Time-dependent drift Hamiltonian

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The lowest-order drift equations are given in a canonical magnetic coordinate form for time-dependent magnetic and electric fields. The advantages of the canonical Hamiltonian form are also discussed.

I. INTRODUCTION

The evaluation of transport coefficients for a low collisionality, asymmetric plasma requires complex particle trajectory integrations. In the simplest class of such calculations, the particles move in given magnetic and electric fields. Even in this case, it is not practical to solve for the particle motion for the required length of time. A charged particle moves rapidly on a nearly circular orbit about a magnetic field line and only slowly drifts from one field line to another. The disparate time scales make exact trajectory integrations difficult, but suggest the use of an asymptotic analysis, the guiding center theory.

The original work on the evaluation of guiding center motion was done by Alfvén[1] in 1940. Alfvén's expression for the particle drift velocity is essentially

$$\mathbf{v}_d = c_{\mathbf{B}}/eB^2 \times (\mu \nabla B + m v_\parallel^2 \mathbf{b}\cdot\nabla\mathbf{b} - e\mathbf{E}). \tag{1}$$

The magnetic and electric fields are $\mathbf{B}$ and $\mathbf{E}$ with $B = |\mathbf{B}|$ and $\mathbf{b} = \mathbf{B}/B$. The particle velocity is resolved into the components parallel to the magnetic field, $\mathbf{v}_\parallel$, and perpendicular, $\mathbf{v}_\perp$. The particle mass is m, the charge is e, and c is the speed of light. The most subtle concept in Alfvén's expression is the adiabatically conserved, magnetic moment μ,

$$\mu \simeq \tfrac{1}{2}(m v_\perp^2/B). \tag{2}$$

Kruskal[2] later showed that the magnetic moment is conserved to all orders in $\rho_\parallel/L$ with $\rho_\parallel = mcv_\parallel/eB$, the parallel gyroradius, and L the system size. Landau and Lifshitz have argued[3] that adiabatic invariants are conserved to exponential accuracy in a small parameter like $\rho_\parallel/L$.

Alfvén's expression became known as "the drift velocity" to plasma physicists and has been the basis for much of the understanding on that subject. Nonetheless, Alfvén's expression has certain fundamental problems since it cannot be put into Hamiltonian form for nonzero parallel current. An example of such a problem is that a toroidal canonical momentum is not conserved by Alfvén's expression in toroidal symmetry for nonzero parallel currents.[4] In Sec. II a heuristic discussion of Hamiltonian guiding center theory is given.

To have a simple and intuitive set of equations for the guiding center trajectories, the coordinate system must be chosen carefully. Since the plasma structure is generally dominated by the magnetic field, a coordinate system determined by this field provides the simplest description of the particle trajectories. At least as important a reason for choosing magnetic coordinates is that these coordinates can be selected so that they are essentially the canonical coordinates of the drift Hamiltonian. Although one can use noncanonical variables in a Hamiltonian drift theory,[5] canonical variables allow one to construct simple Hamiltonian models or approximations. With approximate fields, preservation of the Hamiltonian structure of the drift motion is subtle in noncanonical formulations. A description of the appropriate magnetic coordinates is given in Sec. III.

Previous papers have given a drift Hamiltonian in magnetic coordinates for a time-independent, curl-free magnetic field with perfect surfaces[4] and for a time-independent magnetic field with arbitrary curl and with stochastic regions.[6] In this paper the method is extended to include time-dependent magnetic and electric fields. In addition, a simpler set of canonical variables is introduced. This Hamiltonian system is given in Sec. IV and its relation to Alfvén's drift velocity is discussed. In Sec. V a derivation of this Hamiltonian is given using Taylor's drift Lagrangian.[7] The paper is summarized in Sec. VI.

II. GUIDING CENTER THEORY

Fundamentally, the evaluation of particle trajectories using guiding center theory is based on the adiabatic invariance of the magnetic moment, μ. If μ were an exact invariant instead of an adiabatic invariant, its canonically conjugate coordinate, the gyrophase, would not appear in the Hamiltonian. That is, a canonical transformation could reduce the problem of a particle in magnetic field from six canonical variables to only four. One of the canonical variables, the gyrophase, could be eliminated altogether and another canonical variable, the magnetic moment, would be an isolating constant not a variable. In actuality, one can construct the invariant μ and eliminate the gyrophase only by calculating order by order in the gyroradius to system size $\rho_\parallel/L$. The usual presumption is that the Hamiltonian constructed in this way[5] is only asymptotically correct and may, therefore, most accurately represent the exact particle trajectories in some finite order. Let H_d, the drift Hamiltonian, be the Hamiltonian with only four canonical variables, which most accurately describes the true particle trajectories. An important question, which has very limited treatment in the literature, is how the trajectories of H_d relate to the true particle trajectories over long times with a small but finite $\rho_\parallel/L$.

In this paper we assume that a sufficiently accurate drift Hamiltonian H_d exists. With this assumption an especially important role is played by the lowest-order, in $\rho_\parallel/L$, drift Hamiltonian H. Let q, p stand for the four canonical variables of the Hamiltonian H. One can then presumably find canonical variables for the Hamiltonian H_d, such that if they were denoted by q, p, the function $V(q, p)$,

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$$V(q,p) \equiv H_d(q,p) - H(q,p), \tag{3}$$

would be small when the gyroradius to system size is small. In other words, the most accurate drift Hamiltonian H_d can be considered to be the lowest-order drift Hamiltonian H plus a small perturbation V. The KAM theory[8] demonstrates that under certain conditions if V is small, then the trajectories of the Hamiltonian H lie close to those of H_d for all time. Except in singular circumstances, generally resonances between V and the particle motion, it is plausible that the Hamiltonian H leads to an estimate of diffusion coefficients as reliable as H_d. When higher-order terms are required to break a singularity from the use of H, then presumably only the lowest-order singularity breaking terms in V are required and not all terms of comparable order in gyroradius to system size. An example of a small but important term is the polarization drift which separates electron and ion drift motion in an electric potential. For simplicity we have ignored this drift. However, it could be included by adding the kinetic energy associated with the $\mathbf{E}\times\mathbf{B}$ drift to the Hamiltonian.

Of course, the concept of simplicity or lowest order does not uniquely define $H(q,p)$. The natural division of terms, which form $H_d(q,p)$, between $H(q,p)$ and $V(q,p)$, depends upon the formulation. As long as V in different formulations is nonresonant, the choice between formulations can be based on conceptual or calculational simplicity. This freedom will be used to simplify the canonical Hamiltonian formulation and is a real power of this form of the drift equations.

There is a further subtlety of comparing the trajectories of the most accurate drift Hamiltonian with the lowest-order drift Hamiltonian. In standard Hamiltonian mechanics, one calculates the trajectories in canonical coordinate space q, p. As plasma physicists, the four quantities we want are not q, p, but three spatial coordinates, say x, y, z, and a parallel velocity $v_\parallel$. The four physical variables are functions of the four canonical variables. The functional form of this relationship, of course, depends on the order in gyroradius to system size. However, since one normally needs to know the position of the trajectory to some accuracy relative to global plasma dimensions, this change in functional relation between the physical and canonical variables is of limited interest.

In general, Alfvén's drift velocity, Eq. (1), is a non-Hamiltonian approximation to the most accurate drift Hamiltonian. Consequently, the KAM theory says nothing about the length of time the Alfvén trajectories remain close to the most accurate drift trajectories. Indeed, the two sets of trajectories will generally diverge on a time scale of order $(L/\rho)^2(1/\omega_c)$ with $\omega_c = eB/mc$ the cyclotron frequency.

III. MAGNETIC COORDINATES

There are two basic reasons for using magnetic coordinates as the spatial coordinate system for drift trajectory studies. First, since the magnetic field dominates the structure of many laboratory and astrophysical plasmas, magnetic coordinates simply describe the location of the particle in the plasma. Second, the magnetic coordinates are essentially the canonical coordinates of the lowest-order drift Hamiltonian and simplify the information required to evaluate the trajectories. For toroidal plasmas, the magnetic coordinate system is especially simple if the plasma is in scalar pressure equilibrium. Then the pressure P satisfies the relations

$$\mathbf{B}\cdot\nabla P = 0, \tag{4}$$

and

$$(\nabla\times\mathbf{B})\cdot\nabla P = 0. \tag{5}$$

Unless ∇P is zero over a finite volume, one can show the magnetic field can be written in two forms[9,10]:

$$\mathbf{B} = \nabla\psi\times\nabla\theta + \nabla\phi\times\nabla\psi_p(\psi), \tag{6}$$

and

$$\mathbf{B} = g(\psi)\nabla\phi + I(\psi)\nabla\theta + \beta_*(\psi,\theta,\phi)\nabla\psi. \tag{7}$$

The magnetic coordinates ψ, θ, and ϕ, as well as the associated functions of ψ which are g, I, and ψ_p, are interpreted in Fig. 1. The function β_* will not play a major role in this paper but is closely related to the Pfirsch–Schlüter current.[10] To the mathematically inclined, it should be clear that the coordinates ψ, θ, and ϕ have been chosen to maximize the simplicity of the contravariant, Eq. (6), and covariant, Eq. (7), representations of the magnetic field.

To represent more complicated magnetic fields, the representations, Eqs. (6) and (7), must be generalized. Mathematically stated one must have well-behaved transformation equations $\mathbf{x}(\psi,\theta,\phi)$ between the magnetic coordinates and the ordinary Cartesian coordinates. In the case of tensor pressure equilibria, Eq. (5) no longer holds, but there may be good magnetic surfaces so that an equivalent to Eq. (4) remains valid. In this case, the covariant representation of the magnetic field, Eq. (7), must be generalized[6] to

$$\mathbf{B} = \sigma[g(\psi)\nabla\phi + I(\psi)\nabla\theta + \beta_*(\psi,\theta,\phi)\nabla\psi], \tag{8}$$

with σ, the permeability, chosen so that

$$[\nabla\times(\mathbf{B}/\sigma)]\cdot\nabla\psi = 0. \tag{9}$$

For tensor pressure,

$$\sigma = 1 + 4\pi[(P_\perp - P_\parallel)/B^2]. \tag{10}$$

In the absence of magnetic surfaces, the contravariant representation, Eq. (6), must be generalized[6,11] to

$$\mathbf{B} = \nabla\psi\times\nabla\theta + \nabla\phi\times\nabla\Psi_p(\psi,\theta,\phi). \tag{11}$$

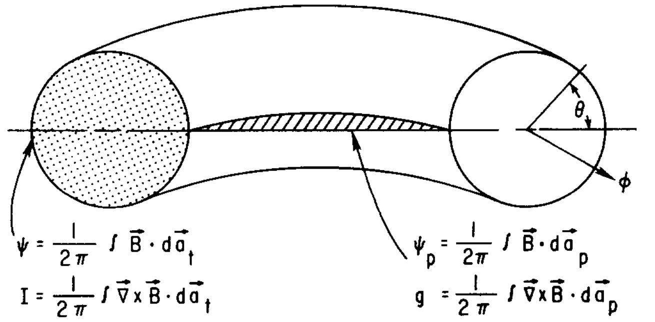

FIG. 1. Magnetic coordinates. A topologically toroidal constant pressure surface has two other surfaces associated with it which are the domain of the toroidal and the poloidal area integrals. The toroidal flux function ψ and the "plasma current" I are defined on one such surface. The poloidal flux function ψ_p and the "coil current" g are defined on the other. The pressure, ψ_p, g, and I are all functions of ψ alone.

That is, the poloidal flux function must be made a function of all three coordinates and not just the toroidal flux function ψ alone. It is easy to show that Ψ_p is the field line Hamiltonian in canonical form.[11] If the Hamiltonian is integrable, then its canonical coordinates ψ, θ can be chosen so Ψ_p is a function of ψ alone, $\psi_p(\psi)$. The toroidal angle ϕ is "time" in the Hamiltonian formulation. In this paper we assume that the breaking of the magnetic surfaces is sufficiently weak so that the ψ variation along a field line is slow compared to the θ or ϕ variation .

If the magnetic field is known as a function of ordinary coordinates, $\mathbf{B}(\mathbf{x})$, then one can evaluate the transformation equations and functions of the magnetic coordinates.[11–13]

IV. DRIFT HAMILTONIAN

The time-dependent drift Hamiltonian has a form which is made plausible by a comparison with the exact particle Hamiltonian. By a rather long and therefore unenlightening exercise in changing the variables of partial differentiation, one can also show the equivalence of this formulation to that of Alfvén to lowest order in gyroradius to system size.[6] In this section the drift Hamiltonian formulation will be given and compared to the exact Hamiltonian formulation as well as other statements of the guiding center drift motion. In the next section, an intuitive but, at least to this physicist, more enlightening derivation based on Taylor's drift Lagrangian will be given.

The drift Hamiltonian should be the particle energy and should depend on four canonical variables θ, p_θ, ϕ, and p_ϕ. Based on the relation between the lowest-order expression for the magnetic moment μ and the perpendicular kinetic energy, Eq. (2), the lowest-order drift Hamiltonian is

$$H(\theta, p_\theta, \phi, p_\phi) = \tfrac{1}{2} m v_\parallel^2 + \mu B + e\Phi, \tag{12}$$

with Φ the electric potential. The two canonical coordinates are θ and ϕ, which are the poloidal and toroidal magnetic coordinate angles discussed in Sec. III. The momenta conjugate to these coordinates are

$$p_\theta = (\sigma I/B) m v_\parallel + (e/c)\psi, \tag{13}$$

and

$$p_\phi = (\sigma g/B) m v_\parallel - (e/c)\Psi_p. \tag{14}$$

The particle trajectories are determined by Hamiltonian's equations:

$$\dot{p}_\theta = -\frac{\partial H}{\partial \theta}, \quad \dot{p}_\phi = -\frac{\partial H}{\partial \phi}, \quad \dot{\theta} = \frac{\partial H}{\partial p_\theta}, \quad \dot{\phi} = \frac{\partial H}{\partial p_\phi}. \tag{15}$$

Now let us compare the exact and the drift Hamiltonians. Both the exact and the drift Hamiltonian are the energy, and the poloidal and toroidal angles can be used as canonical coordinates for both. In toroidal symmetry, the toroidal canonical momentum of the exact Hamiltonian is

$$P_\phi = (\sigma g/B_\phi) m v_\phi - (e/c)\Psi_p, \tag{16}$$

since one can show $\sigma g/B_\phi$ is the major radius. Consequently, the relation between the exact and approximate toroidal canonical momentum is just the replacement of v_ϕ by $(B_\phi/B)v_\parallel$, a very intuitive result. Equivalent statements can, of course, be made about the poloidal canonical momentum. There are six canonical coordinates in the exact Hamiltonian formulation. Specifically, ψ and P_ψ are required. However, even in the exact problem, four canonical variables θ, P_θ, ϕ, and P_ϕ locate the particle within a gyroradius and determine the parallel velocity to within the gyroradius to system size.

The relation between the drift Hamiltonian of this paper and an earlier drift Hamiltonian[6] is more subtle than one might suppose. In a globally steady-state magnetic field, the guiding center velocity can be written[4,5]

$$v = \frac{v_\parallel}{B} \frac{1}{1 + \rho_\parallel (\mathbf{B}\cdot\nabla\times\mathbf{B})/B} (\mathbf{B} + \nabla\times\rho_\parallel \mathbf{B}), \tag{17}$$

with $\rho_\parallel(E, \mu, \mathbf{x})$ the parallel gyroradius, $mcv_\parallel/eB$, taken to be a function of energy, magnetic moment, and position. This velocity is consistent with a Hamiltonian, and a canonical Hamiltonian formulation has been given.[6] However, for β_* not equal to zero [see Eq. (8)], the Hamiltonian formulation of this paper is not exactly equivalent to Eq. (17). To understand the difference, we note that the drift velocity Eq. (17) consists of two factors. The last factor

$$\mathbf{H} = \mathbf{B} + \nabla\times\rho_\parallel \mathbf{B} \tag{18}$$

determines the trajectory. The first factor, which is a velocity space Jacobian,[4] determines how long a particle takes to cover the different parts of its trajectory. The first factor must be chosen appropriately to make the drift velocity Hamiltonian. The vector $\mathbf{H}$ for the Hamiltonian of this paper can be shown to be

$$\mathbf{H}' = \mathbf{B} + \nabla\times[\rho_\parallel(\mathbf{B} - \sigma\beta_* \nabla\psi)]. \tag{19}$$

In the Appendix it is shown that the difference between $\mathbf{H}'$ and $\mathbf{H}$ just leads to an order gyroradius to system size correction to the θ and ϕ position of a particle and can therefore be ignored in the lowest-order drift Hamiltonian. As noted in the introduction, such nonresonant differences between Hamiltonian formulations are not important for determining which formulation is better.

V. LAGRANGIAN DERIVATION

The behavior of particles in a time-dependent magnetic field is clarified by studying the exact particle Lagrangian. The importance of the Lagrangian is that the form of Lagrange's equations is independent of the spatial coordinate system. Therefore, using the Lagrangian, one can transform from inertial Cartesian to the noninertial magnetic coordinates of a time-dependent magnetic field. That is, the exact particle Lagrangian can be written in a coordinate system tied to the magnetic field lines. This Lagrangian leads intuitively to a drift Lagrangian first given by Taylor which in turn yields the Hamiltonian equations of the last section.

First consider the exact Lagrangian in a stationary frame of reference,

$$L = \tfrac{1}{2} m V^2 + (e/c)\mathbf{A}\cdot\mathbf{V} - e\Phi_s. \tag{20}$$

If the magnetic field is time dependent, then the exact particle velocity $\mathbf{V} = d\mathbf{x}/dt$ can be written, using the transformation equations $\mathbf{x}(\psi, \theta, \phi, t)$, as

$$\mathbf{V} = \frac{\partial \mathbf{x}}{\partial t} + \mathbf{e}_\alpha \dot{\xi}^\alpha \tag{21}$$

with $\xi^1 = \psi, \xi^2 = \theta, \xi^3 = \phi$, and $\mathbf{e}_\alpha = \partial \mathbf{x}/\partial \xi^\alpha$. The expression

$$\mathbf{V}_B = \frac{\partial \mathbf{x}}{\partial t} \tag{22}$$

is the velocity of the magnetic field, and

$$\mathbf{v} = \mathbf{e}_\alpha \dot{\xi}^\alpha \tag{23}$$

is the velocity of the particle relative to the field in contravariant form. The exact Lagrangian can, therefore, be written

$$L_e = \tfrac{1}{2}mv^2 + (e/c)\mathbf{a}\cdot\mathbf{v} - e\Phi, \tag{24}$$

with

$$\mathbf{a} = \mathbf{A} + (mc/e)\mathbf{V}_B, \quad \Phi = \Phi_s - \mathbf{A}\cdot\mathbf{v}_B/c - \tfrac{1}{2}(m/e)V_B^2. \tag{25}$$

The total field $\nabla\times\mathbf{a}$, which is the sum of the magnetic and the vorticity field, is usually indistinguishable from the magnetic field in plasma problems. The point is

$$\left|\frac{mc\mathbf{V}_B}{e\mathbf{A}}\right| \simeq \frac{V_B}{V}\frac{\rho}{L},$$

with ρ the gyroradius and L a typical plasma dimension. In plasmas, both V_B/V and ρ/L are generally quite small. Consequently, we will not distinguish between **a** and **A**, although technically one should. This distinction is important for Galilean invariance. Similarly, the potential Φ can be considered to obey $\Phi = \Phi_s - \mathbf{A}\cdot\mathbf{V}_B/c$, which is just the Lorentz transformed potential. The term $\mathbf{A}\cdot\mathbf{V}_B/c$ contains the so-called $\mathbf{E}\times\mathbf{B}$ motion of the field lines.

If the distinction between **a** and **A** were ignored, the canonical momentum of the exact particle motion, $P_\alpha = \partial L_e/\partial\dot{\xi}^\alpha$, would be

$$P_\alpha = [m\mathbf{v} + (e/c)\mathbf{A}]\cdot\mathbf{e}_\alpha, \tag{26}$$

and the exact Hamiltonian would be $H_e = P_\alpha\dot{\xi}^\alpha - L_e$, which gives

$$H_e = (1/2m)[\,P_\alpha - (e/c)A_\alpha\,]g^{\alpha\beta}[\,P_\beta - (e/c)A_\beta\,] + e\Phi. \tag{27}$$

The adiabatic conservation of the magnetic moment $\mu \simeq mv_\perp^2/2B$ suggests that the kinetic energy associated with the perpendicular motion behaves as potential energy μB. This substitution would follow from standard mechanics if μ were an exact invariant. This assumption together with the identification of **a** with **A** implies that the Lagrangian, Eq. (24), can be written in a form first given by Taylor,[7]

$$L = \tfrac{1}{2}mv_\parallel^2 + (e/c)\mathbf{A}\cdot\mathbf{v} - \mu B - e\Phi. \tag{28}$$

The Taylor Lagrangian depends in principle on six variables, three spatial coordinates, and their time derivatives.

To proceed we need the Taylor Lagrangian in magnetic coordinates. Using Eq. (8) for **B** and Eq. (23) for **v**, one finds

$$v_\parallel = (\sigma/B)(g\dot{\phi} + I\dot{\theta}), \tag{29}$$

with the neglect of the term $\beta_* \dot{\psi}$, which *a posteriori* is small because of the smallness of $\dot{\psi}$ in comparison to either $\dot{\theta}$ or $\dot{\phi}$. The vector potential of the magnetic field, as given in Eq. (11), is

$$\mathbf{A} = \psi\nabla\theta - \Psi_p\nabla\phi, \tag{30}$$

which implies

$$\mathbf{A}\cdot\mathbf{v} = \psi\dot{\theta} - \Psi_p\dot{\phi}. \tag{31}$$

The lowest-order Taylor Lagrangian in magnetic coordinates is then

$$L = \tfrac{1}{2}m(\sigma/B)^2(g\dot{\phi} + I\dot{\theta})^2 + (e/c)(\psi\dot{\theta} - \Psi_p\dot{\phi}) - \mu B - e\Phi. \tag{32}$$

Although the Taylor Lagrangian is, in principle, a function of six variables, an important feature of magnetic coordinates is that only five variables explicitly appear. That is, $\dot{\psi}$ is absent. The generalized momenta are given by the partial derivatives of L with respect to $\dot{\psi}$, $\dot{\theta}$, and $\dot{\phi}$. The momentum p_ψ is identically zero while p_θ and p_ϕ agree with Eqs. (13) and (14). The Hamiltonian is defined by

$$H = p_\psi\dot{\psi} + p_\theta\dot{\theta} + p_\phi\dot{\phi} - L. \tag{33}$$

Since p_ψ is identically zero, $\partial H/\partial p_\psi$ is ill defined, and the equation for the time derivative of p_ψ, which is $-\partial H/\partial\psi$, vanishes. This just means θ, ϕ, p_θ, and p_ϕ specify ψ, which is obvious from the functional form of p_θ and p_ϕ, and give a complete description of the mechanics problem. A simple calculation demonstrates that the Hamiltonian of Eq. (33) agrees with that of Eq. (12).

VI. SUMMARY

The lowest-order drift Hamiltonian can be expressed simply and intuitively in a canonical, magnetic coordinate representation. In this representation, three functions of the three magnetic coordinates ψ, θ, and ϕ determine the particle behavior. The cross-field magnetic drifts are determined by $B(\psi,\theta,\phi)$, the magnetic field strength. The structure of the magnetic field is determined by the poloidal flux function $\Psi_p(\psi,\theta,\phi)$, which is also the Hamiltonian of the magnetic field lines. The rapid radial or ψ motion of passing particles in stochastic magnetic fields[15] is determined by Ψ_p. The time derivation of Ψ_p, with the magnetic coordinates held fixed, is proportional to the loop voltage.[10] The electric potential $\Phi(\psi,\theta,\phi)$ is a function of ψ alone in the simplest models of toroidal plasmas with good surfaces. However, electrostatic instabilities can give Φ a complex structure.

KAM theory[8] implies that a small nonresonant perturbation to a Hamiltonian problem does not produce a large change in the particle trajectories. Consequently, a Hamiltonian problem can be simplified so long as the canonical structure is maintained and the difference between the simplified and the accurate Hamiltonian is small and nonresonant. This is a justification for the use of lowest-order guiding center theory since the difference between the most accurate and the lowest-order drift Hamiltonian is small whenever the gyroradius to system size is small. In particular problems, the drift Hamiltonian, Eqs. (12)–(15), can be further simplified by the same argument. This feature makes the canonical representation of the drift problem particularly attractive.

The most subtle feature of guiding center theory is the reduction from the six canonical variables required for an exact Hamiltonian treatment to the four canonical variables of any drift Hamiltonian. This reduction is accomplished by finding a canonical variable which is adiabatically conserved, the magnetic moment μ, and eliminating its canonically conjugate coordinate, the gyrophase, from the Hamiltonian. The practical limits on this procedure, when the long-time trajectories are required for transport studies, remain obscure.

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APPENDIX: THE β_* CORRECTION TO THE DRIFT

The standard Alfvén guiding center drift equations are equivalent to the guiding center velocity[16]

$$\mathbf{v}_g = (v_\parallel / B)[\mathbf{B} + (\nabla\times\rho_\parallel \mathbf{B})_\perp], \tag{A1}$$

with the perpendicular sign meaning the component orthogonal to the magnetic field. In this paper it proved more convenient to use an expression for the drift velocity which is equivalent to adding the velocity

$$\mathbf{v}_\beta = -\nabla\times(\sigma\rho_\parallel \beta_* \nabla\psi) \tag{A2}$$

to the lowest-order drift velocity. Here we show that the velocity $\mathbf{v}_\beta$ corresponds, in lowest order, to a shift in the guiding center position which is negligible compared to systematic drift produced by $\mathbf{v}_g$. In other words, there is a certain arbitrariness as to the location of the guiding center of a given particle. The presence or absence of the $\mathbf{v}_\beta$ drift just corresponds to a difference choice for the guiding center position. With either choice the particle stays, in lowest-order theory, within roughly a gyroradius of its guiding center.

In lowest-order drift theory, the parallel component of $\mathbf{v}_\beta$ is irrelevant. The perpendicular components can be simply studied in a Clebsch representation of the magnetic field

$$\mathbf{B} = \nabla\psi\times\nabla_0. \tag{A3}$$

In addition to the coordinates ψ and θ_0, we will use the toroidal angle ϕ. Since $\mathbf{v}_\beta\cdot\nabla\psi$ vanishes, $\mathbf{v}_\beta$ does not directly change the ψ position of a particle. The only important effect is the change in the θ_0 position.

The direct effect of the velocity $\mathbf{v}_\beta$ on the θ_0 position of a particle is given by

$$\left(\frac{d\theta_0}{dt}\right)_\beta \equiv \mathbf{v}_\beta\cdot\nabla\theta_0. \tag{A4}$$

From Eqs. (A2) and (A3),

$$\mathbf{v}_\beta\cdot\nabla\theta_0 = -\frac{v_\parallel}{B}\mathbf{B}\cdot\nabla\phi\,\frac{\partial(\sigma\rho_\parallel \beta_*)}{\partial\phi}. \tag{A5}$$

To zeroth order in the gyroradius to system size, a particle just moves along the field lines. The first-order perpendicular displacement of a particle in the θ_0 direction because $\mathbf{v}_\beta$ is therefore

$$[\theta_0]_\beta = \int \frac{dl}{v_\parallel}\mathbf{v}_\beta\cdot\nabla\theta_0. \tag{A6}$$

The differential distance dl along a field line is

$$dl = (B/\mathbf{B}\cdot\nabla\phi)d\phi. \tag{A7}$$

Using Eq. (A5) one finds

$$[\theta_0]_\beta = -[\sigma\rho_\parallel \beta_*], \tag{A8}$$

with [⋯] meaning the change in the quantity along the trajectory.

In lowest-order guiding center theory, the θ_0 motion caused by $\mathbf{v}_\beta$ is bounded by $\sigma\rho_\parallel \beta_*$. In practice σ is near unity, and β_* is roughly given by

$$\beta_* \approx (1/B)(4\pi/c)J_{ps}, \tag{A9}$$

with J_{ps} for the Pfirsch–Schlüter current.[10] Consequently, $\sigma\rho_\parallel \beta_*$ is always smaller than the gyroradius divided by the system size. The guiding center drift velocity gives a systematic precession of the particle in θ_0. That is, $\mathbf{v}_g$ gives a larger change in θ_0 compared to $\mathbf{v}_\beta$. To see this, we assume the magnetic field is curl free so that

$$\mathbf{B} = g\nabla\phi, \tag{A10}$$

with g a constant. The relevant change in θ_0 is given by

$$\left(\frac{d\theta_0}{dt}\right)_g = \mathbf{v}_g\cdot\nabla\theta_0. \tag{A11}$$

Using Eqs. (A1), (A3), and (A10),

$$\mathbf{v}_g\cdot\nabla\theta_0 = \frac{v_\parallel}{B}g\mathbf{B}\cdot\nabla\phi\,\frac{\partial\rho_\parallel}{\partial\psi}.$$

The arguments used to derive Eq. (A8) then imply

$$[\theta_0]_g = g\int\frac{\partial}{\partial\psi}\rho_\parallel\, d\phi, \tag{A12}$$

with the integral over the ϕ or parallel motion of the particle. The integral can also be written

$$[\theta_0]_g = \frac{\partial}{\partial\psi}\left(\frac{mc}{e}\int v_\parallel\, dl\right), \tag{A13}$$

which is trivially transformed into the well-known action form for the precession of trapped particles.

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