{\partial t} \right\rangle_0$$

But  $\frac{\partial \bar{g}}{\partial \theta} = -\hat{f} \times \bar{g}$  so

$$= -\bar{n} \left\langle \hat{g} \times (\hat{f} \times \bar{g}) \cdot \frac{\partial \bar{g}}{\partial t} \right\rangle_0$$

$$= \bar{n} \left\langle \bar{g} \cdot \frac{\partial \bar{g}}{\partial t} \right\rangle_0$$

$$= \bar{n} \frac{2}{\partial t} \left\langle \frac{1}{2} g^2 \right\rangle_0 = \frac{\bar{n}}{2} \frac{2}{\partial t} (g^2)$$

# Step 3

PUTTING IT ALL TOGETHER

$$\text{d} \left( r \rho^2 \right) / \text{d}t = \frac{\rho^2}{2} \underbrace{(\vec{\ell}_1 \cdot \vec{\ell}_2 - \vec{\ell}_2 \cdot \vec{\ell}_1)}_{-\vec{f} \times \vec{\nabla}} : \vec{\nabla} (\bar{a} + \bar{v} \times \bar{r}) + \frac{r}{2} \frac{\text{d} \rho^2}{\text{d}t}$$

so

$$\underbrace{\frac{\text{d}}{\text{d}t} (r \rho^2) - \frac{r}{2} \frac{\text{d} \rho^2}{\text{d}t}}_{(Y=r)} = -\frac{\rho^2}{2} (\vec{f} \times \vec{\nabla}) \cdot (\bar{a} + \bar{v} \times \bar{r})$$

$$(Y=r) \quad \frac{\text{d}}{\text{d}t} \left( \frac{1}{2} r \rho^2 \right) + \frac{1}{2} \rho^2 \frac{\text{d}r}{\text{d}t}$$

so

$$\frac{\text{d}}{\text{d}t} \left( \frac{1}{2} r \rho^2 \right) = -\frac{\rho^2}{2} \left[ (\vec{f} \times \vec{\nabla}) \cdot (\bar{a} + \bar{v} \times \bar{r}) + \frac{\text{d}r}{\text{d}t} \right]$$

$$\text{BUT} \quad (\vec{f} \times \vec{\nabla}) \cdot (\bar{a} + \bar{v} \times \bar{r}) = \vec{f} \cdot \vec{\nabla} \times (\bar{a} + \bar{v} \times \bar{r})$$

∴

$$\frac{\text{d}}{\text{d}t} \left( \frac{1}{2} r \rho^2 \right) = -\frac{\rho^2}{2} \left[ \vec{f} \cdot \vec{\nabla} \times (\bar{a} + \bar{v} \times \bar{r}) + \frac{\text{d}r}{\text{d}t} \right]$$

# Examples and Consequences

Now, For us, IF  $\bar{a} = \frac{q}{m} \bar{E}$  And  $\bar{\tau} = \frac{q\bar{B}}{mc}$ ,  
WE CAN SAY

$$\frac{1}{2} \bar{r} p^2 = \frac{\frac{1}{2} m \omega^2}{B} \cdot \frac{1}{\frac{q}{mc}} = \mu \cdot \frac{1}{(\frac{q}{mc})}$$

WHERE  $\mu$  = MAGNETIC MOMENT.

THE R.H.S. OF THE EQ. ON THE BOTTOM OF P. 5 IS  
 $E_F \rho_0$ . WE CAN SHOW THIS IN TWO WAYS.

$$c \left( \frac{m}{q} \right)^2 \frac{dH}{dt} = - \frac{p^2}{2} \cdot \hat{r} \cdot \left[ \vec{\nabla} \times [E + \frac{\vec{v}}{c} \times \vec{B}] + \frac{1}{c} \frac{d\vec{B}}{dt} \right]$$

# Examples and Consequences

FIRST, WE CAN TRANSFORM INTO THE FRAME OF REFERENCE AS SEEN BY THE PARTICLE moving with velocity,  $\bar{V}$ . IN THIS CASE

$$\bar{\nabla} \times [E + \frac{V}{c} \times \bar{B}] + \frac{1}{c} \frac{d\bar{B}}{dt} \Rightarrow \bar{\nabla} \times E' + \frac{1}{c} \frac{d\bar{B}'}{dt} = 0$$

SECOND, WE COULD WRITE

$$\frac{d\bar{B}}{dt} = \frac{2\bar{B}}{ct} + (\bar{V} \cdot \bar{\nabla}) \bar{B}$$

AND USE

$$\bar{\nabla} \times (\frac{\bar{V}}{c} \times \bar{B}) = -\frac{\bar{B}}{c}(\bar{\nabla} \cdot \bar{V}) + (\bar{B} \cdot \bar{\nabla}) \bar{V} - (\bar{V} \cdot \bar{\nabla}) \bar{B}$$

AND RECOGNIZE THAT ...  $\bar{\nabla} \cdot \bar{V} - \bar{B} \cdot (\bar{B} \cdot \bar{\nabla}) \bar{V} = 0$

SO

$$\boxed{\frac{d\bar{B}}{dt} = 0}$$

Q.E.D.