Guiding center drift equations

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(Received 16 March 1979; accepted 18 January 1980)

The equations for particle drift orbits are given in a new magnetic coordinate system. This form of the equations separates the fast motion along the magnetic field lines from the slow motion across the lines. In addition, less information is required about the magnetic field structure than in alternative forms of the drift equations.

I. INTRODUCTION

The close relation between particle drift orbits and transport in high temperature plasmas is well known. However, the evaluation of the drift orbits is not easily accomplished, even computationally, if there are no symmetry directions. Even in ideally symmetric systems, like the tokamak, small symmetry breaking terms which occur in real devices have significant transport effects. These effects are larger and often more subtle in nonsymmetric systems like the stellarator.

In this paper a simple form for the drift velocity in steady-state fields is given. A new magnetic coordinate system is developed and the drift orbit equations are given in this system. These drift orbit equations not only separate the slow and fast particle motion, but also require minimal information about the magnetic field.

II. DRIFT VELOCITY

The expression for the drift velocity across a steady-state magnetic field is well known

$$\mathbf{v}_\perp = (c\mathbf{B}/eB^2) \times (\mu\nabla B + e\nabla\Phi + mv_\parallel^2 \hat{b}\cdot\nabla\hat{b}), \tag{1}$$

with $\hat{b} = \mathbf{B}/B$ a unit vector along the magnetic field $\mathbf{B}$. The electric potential is Φ, μ is the magnetic moment, and $v_\parallel$ is the velocity of the particle along the magnetic field. In a steady-state field, energy conservation

$$E = \tfrac{1}{2} mv_\parallel^2 + \mu B + e\Phi \tag{2}$$

permits the drift velocity to be written in a simple and often more useful form.

To derive the desired expression, we note that

$$\nabla(\mu B + e\Phi) = -\nabla \tfrac{1}{2} m v_\parallel^2 \tag{3}$$

with the energy of the particle E taken to be a constant. The expression $\hat{b}\times(\hat{b}\cdot\nabla\hat{b})$ can be rewritten using a vector identity for unit vectors,

$$\hat{b}\cdot\nabla\hat{b} = -\hat{b}\times(\nabla\times\hat{b}) \tag{4}$$

and

$$\nabla\times\mathbf{B} = B\nabla\times\hat{b} - \hat{b}\times\nabla B, \tag{5}$$

to yield

$$\hat{b}\times(\hat{b}\cdot\nabla\hat{b}) = B^{-1}\hat{b}\times\nabla B + B^{-1}(\nabla\times\mathbf{B})_\perp. \tag{6}$$

It is then simple algebra to show

$$\begin{aligned}\mathbf{v}_\perp &= -\frac{c\mathbf{B}}{eB^2}\times\left(mv_\parallel\nabla v_\parallel - mv_\parallel^2\frac{1}{B}\nabla B\right) + mv_\parallel^2\frac{c}{eB^2}(\nabla\times\mathbf{B})_\perp \\ &= (v_\parallel/B)[\nabla\times(\rho_\parallel\mathbf{B})]_\perp\end{aligned} \tag{7}$$

with

$$\rho_\parallel \equiv mcv_\parallel/eB. \tag{8}$$

To obtain an expression for the drift velocity of a particle, we must add a parallel component to the expression for the perpendicular drift velocity. It is tempting to write the parallel component as $v_\parallel\mathbf{B}/B$, but this is not quite consistent with the use of $\mathbf{v}$ in drift kinetic theory. Consider the ideal drift kinetic equation with the distribution function depending only on the constants of motion E and μ, then for a valid expression for $\mathbf{v}$

$$\mathbf{v}\cdot\nabla f = 0. \tag{9}$$

This is just a statement that constants of motion are indeed constants of the motion. For $\mathbf{v}$ to be valid, Eq. (9) must also predict particle conservation, which, for f depending only on E and μ, just implies that the divergence of the particle flux is zero

$$\nabla\cdot\left(\int \mathbf{v} f\, d^3v\right) = 0. \tag{10}$$

To make Eq. (10) meaningful, d^3v must be transformed to the velocity coordinates of drift kinetic theory, E and μ, and to the particle phase ϕ_v in its cyclotron motion. This transformation gives a Jacobian with $d^3v = J\,dE\,d\mu\,d\phi_v$. Under the assumptions of drift kinetic theory, f is independent of ϕ_v as is $\mathbf{v}$; so the integration over ϕ_v can be performed to give $d^3v = 2\pi\langle J\rangle dE\,d\mu$. An expression is derived in the Appendix for the average of J, $\langle J\rangle$, which is valid through first order in the gyroradius

$$\langle J\rangle = (1/m^2)(B/v_\parallel)(1 + \rho_\parallel\hat{b}\cdot\nabla\times\hat{b}). \tag{11}$$

Since Eq. (10) must be valid for arbitrary $f(E,\mu)$ it implies

$$\nabla\cdot(\langle J\rangle\mathbf{v}) = 0. \tag{12}$$

The fact that $\langle J\rangle\mathbf{v}$ is divergence-free, which is essentially Liouville's theorem, is closely connected with other conservation laws as will be shown.

The exact expression one uses for the Jacobian $\langle J\rangle$ is generally not important in drift kinetic theory. The point is that $\langle J\rangle$ is a dividing factor throughout the drift kinetic equation, $\mathbf{v}\cdot\nabla f = C(f)$, for time independent

 0031-9171/80/050904-05$00.90

problems. Equation (12) implies that this is true for the left-hand side of the drift kinetic equation. The particle conserving feature of the collision operator implies that $\langle J\rangle C(f)$ is the fundamental operator. Since the exact expression for J, in practice, is not important, we use only the zeroth-order expression to obtain

$$\mathbf{v}=(v_{\parallel}/B)[\mathbf{B}+\nabla\times(\rho_{\parallel}\mathbf{B})]. \tag{13}$$

This expression for the drift velocity satisfies Eqs. (7), (9), and (12). However, the neglect of the first-order term in $\langle J\rangle$ does cause Eq. (13) to give an incorrect first-order correction to $\mathbf{v}\cdot\hat{b}$. A discussion of Eq. (13) with a comparison to other work is given in the Appendix. Expressions for the drift velocity similar to Eq. (13) have been given by several authors[1-3] for $\nabla\times\mathbf{B}=0$. However, the divergence condition on a physically reasonable $\mathbf{v}$ and its implications in the form for the drift velocity appear to be new.

The expression for the drift velocity can be written as

$$\mathbf{v}=(v_{\parallel}/B)\mathbf{H}\,, \quad \text{with } \mathbf{H}=\mathbf{B}+\nabla\times(\rho_{\parallel}\mathbf{B}) \tag{14}$$

in some sense a "real" magnetic field, that is, $\nabla\cdot\mathbf{H}=0$. The field $\mathbf{H}$ does have the unfortunate feature of being singular at turning points ($v_{\parallel}=0$). At these points, the $\mathbf{H}$ field with $v_{\parallel}>0$ is to be joined to the $\mathbf{H}$ field with $v_{\parallel}<0$ to obtain the drift orbit. The $\mathbf{H}$ field does allow a simple evaluation of constants of the drift motion in symmetric fields. Let $\boldsymbol{\zeta}$ be a vector pointing in the direction of symmetry chosen so that $\nabla\cdot\boldsymbol{\zeta}=0$. Then, if the curl of $\boldsymbol{\zeta}$ is of the form $\nabla\times\boldsymbol{\zeta}=\gamma\boldsymbol{\zeta}$, the magnetic field and its vector potential can be written in the form

$$\mathbf{B}=g\boldsymbol{\zeta}+\boldsymbol{\zeta}\times\nabla\psi\,, \quad \mathbf{A}=-\psi\boldsymbol{\zeta}+\mathbf{a}\,, \tag{15}$$

with $\mathbf{a}$ defined so $\boldsymbol{\zeta}\cdot\mathbf{a}=0$. The vector $\mathbf{H}$ can, of course, be written in a similar form with its vector potential $\mathbf{A}_*=-\psi_*\boldsymbol{\zeta}+\mathbf{a}_*$. Using the definition of $\mathbf{H}$, one finds

$$\psi_*=\psi-g\rho_{\parallel}\,. \tag{16}$$

Since $\mathbf{H}\cdot\nabla\psi_*=0$, ψ_* is a constant of the motion. In toroidal symmetry, $\boldsymbol{\zeta}=\nabla\phi=\hat{\phi}/R$ and ψ_* conservation is essentially p_ϕ conservation. In helical symmetry,

$$\boldsymbol{\zeta}=\frac{hr\hat{\theta}+l\hat{z}}{l^2+h^2r^2}\,. \tag{17}$$

III. MAGNETIC GEOMETRY

When solving the drift orbit equations, it is clearly advantageous to go to a magnetic coordinate system. By a magnetic coordinate system we mean one in which magnetic field lines serve as coordinate lines. In this coordinate system the rapid particle motion along the lines is separated from the slow motion across the lines. The magnetic coordinates used in this paper are α, ψ, and χ, which are three functions of position chosen so that the magnetic field can be written in a contravariant and a covariant form

$$\mathbf{B}=\nabla\alpha\times\nabla\psi\,, \tag{18}$$

$$\mathbf{B}=\nabla\chi+\beta\nabla\psi+\gamma\nabla\alpha\,. \tag{19}$$

The first or contravariant form for $\mathbf{B}$ is well-known as the Clebsch representation and can be used to describe any divergence-free field. The second or covariant form is not so familiar, but is completely general so long as $\nabla\chi\cdot(\nabla\alpha\times\nabla\psi)$, the inverse of the Jacobian, is not zero. The inverse of the Jacobian of the α, ψ, and χ coordinates is especially simple

$$\nabla\chi\cdot(\nabla\alpha\times\nabla\psi)=B^2\,. \tag{20}$$

When there is a scalar pressure, the physical interpretation of ψ, χ, and α is simple. The coordinate ψ labels the constant pressure surfaces and is essentially the magnetic flux within a surface. The coordinate α is an angle within a pressure surface labeling the various field lines, and χ is, in some sense, the distance along a line. Actually, the differential distance along a field line is $d\chi/B$.

The equations for the guiding center drift orbits can be simply expressed for the general magnetic field. However, the equations are simpler and the covariant form of the magnetic field has an interesting structure in the scalar pressure case

$$\nabla P=c^{-1}(\mathbf{j}\times\mathbf{B})\,. \tag{21}$$

Since $\mathbf{B}\cdot\nabla P=0$, one can always choose ψ so that the pressure is a function of ψ alone. This choice of ψ will be assumed. Using $\mathbf{j}\cdot\nabla P=0$, it is easy to show that one can redefine β and γ so that $\gamma=0$. The current density in a plasma with a scalar pressure is then

$$\mathbf{j}=\frac{c}{4\pi}\nabla\times\mathbf{B}=\frac{c}{4\pi}\left(\mathbf{B}\frac{\partial\beta}{\partial\alpha}-(\nabla\psi\times\nabla\chi)\frac{\partial\beta}{\partial\chi}\right), \tag{22}$$

which implies

$$\frac{\partial\beta}{\partial\alpha}=\frac{4\pi}{c}\frac{j_{\parallel}}{B} \tag{23}$$

with $j_{\parallel}$ the parallel current density. Evaluating $\mathbf{j}\times\mathbf{B}$ using the contravariant form of $\mathbf{B}$, Eq. (18), one finds

$$\frac{\partial\beta}{\partial\chi}=\frac{4\pi}{B^2}\frac{dP}{d\psi}\,. \tag{24}$$

There is some arbitrariness in the definition of β and χ which can be eliminated by the boundary condition

$$\beta(\alpha=0,\ \psi,\ \chi=0)=0\,. \tag{25}$$

With this boundary condition, $\mathbf{B}=\nabla\chi$ when the magnetic field is curl-free on a constant ψ surface.

IV. DRIFT ORBIT EQUATIONS

The expressions derived in the last two sections for the magnetic field and the drift velocity permit a very simple derivation of the drift orbit equations. These equations are expressions for $d\alpha/dt$, $d\psi/dt$, and $d\chi/dt$ along the drift trajectory of a particle and are derived using $d\alpha/dt=\mathbf{v}\cdot\nabla\alpha$, $d\psi/dt=\mathbf{v}\cdot\nabla\psi$, and $d\chi/dt=\mathbf{v}\cdot\nabla\chi$.

To derive the drift orbit equations, we note that the drift velocity, Eq. (13), can be expressed as

$$\mathbf{v}=(v_{\parallel}/B)[\nabla\alpha\times\nabla\psi+\nabla\times(\rho_{\parallel}\nabla\chi+\beta\rho_{\parallel}\nabla\psi+\gamma\rho_{\parallel}\nabla\alpha)]\,, \tag{26}$$

using Eqs. (18) and (19) for the magnetic field. Expressions like $\nabla\times(\rho_{\parallel}\nabla\chi)$ are rewritten as

$$\nabla\times(\rho_{\parallel}\nabla\chi)=-(\nabla\chi\times\nabla\alpha)\frac{\partial\rho_{\parallel}}{\partial\alpha}-(\nabla\chi\times\nabla\psi)\frac{\partial\rho_{\parallel}}{\partial\psi}\,. \tag{27}$$

Then with repeated use of $\nabla\chi\cdot(\nabla\alpha\times\nabla\psi)=B^2$, one finds

$$\frac{d\alpha}{dt}=v_{\parallel}B\left(\frac{\partial\rho_{\parallel}}{\partial\psi}-\frac{\partial\beta\rho_{\parallel}}{\partial\chi}\right), \tag{28}$$

$$\frac{d\psi}{dt}=-v_{\parallel}B\left(\frac{\partial\rho_{\parallel}}{\partial\alpha}-\frac{\partial\gamma\rho_{\parallel}}{\partial\chi}\right), \tag{29}$$

$$\frac{d\chi}{dt}=v_{\parallel}B\left(1-\frac{\partial\gamma\rho_{\parallel}}{\partial\psi}+\frac{\partial\beta\rho_{\parallel}}{\partial\alpha}\right). \tag{30}$$

These equations determine the drift orbit of a particle of given energy E and magnetic moment μ once the four functions of α, ψ, and χ are specified. These functions are B, β, γ, and Φ.

At first there appear to be a number of difficulties with integrating Eqs. (28)–(30) in order to obtain drift orbits, due to the question of the sign of $v_{\parallel}$ and the singular nature of its derivatives at the turning points. Actually, these problems can easily be dealt with. The problem of the sign of $v_{\parallel}$ is of importance only in the equation for $d\chi/dt$ and then only near turning points. This problem comes from evaluating $v_{\parallel}$ using energy conservation [Eq. (2)] and is avoided by evaluating $v_{\parallel}$, at least near turning points, using a differential equation for $dv_{\parallel}/dt$. It is simpler to use the differential equation for $d\rho_{\parallel}/dt$ which clearly serves the same purpose ($\rho_{\parallel}=mcv_{\parallel}/eB$)

$$\frac{d\rho_{\parallel}}{dt}=\mathbf{v}\cdot\nabla\rho_{\parallel}=v_{\parallel}B\left[\frac{\partial\rho_{\parallel}}{\partial\chi}-\rho_{\parallel}\left(\frac{\partial\rho_{\parallel}}{\partial\alpha}\frac{\partial\beta}{\partial\chi}-\frac{\partial\rho_{\parallel}}{\partial\chi}\frac{\partial\beta}{\partial\alpha}\right)+\rho_{\parallel}\left(\frac{\partial\rho_{\parallel}}{\partial\psi}\frac{\partial\gamma}{\partial\chi}-\frac{\partial\rho_{\parallel}}{\partial\chi}\frac{\partial\gamma}{\partial\psi}\right)\right]. \tag{31}$$

This expression is especially simple in the curl-free field case with $d\rho_{\parallel}/dt=v_{\parallel}B(\partial\rho_{\parallel}/\partial\chi)$ of a form similar to the other orbit equations. The problem of singular derivatives of $v_{\parallel}$ at turning points is of no fundamental importance to the drift orbit equations since derivatives of $v_{\parallel}$ are always multiplied by a $v_{\parallel}$ factor which gives a finite product. The following easily derived, but useful, expression illustrates this with ξ equal to α, ψ, or χ:

$$v_{\parallel}B\frac{\partial\rho_{\parallel}}{\partial\xi}=-c\frac{\partial\Phi}{\partial\xi}-\left(\frac{c}{e}\mu+\frac{eB}{mc}\rho_{\parallel}^2\right)\frac{\partial B}{\partial\xi}. \tag{32}$$

In symmetric systems, the drift equations conserve $\psi_*=\psi-g\rho_{\parallel}$ as demonstrated in Sec. II. The relation between this conservation law and Eqs. (28)–(30) is quite fascinating in the scalar pressure case. Remembering that $\boldsymbol{\zeta}$ is a vector in the symmetry direction and $\boldsymbol{\zeta}\cdot\nabla\psi=0$, one has $\boldsymbol{\zeta}\cdot\mathbf{B}=\boldsymbol{\zeta}\cdot\nabla\chi$ using Eq. (19) for B. Using Eq. (18) for B, one obtains $\boldsymbol{\zeta}\times\mathbf{B}=-(\boldsymbol{\zeta}\cdot\nabla\alpha)\nabla\psi$. Finally, using Eq. (15) for B, one finds $\boldsymbol{\zeta}\cdot\mathbf{B}=g\zeta^2$ and $\boldsymbol{\zeta}\times\mathbf{B}=-\zeta^2\nabla\psi$. These results imply $\boldsymbol{\zeta}\cdot\nabla\chi=g\boldsymbol{\zeta}\cdot\nabla\alpha$ or if f is any function such that $\boldsymbol{\zeta}\cdot\nabla f=0$, then

$$\frac{\partial f}{\partial\alpha}=-g\frac{\partial f}{\partial\chi}. \tag{33}$$

The conservation of ψ_* means $d\psi_*/dt=\mathbf{v}\cdot\nabla\psi_*=0$, but evaluating $d\psi_*/dt$ one finds

$$\frac{d\psi_*}{dt}=v_{\parallel}B\rho_{\parallel}\left[-g\frac{\partial\rho_{\parallel}}{\partial\chi}\left(\frac{\partial g}{\partial\psi}+g\frac{\partial\beta}{\partial\chi}+\frac{\partial\beta}{\partial\alpha}\right)+g\frac{\partial\rho_{\parallel}}{\partial\psi}\frac{\partial g}{\partial\chi}-\rho_{\parallel}\frac{\partial g}{\partial\chi}\left(g\frac{\partial\beta}{\partial\chi}+\frac{\partial\beta}{\partial\alpha}\right)-\frac{\partial g}{\partial\chi}\right]. \tag{34}$$

This expression must be zero for any $\rho_{\parallel}$ which obeys the symmetry ($\boldsymbol{\zeta}\cdot\nabla\rho_{\parallel}=0$), which in turn implies that g is only a function of ψ alone and

$$\frac{dg}{d\psi}+g\frac{\partial\beta}{\partial\chi}+\frac{\partial\beta}{\partial\alpha}=0. \tag{35}$$

This is equivalent to

$$j_{\parallel}=-\frac{c}{4\pi}B\frac{dg}{d\psi}-c\frac{g}{B}\frac{dP}{d\psi}. \tag{36}$$

These results appear quite remarkable. The symmetry conditions applied to the drift orbit equations give us information about the magnetic field rather than the other way around. Actually, the results are not so amazing if one looks at the formal operations involved. The result comes from being able to write any divergence-free field as $\mathbf{B}=g\boldsymbol{\zeta}+\boldsymbol{\zeta}\times\nabla\psi$. Consequently, $\mathbf{H}=\mathbf{B}+\nabla\times(\rho_{\parallel}\mathbf{B})$ can be written in this form with the only condition on $\rho_{\parallel}$ being $\boldsymbol{\zeta}\cdot\nabla\rho_{\parallel}=0$.

The longitudinal adiabatic invariant J is, of course, conserved by Eqs. (28)–(30) provided that the parallel motion is fast enough compared with the cross-field drifts. To prove this, let us construct a function f such that $\mathbf{v}\cdot\nabla f=0$. We let $f=f_0+f_1+\cdots$ with the subscripts representing orders in the small parameter $\rho_{\parallel}$. The zeroth order is

$$\mathbf{v}\cdot\nabla f_0=v_{\parallel}B\frac{\partial f_0}{\partial\chi}=0, \tag{37}$$

and $f_0=f_0(\alpha,\psi)$. In first order

$$\frac{\partial f_1}{\partial\chi}+\left(\frac{\partial\rho_{\parallel}}{\partial\psi}-\frac{\partial\beta\rho_{\parallel}}{\partial\chi}\right)\frac{\partial f_0}{\partial\alpha}-\left(\frac{\partial\rho_{\parallel}}{\partial\alpha}-\frac{\partial\gamma\rho_{\parallel}}{\partial\chi}\right)\frac{\partial f_0}{\partial\psi}=0. \tag{38}$$

The consistency condition on this equation for trapped particles is

$$\left(\oint\frac{\partial\rho_{\parallel}}{\partial\psi}d\chi\right)\frac{\partial f_0}{\partial\alpha}=\left(\oint\frac{\partial\rho_{\parallel}}{\partial\alpha}d\chi\right)\frac{\partial f_0}{\partial\psi} \tag{39}$$

with the loop integral implying an integral at constant α and ψ from a point where $\rho_{\parallel}=0$ to another where $\rho_{\parallel}=0$ and back, but

$$\oint\frac{\partial\rho_{\parallel}}{\partial\psi}d\chi=\frac{\partial}{\partial\psi}\oint\rho_{\parallel}\,d\chi=\frac{\partial}{\partial\psi}\frac{c}{e}m\oint v_{\parallel}\,dl=\frac{c}{e}\frac{\partial J}{\partial\psi} \tag{40}$$

with J the longitudinal invariant (we used $dl=d\chi/B$). It is then obvious that $f_0=f_0(J)$ so J is conserved at least to lowest order.

V. CONCLUSION

In a reactor grade plasma, even thermal particles can travel 10 km between collisions, the ratio of the cross field to the parallel velocity can be 10^{-3}, and a particle can have 10^7 cyclotron orbits per collision. These parameters imply that a brute force technique may not be sufficiently accurate to calculate particle orbits. In this paper a method of finding particle orbits has been developed which is based on the drift kinetic equation and magnetic field line coordinates. This system has a number of advantages. First, the fast particle motion along the field lines is separated from the slow drift across the lines. Second, less information is required about the magnetic field than in other formulations. Third, if there is a scalar pressure, the constant pressure surfaces can be used as a coordinate. Since the transport is determined by the distance particles stray from the constant pressure surfaces, the use of these surfaces as a coordinate greatly simplifies the interpretation of the results.

Although the drift equations are given in the paper for an arbitrary magnetic field, the most important case is the locally curl-free field due to its simplicity. For this case the magnetic field can be written

$$\mathbf{B}=\nabla\alpha\times\nabla\psi=\nabla\chi .$$

Using α, ψ, and χ as coordinates, the particle equations of motion are

$$\frac{d\alpha}{dt}=-c\,\frac{\partial\Phi}{\partial\psi}-\left(\frac{c}{e}\mu+\frac{eB}{mc}\rho_{\parallel}^{2}\right)\frac{\partial B}{\partial\psi},$$

$$\frac{d\psi}{dt}=c\,\frac{\partial\Phi}{\partial\alpha}+\left(\frac{c}{e}\mu+\frac{eB}{mc}\rho_{\parallel}^{2}\right)\frac{\partial B}{\partial\alpha},$$

$$\frac{d\chi}{dt}=\frac{e}{mc}\rho_{\parallel}B^{2},$$

$$\frac{d\rho_{\parallel}}{dt}=-c\,\frac{\partial\Phi}{\partial\chi}-\left(\frac{c}{e}\mu+\frac{eB}{mc}\rho_{\parallel}^{2}\right)\frac{\partial B}{\partial\chi}.$$

In these equations $\Phi(\alpha,\psi,\chi)$ is the electric potential, $B(\alpha,\psi,\chi)$ is the magnetic field strength, and e, m, and μ are the charge, mass, and magnetic moment of the particle. The quantity $\rho_{\parallel}=mcv_{\parallel}/eB$. Actually, the equations for a curl-free field can be stated more elegantly by defining a Hamiltonian as

$$H(\rho_{\parallel},\alpha,\psi,\chi)=\frac{1}{2}\rho_{\parallel}^{2}\frac{eB^{2}}{mc}+\frac{\mu c}{e}B+c\Phi;$$

then

$$\frac{d\chi}{dt}=\frac{\partial H}{\partial\rho_{\parallel}},\quad \frac{d\rho_{\parallel}}{dt}=-\frac{\partial H}{\partial\chi},\quad \frac{d\psi}{dt}=\frac{\partial H}{\partial\alpha},\quad \frac{d\alpha}{dt}=-\frac{\partial H}{\partial\psi}.$$

The adiabatic invariance of J,

$$J=\frac{e}{c}\int\rho_{\parallel}\,d\chi=\int mv_{\parallel}\,dl,$$

then follows from the standard classical mechanics[4] treatment. The Hamiltonian is just the energy E times c/e and it is conserved.

ACKNOWLEDGMENT

This work was supported by the United States Department of Energy, Contract No. EY-76-C-02-3073.

APPENDIX

A number of questions have been raised relative to Eq. (13) and its relation to the work of Northrop and Rome[5] and Hazeltine.[6] These questions actually turn out to involve higher-order corrections in gyroradius to system size. Although these higher-order corrections were not in the original scope of the work, they are, nonetheless, of interest.

To calculate to higher order in the gyroradius, a more accurate expression is required for the magnetic moment. Northrop and Rome give expressions for the higher-order effects and we follow their results. In their expression for the magnetic moment, they use a hybrid velocity $\mathbf{u}$ with $u_{\parallel}$ the parallel drift velocity $v_{\parallel}$ and with $\mathbf{u}_{\perp}$ the perpendicular particle velocity $V_{\perp}$. They give

$$\mu=\frac{1}{2}\frac{mu_{\perp}^{2}}{B}-\frac{m^{2}}{2eB}\left[(u^{2}\hat{b}+\mathbf{u}u_{\parallel})\cdot(\mathbf{u}\times\hat{b}\cdot\nabla B)\right.$$
$$\left.+u_{\parallel}(\nabla\times\mathbf{B})\cdot(\tfrac{1}{2}u_{\perp}^{2}\hat{b}+2\mathbf{u}_{\perp}u_{\parallel})\right].\qquad(\mathrm{A1})$$

The phase average of μ turns out to play an important role. This is

$$\langle\mu\rangle=\mu_{0}(1-\rho_{\parallel}\hat{b}\cdot\nabla\times\hat{b})\qquad(\mathrm{A2})$$

with $\mu_{0}=mV_{\perp}^{2}/2B$, the lowest-order magnetic moment, and $\rho_{\parallel}=v_{\parallel}(eB/mc)^{-1}$.

The derivation of the first-order Jacobian for the transformation from particle velocity coordinates $\mathbf{V}$ to E, μ, ϕ_{v} coordinates goes as follows: First, we note that $d^{3}V=d\langle V_{\parallel}\rangle V_{\perp}dV_{\perp}d\phi_{v}$ with $\langle V_{\parallel}\rangle$ the phase averaged parallel velocity. We use the result of Northrop and Rome

$$\langle V_{\parallel}\rangle=v_{\parallel}-(\mu_{0}/m)\hat{b}\cdot\nabla\times\hat{b}\qquad(\mathrm{A3})$$

to show $d^{3}V=d^{3}u$, or

$$d^{3}V=(B/m)dv_{\parallel}d\mu_{0}d\phi_{v}.\qquad(\mathrm{A4})$$

The next transformation is to E, μ, ϕ_{v} coordinates with $E=\frac{1}{2}mv_{\parallel}^{2}+\mu B+e\Phi$. If one lets $\mu=\mu_{0}/(1+\epsilon)$, one finds

$$J=\frac{B}{m^{2}v_{\parallel}}\left(1+\epsilon+\mu\frac{\partial\epsilon}{\partial\mu}+\mu B\frac{\partial\epsilon}{\partial E}\right);\qquad(\mathrm{A5})$$

so the gyrophase averaged Jacobian is

$$\langle J\rangle=(B/m^{2}v_{\parallel})(1+\rho_{\parallel}\hat{b}\cdot\nabla\times\hat{b}).\qquad(\mathrm{A6})$$

A comparison can now be made between Eq. (13) and the results of Northrop and Rome. Rewriting Eq. (13) using the first-order gyrophase averaged Jacobian, we have

$$\mathbf{v}=\frac{v_{\parallel}}{B}\frac{1}{1+\rho_{\parallel}\hat{b}\cdot(\nabla\times\hat{b})}(\mathbf{B}+\nabla\times\rho_{\parallel}\mathbf{B})\qquad(\mathrm{A7})$$

and $\mathbf{v}\cdot\hat{b}=v_{\parallel}$, which is the result of Northrop and Rome. For the perpendicular particle drift, they give an expression which can be written as

$$\mathbf{v}_{\perp}=(c\mathbf{B}/eB^{2})\times(\mu\nabla B+mv_{\parallel}^{2}\hat{b}\cdot\nabla\hat{b})$$
$$+\rho_{\parallel}[(\nabla\times\mathbf{v})_{\perp}-\mathbf{v}_{\perp}\hat{b}\cdot\nabla\times\hat{b}]+\mathbf{W}_{\perp}\qquad(\mathrm{A8})$$

with $\mathbf{W}_{\perp}$ depending on the definition of the guiding center. This expression without $\mathbf{W}_{\perp}$ would agree with Eq. (A7) if we replace $\rho_{\parallel}B=mcv_{\parallel}/e$ by $mc\mathbf{v}/e$. A plausible argument on this replacement is given at the end of the Appendix. Clearly, the results of this paper are in close agreement with the work of Northrop and Rome.

Hazeltine has derived a drift-kinetic equation in which the steady-state case can be written as

$$\mathbf{v}\cdot\nabla f+\mu_{0}v_{\parallel}\hat{b}\cdot\nabla(\rho_{\parallel}\hat{b}\cdot\nabla\hat{b})\frac{\partial f}{\partial\mu_{0}}=C(f).\qquad(\mathrm{A9})$$

His parallel velocity was written in the paper as $\langle V_{\parallel}\rangle$

$+(\mu_0/m)\hat{b}\cdot(\nabla\times\hat{b})$, but as shown by Northrop and Rome this is just $v_\parallel$. The term in Hazeltine's equation which goes as $\partial f/\partial\mu_0$ arises from his use of the lowest-order magnetic moment as the velocity coordinate rather than the exact magnetic moment μ. To show this, one notes

$$(\mathbf{v}\cdot\nabla f)_\mu=(\mathbf{v}\cdot\nabla f)_{\mu_0}+(\mathbf{v}\cdot\nabla\mu_0)_\mu\frac{\partial f}{\partial\mu_0}. \tag{A10}$$

In a gyrophase averaged equation, only the gyrophase average correction to μ is important, so

$$(\mathbf{v}\cdot\nabla\mu_0)_\mu\simeq\mu_0 v_\parallel\hat{b}\cdot\nabla(\rho_\parallel\hat{b}\cdot\nabla\hat{b}). \tag{A11}$$

Consequently, his result agrees with ours.

Finally, we will give an interesting derivation of the drift kinetic equation which illustrates the fundamental nature of Eq. (13). The exact particle velocity obeys

$$m\frac{dV}{dt}=\frac{e}{c}\mathbf{V}\times\mathbf{B}-e\nabla\Phi. \tag{A12}$$

Suppose, instead of using time as the independent variable, we use position; so

$$\frac{d\mathbf{V}}{dt}=\mathbf{V}\cdot\nabla\mathbf{V}=\nabla\frac{V^2}{2}-\mathbf{V}\times(\nabla\times\mathbf{V}). \tag{A13}$$

One can then write

$$\frac{e}{c}\mathbf{V}\times\left(\mathbf{B}+\frac{mc}{e}\nabla\times\mathbf{V}\right)=\nabla\left(\frac{1}{2}mv^2+e\Phi\right). \tag{A14}$$

If we describe $\mathbf{V}$ by the position of the particle, energy, exact magnetic moment, and gyrophase at a definite position, we have

$$\mathbf{V}\times\mathbf{h}=0,\quad \mathbf{h}=\mathbf{B}+(mc/e)\nabla\times\mathbf{V}. \tag{A15}$$

Consequently, $\mathbf{V}$ is always parallel to $\mathbf{h}$, that is, $\mathbf{V}=\lambda\mathbf{h}$. By particle conservation arguments like those of the paper, one can show that $\lambda\propto 1/J$ with J the Jacobian of the transformation to the new velocity coordinates E, μ, and ϕ_v. The exact kinetic equation is

$$\mathbf{V}\cdot\nabla f=C(f). \tag{A16}$$

Let us spatially average this equation after division by λ. By a spatial average of $F(\mathbf{x})$, we mean

$$\langle F(\mathbf{x})\rangle=\int s(\mathbf{x}-\mathbf{y})F(\mathbf{y})\,d^3y \tag{A17}$$

with $s(\mathbf{x})$ constant for $|\mathbf{x}|<l$, zero for $|\mathbf{x}|\gg l$, and

$$\int s(\mathbf{x})\,d^3x=1. \tag{A18}$$

The scale l is the smoothing scale which we assume is large compared to gyroradii, but small compared with the system size. One finds

$$\begin{aligned}\langle\lambda^{-1}C(f)\rangle&=\langle\mathbf{h}\cdot\nabla f\rangle,\\ \langle\lambda^{-1}C(f)\rangle&=\nabla\cdot\langle\mathbf{h}f\rangle.\end{aligned} \tag{A19}$$

If f is slowly varying compared with the gyroradius scale,

$$\begin{aligned}\langle hf\rangle&=\int s(\mathbf{x}-\mathbf{y})\mathbf{h}(\mathbf{y})f(\mathbf{y})\,d^3y\\ &\simeq f(\mathbf{x})\int s(\mathbf{x}-\mathbf{y})h(\mathbf{y})\,d^3y;\end{aligned} \tag{A20}$$

so one can write $\langle\mathbf{h}f\rangle\simeq\mathbf{H}f$ with $\mathbf{H}\equiv\langle\mathbf{h}\rangle$. Now $\langle\nabla\times\mathbf{V}\rangle=\nabla\times\langle\mathbf{V}\rangle$ so letting $\mathbf{v}=\langle\mathbf{V}\rangle$, we have

$$\mathbf{H}=\mathbf{B}+(mc/e)\nabla\times\mathbf{v}, \tag{A21}$$

and

$$\mathbf{H}\cdot\nabla f=\langle\lambda^{-1}C(f)\rangle. \tag{A22}$$

Identifying $\mathbf{v}$ with the drift velocity, one has the drift kinetic equation.

[1] A. I. Morozov and L. S. Solov'ev, in *Review of Plasma Physics*, edited by M. A. Leontovich (Consultants Bureau, New York, 1966), Vol. 2, p. 228.
[2] P. H. Rutherford, Phys. Fluids **13**, 482 (1970).
[3] D. Dobrott and E. A. Frieman, Phys. Fluids **14**, 349 (1971).
[4] L. D. Landau and E. M. Lifshitz, *Mechanics* (Pergamon, New York, 1976), 3rd ed., p. 154.
[5] T. G. Northrop and J. A. Rome, Phys. Fluids **21**, 384 (1978).
[6] R. D. Hazeltine, Plasma Phys. **15**, 77 (1973).