Homework #3 (Ch. 3)

APPH E6101x
Plasma Physics 1

- Fitzpatrick: Exercise #1 in Chapter 2
- Piel: All seven problems in Ch. 3 (answers in back of text)

1. Given that $\boldsymbol{\rho} = \rho\,(-\cos\gamma\,\mathbf{e}_1 + \sin\gamma\,\mathbf{e}_2)$, and $\mathbf{u} = \Omega\,\boldsymbol{\rho}\times\mathbf{b}$, where $\rho = u_\perp/\Omega$, and $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{b} \equiv \mathbf{B}/B$ are a right-handed set of mutually perpendicular unit basis vectors, demonstrate that:

(a)

$$\langle\boldsymbol{\rho}\,\boldsymbol{\rho}\rangle = \frac{u_\perp^2}{2\,\Omega^2}\,(\mathbf{I}-\mathbf{b}\,\mathbf{b})\,.$$

(b)

$$e\,\langle\mathbf{u}\times(\boldsymbol{\rho}\cdot\nabla)\,\mathbf{B}\rangle = -\mu\,\nabla B.$$

(c)

$$e\,\langle\mathbf{u}\cdot(\boldsymbol{\rho}\cdot\nabla)\,\mathbf{E}\rangle = \mu\,\frac{\partial B}{\partial t}\,.$$

(d)

$$e\,\langle\mathbf{u}\cdot(\boldsymbol{\rho}\cdot\nabla)\,\mathbf{A}\rangle = -\mu\,B.$$

Here, $\mu = m\,u_\perp^2/(2\,B)$, and $\langle\cdots\rangle \equiv \oint(\cdots)\,d\gamma/2\pi$.

3.1 Consider a cylindrical straight wire of radius a with a homogeneous distribution of current density inside. Use Ampere's law to derive the azimuthal magnetic field $H_\varphi(r)$ for $r < a$ and $r \geq a$.

3.2 Consider now a cylindrical discharge tube, in which the plasma density profile and the associated current distribution is parabolic:

$$j(r) = j_0 \left(1 - \frac{r^2}{a^2}\right) .$$

What is the magnetic field distribution $H_\varphi(r)$ for $r < a$ in this case?

3.3 (a) What is the electron cyclotron frequency resulting from the Earth magnetic field at the author's location? (c.f. Sect. 3.1.3).
(b) What is the gyroradius of an electron with 10 eV kinetic energy in this field?

3.4 (a) The Earth magnetic field is assumed to be created by a dipole of magnetic moment $\mathbf{M} = -M\mathbf{e}_z$, which reads in cartesian coordinates:

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \frac{3\mathbf{r}(\mathbf{r}\cdot\mathbf{M}) - r^2\mathbf{M}}{r^5} .$$

Find the corresponding components B_r and B_θ in spherical coordinates (r, θ).
(b) In the equatorial ionosphere the horizontal component of the Earth magnetic field is approximately 30 µT. Calculate the dipole moment at the Earth center that would generate such a magnetic field.

3.5 (a) Calculate the gradient of the Earth magnetic field at the magnetic equator at an altitude of 500 km and the radius of curvature of a magnetic field line, $R_c = |B_\theta/(\mathrm{d}B_\theta/\mathrm{d}r)|$.
(b) What is the speed of the gradient drift and curvature drift for electrons, which have 3 eV kinetic energy in parallel and perpendicular motion?

3.6 Determine the trajectory $[x(t), y(t)]$ of an electron in crossed fields $\mathbf{B} = (0, 0, B_z)$ and $\mathbf{E} = (E_x, 0, 0)$, when the electron is initially at rest, $\mathbf{v}(t = 0) = 0$.

3.7 The vector of the magnetic field is tangent to the field line. Therefore, the differential equation for a magnetic field line is

$$\frac{\mathrm{d}\mathbf{s}}{\mathrm{d}t} = \mathbf{e}_B .$$

Here, $\mathbf{s} = (x, y, z)$ is a point on the field line and t a parameter, which makes tick-marks along the trajectory. Write the defining equation for the field line in components, eliminate t, and show that the equation for a magnetic field line in the x–z plane reads

$$\frac{\mathrm{d}z}{\mathrm{d}x} = \frac{B_z}{B_x} .$$

Solve this differential equation for the dipole field given in problem 3.4 by separating the variables and show that the field line is given as

$$z(x) = \sqrt{x_0^{2/3} x^{4/3} - x^2} ,$$

where x_0 marks the intersection of the field line with the x-axis.