Homework #1 (Ch. 2)

APPH E6101x
Plasma Physics 1
2.1 Prove that the electron Debye length can be written as

\[ \lambda_{De} = 69 \text{ m} \left[ \frac{T (\text{K})}{n_e (\text{m}^{-3})} \right]^{1/2} \]

2.2 Calculate the electron and ion Debye length
(a) for the ionospheric plasma \( (T_e = T_i = 3000 \text{ K}, n = 10^{12} \text{ m}^{-3}) \).
(b) for a neon gas discharge \( (T_e = 3 \text{ eV}, T_i = 300 \text{ K}, n = 10^{16} \text{ m}^{-3}) \).

2.3 Consider an infinitely large homogeneous plasma with \( n_e = n_i = 10^{16} \text{ m}^{-3} \).
From this plasma, all electrons are removed from a slab of thickness \( d = 0.01 \text{ m} \) extending from \( x = -d \) to \( x = 0 \) and redeposited in the neighboring slab from \( x = 0 \) to \( x = d \). (a) Calculate the electric potential in this double slab using Poisson’s equation. What are the boundary conditions at \( x = \pm d \)? (b) Draw a sketch of space charge, electric field and potential for this situation. What is the potential difference between \( x = -d \) and \( x = d \)?

2.4 Show that the equation for the shielding contribution (2.24) results from (2.21) and (2.23).

2.5 Derive the relationship between the coupling parameter for ion-ion interaction \( \Gamma \) Eqs. (2.15) and \( N_D \) (2.33) under the assumption that \( T_e = T_i \).

2.6 Show that the second Lagrange multiplier in Eq. (2.6) is \( \lambda = (k_B T)^{-1} \).
Hint: Start from

\[ \frac{1}{T} = \frac{\partial S}{\partial \lambda} \frac{\partial \lambda}{\partial U} \]

and use \( \sum n_i = 1 \).
2.1

\[ \lambda_{De} = \sqrt{\frac{e_0 k_B T_e}{n_e e^2}} = \sqrt{\frac{e_0 k_B T_e}{n_e}} = 69.0 \text{ m} \sqrt{\frac{T_e(K)}{n_e(\text{m}^{-3})}} \]

2.2 (a) \(\lambda_{De} = \lambda_{Di} = 69 \sqrt{3000} \times 10^{-12} \text{ m} = 3.8 \times 10^{-3} \text{ m}.\)

(b) Note that 3 eV \(\sim 3 \times 11600 \text{ K}. \lambda_{De} = 1.3 \times 10^{-4} \text{ m}, \lambda_{Di} = 1.2 \times 10^{-5} \text{ m}.\)

2.3 Poisson’s equation states that the curvature of the potential is proportional to the (negative) space charge. \(\Phi'' = -e n_i / e_0.\) (a) The electric field increases linearly in the positive space charge region from \(E(-d) = 0\) to \(E(0) = E_{\max}\) and decreases in the negative space charge region to \(E(d) = 0.\) Hence, there is no electric field at the edges of the quasineutral plasma. The potential decreases in the positive space charge region as \(\Phi(x) = -\frac{1}{2} n_i e (x + d)^2 / e_0\) and reaches \(\Phi(0) = -9.0 \times 10^3 \text{ V}.\) In the negative space charge region, \(\Phi(x)\) has the opposite curvature and reaches a final value \(\Phi(d) = -18.0 \times 10^3 \text{ V}.\)

![Diagram](image)

2.4 Calculate the derivatives

\[ \Phi(r) = \frac{a}{r} f(r), \quad \Phi' = -\frac{a}{r^2} f(r) + \frac{a}{r} f', \quad \Phi'' = \frac{2a}{r^3} f - \frac{2a}{r^2} f' + \frac{a}{r} f'' \]

Inserting into (2.21) gives

\[ \frac{a}{r} \left[ f'' - \frac{1}{\lambda_S^2} f \right] = 0. \]

2.5 Inserting the Wigner-Seitz radius \(a_{WS} = [3/(4\pi n_i)]^{1/3}\) into (2.15) gives

\[ I_1 = \frac{(4\pi/3)^{1/3}}{4\pi} \frac{e^2 n_i^{1/3}}{e_0 k_B T_i} = \frac{1}{3} \left( \frac{4\pi}{3} n_i^{1/3} \right)^{-2/3} = \frac{1}{3} N^{-2/3} \]

2.6 Starting from the definitions \(S = -k_B \sum_i n_i \ln n_i\) and \(U = \sum_i n_i W_i\) we use the thermodynamic definition of temperature \(1/T = \partial S / \partial U.\)

\[ \frac{1}{T} = \frac{\partial S}{\partial \lambda} \frac{\partial \lambda}{\partial U} = -k_B \sum_i \left[ \frac{\partial n_i}{\partial \lambda} (\lambda W_i - \ln Z) + \frac{\partial n_i}{\partial \lambda} \right] \]

\[ = -k_B \lambda \sum_i \frac{\partial n_i}{\partial \lambda} W_i + k_B (\ln Z - 1) \sum_i \frac{\partial n_i}{\partial \lambda} W_i \]

Using \(\sum_i n_i = 1\), we obtain the result \(\lambda = (k_B T)^{-1}.\)