More Practice Problems

AP 6101
Practice for Quiz #1
From Fitzpatrick, and Gurnett and Bhattacharjee

Dielectric of a Magnetized Plasma

2. A quasi-neutral slab of cold (i.e., $\lambda_D \to 0$) plasma whose bounding surfaces are normal to the x-axis consists of electrons of mass m_e, charge $-e$, and mean number density n_e, as well as ions of mass m_i, charge e, and mean number density n_e. The slab is fully magnetized by a uniform y-directed magnetic field of magnitude B. The slab is then subject to an externally generated, uniform, x-directed electric field that is gradually ramped up to a final magnitude E_0. Show that, as a consequence of ion polarization drift, the final magnitude of the electric field inside the plasma is

$$E_1 \simeq \frac{E_0}{\epsilon},$$

where

$$\epsilon = 1 + \frac{c^2}{V_A^2},$$

and $V_A = B/\sqrt{\mu_0\, n_e\, m_i}$ is the so-called Alfvén velocity.

Thermal Equilibrium

5. A uniform isothermal quasi-neutral plasma with singly-charged ions is placed in a relatively weak gravitational field of acceleration $\mathbf{g} = -g\,\mathbf{e}_z$. Assuming, first, that both species are distributed according to the Maxwell-Boltzmann statistics; second, that the perturbed electrostatic potential is a function of z only; and, third, that the electric field is zero at $z = 0$ (and well behaved as $z \to \infty$), demonstrate that the electric field in the region $z > 0$ takes the form $\mathbf{E} = E_z\,\mathbf{e}_z$, where

$$E_z(z) = E_0\left[1 - \exp\left(\frac{\sqrt{2}\,z}{\lambda_D}\right)\right],$$

and

$$E_0 = \frac{m_i\,g}{2\,e}.$$

Here, λ_D is the Debye length, e the magnitude of the electron charge, and m_i the ion mass.

Adiabatic Invariants

6. A particle of charge e, mass m, and energy $\mathcal{E}$, is trapped in a one-dimensional magnetic well of the form

$$B(x,t) = B_0\,(1 + k^2\,x^2),$$

where B_0 is constant, and $k(t)$ is a very slowly increasing function of time. Suppose that the particle's mirror points lie at $x = \pm x_m(t)$, and that its bounce time is $\tau_b(t)$. Demonstrate that, as a consequence of the conservation of the first and second adiabatic invariants,

$$x_m(t) = x_m(0)\left[\frac{k(0)}{k(t)}\right]^{1/2},$$

$$\tau_b(t) = \tau_b(0)\left[\frac{k(0)}{k(t)}\right],$$

$$\mathcal{E}(t) = \mathcal{E}_{0\perp} + \left[\frac{k(t)}{k(0)}\right]\mathcal{E}_{0\parallel}.$$

Here, $\mathcal{E}_{0\perp}$ is the perpendicular energy [i.e., $(1/2)\,m\,v_\perp^2$], and $\mathcal{E}_{0\parallel}$ is the parallel energy [i.e., $(1/2)\,m\,v_\parallel^2$], both evaluated at $x = 0$ and $t = 0$. Assume that the particle's gyroradius is relatively small, and that the electric field-strength is negligible.

Drift Velocity w Collisions

8. Consider a spatially uniform, unmagnetized plasma in which both species have zero mean flow velocity. Let n_e and T_e be the electron number density and temperature, respectively. Let $\mathbf{E}$ be the ambient electric field. The electron distribution function f_e satisfies the simplified kinetic equation

$$-\frac{e}{m_e}\,\mathbf{E}\cdot\nabla_v f_e = C_e.$$

We can crudely approximate the electron collision operator as

$$C_e = -\nu_e\,(f_e - f_0)$$

where ν_e is the effective electron-ion collision frequency, and

$$f_0 = \frac{n_e}{\pi^{3/2}\,v_{te}^3}\exp\left(-\frac{v^2}{v_{te}^2}\right).$$

Here, $v_{te} = \sqrt{2\,T_e/m_e}$. Suppose that $E \ll m_e\,\nu_e\,v_{te}/e$. Demonstrate that it is a good approximation to write

$$f_e = f_0 + \frac{e}{m_e\,\nu_e}\,\mathbf{E}\cdot\nabla_v f_0.$$

Hence, show that

$$\mathbf{j} = \sigma\,\mathbf{E},$$

where

$$\sigma = \frac{e^2\,n_e}{m_e\,\nu_e}.$$

Static MHD Equilibrium

7.6. For a force-balanced MHD equilibrium in a cylindrical geometry with $\mathbf{B} = [0, B_\phi(\rho), B_z(\rho)]$ the radial component of the pressure balance condition $\mathbf{J} \times \mathbf{B} = \nabla P$ can be written

$$\frac{\partial}{\partial \rho}\left(P + \frac{B_\phi^2}{2\mu_0} + \frac{B_z^2}{2\mu_0}\right) = [(\mathbf{B} \cdot \nabla)\mathbf{B}]_\rho.$$

Show that $[(\mathbf{B} \cdot \nabla)\mathbf{B}]_\rho = -B_\phi^2/\rho$.
Hint: Use the identity $\nabla(\mathbf{F} \cdot \mathbf{G}) = (\mathbf{F} \cdot \nabla)\mathbf{G} + (\mathbf{G} \cdot \nabla)\mathbf{F} + \mathbf{F} \times (\nabla \times \mathbf{G}) + \mathbf{G} \times (\nabla \times \mathbf{G})$.

Alfvén Waves with Collisions/Viscosity

1. We can add viscous effects to the MHD momentum equation by including a term $\mu\,\nabla^2\mathbf{V}$, where μ is the dynamic viscosity, so that

$$\rho\,\frac{d\mathbf{V}}{dt} = \mathbf{j}\times\mathbf{b} - \nabla p + \mu\,\nabla^2\mathbf{V}.$$

Likewise, we can add finite conductivity effects to the Ohm's law by including the term $(1/\mu_0\,\sigma)\,\nabla^2\mathbf{B}$, to give

$$\frac{\partial\mathbf{B}}{\partial t} = \nabla\times(\mathbf{V}\times\mathbf{B}) + \frac{1}{\mu_0\,\sigma}\,\nabla^2\mathbf{B},$$

Show that the modified dispersion relation for Alfvén waves can be obtained from the standard one by multiplying both ω^2 and V_S^2 by a factor

$$[1 + \mathrm{i}\,k^2/(\mu_0\,\sigma\,\omega)],$$

and ω^2 by an additional factor

$$[1 + \mathrm{i}\,\mu\,k^2/(\rho_0\,\omega)].$$

If the finite conductivity and viscous corrections are small (i.e., $\sigma\to\infty$ and $\mu\to 0$), show that, for parallel ($\theta = 0$) propagation, the dispersion relation for the shear-Alfvén wave reduces to

$$k \simeq \frac{\omega}{V_A} + \mathrm{i}\,\frac{\omega^2}{2\,V_A^3}\left(\frac{1}{\mu_0\,\sigma} + \frac{\mu}{\rho_0}\right).$$