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The Motion of a Charged Particle in a Strong Magnetic Field

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Abstract

The reduced description in terms of drifts and adiabatic invariants of the motion of a charged particle in a strong magnetic field is derived. The demonstration employs systematically two time scales and an iteration scheme for each quasiperiodicity. This leads to a particularly expeditious derivation, as well as the details of the rapid oscillations at each stage. Moreover the motivation of each part is clear, as is the relation to simple problems in dynamics. The small parameters, the existence of which underlines the method, are displayed explicitly.

Introduction

A central problem in plasma physics is the derivation of a tractable description of the motion of a charged particle in a strong magnetic field. This task was initiated by Alfvén (1) on a physical basis, and carried to a high degree of mathematical sophistication by Kruskal (2). The formal considerations of the latter author provide a constructive technique for the development of a so-called "reduced description" of the motion in powers of an appropriate small parameter. The method applies to all dynamical system that exhibit one or more, almost periodic motions. A partial summary of both points of view has been given by Northrup (3), who combines several points of view. The basic notion underlying all treatments is that of the existence for each almost periodic motion of two time scales, one of which describes the rapid periodic aspect, and the other any slow perturbation of this. When this notion is applied systematically to the problem of the motion of a charged particle in a strong magnetic field, coupled with an appropriate iteration scheme, it is possible to derive, in a very efficient manner, all of the well-known results and, in addition, indicate explicitly what are the small parameters and the details of the reduced description.

Section I is devoted to developing the guiding-center description and the associated approximate constant of the motion or adiabatic invariant, the magnetic moment, to the lowest significant order. Section II is concerned with the derivation of the reduced description when the motion of the guiding center along the lines of force is periodic and the particle does not move much perpendicular to the line in one period. The second adiabatic invariant, the so-called longitudinal invariant, is found to the lowest significant order in this second small parameter, as well as a description of the rapid oscillation. In Section III, a final reduction is affected in the description when the energy of the guiding-center particle changes but little in the time required to circulate once on a magnetic surface. Here, in addition to a third adiabatic invariant the magnetic flux, the details of the motion in the constant flux surface are found.

The derivations presented here have the virtue of considerable analytical simplicity and conceptual unity. It is also clear from them explicitly what the small parameter in question is, and also how to proceed to the next order. Moreover, at all stages the treatment is simply related to a familiar problem in classical mechanics.

I. The Gyrating Particle and the Magnetic Moment Adiabatic Invariant

The equation of motion of a particle of charge q and mass m acted on by an electric field $\mathbf{E}(\mathbf{r},t)$, a magnetic field $\mathbf{B}(\mathbf{r},t)$ and a gravitational potential $G(\mathbf{r},t)$ is

$$\ddot{\mathbf{r}} = \mathbf{a} - \mathbf{\Omega} \times \dot{\mathbf{r}} \tag{1-1}$$

where

$$\mathbf{a}(\mathbf{r},t) = q\mathbf{E}(\mathbf{r},t)/m - \nabla G(\mathbf{r},t) \tag{1-2}$$

$$\mathbf{\Omega}(\mathbf{r},t) = q\mathbf{B}(\mathbf{r},t)/mc \tag{1-3}$$

When $\mathbf{a}$ and $\mathbf{\Omega}$ are constant in space and time, the solution of eq. 1-1 can be written (4):

$$\mathbf{r} = \mathbf{R} + \boldsymbol{\rho} \tag{1-4}$$

where

$$\mathbf{R}(t) = \mathbf{R}(0) + \mathbf{bb}\cdot[\dot{\mathbf{R}}(0)t + \tfrac{1}{2}\,\mathbf{a}t^2] + \mathbf{b} \times \mathbf{a}\, t/\Omega \tag{1-5}$$

$$\boldsymbol{\rho}(t) = [\mathbf{e}_2 \cos(\Omega t + \phi) + \mathbf{e}_1 \sin(\Omega t + \phi)]w/\Omega \tag{1-6}$$

In the above expressions the perpendicular speed w and the phase ϕ are constants, and we have introduced

$$\Omega = qB/mc \tag{1-7}$$

$$\mathbf{b} = \mathbf{B}/B \tag{1-8}$$

and the orthonormal right-handed set of Cartesian unit vectors, $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3 = \mathbf{b}$. Clearly,

$$\dot{\mathbf{R}}(t) = \mathbf{bb}\cdot[\dot{\mathbf{R}}(0) + \mathbf{a}t] + \mathbf{b} \times \mathbf{a}/\Omega \tag{1-9}$$

$$\dot{\boldsymbol{\rho}} = w[-\mathbf{e}_2 \sin(\Omega t + \phi) + \mathbf{e}_1 \cos(\Omega t + \phi)] \tag{1-10}$$
$$= -\mathbf{\Omega} \times \boldsymbol{\rho}$$

The solution is readily verified by substitution in eq. 1-1. The vector $\mathbf{R}(t)$ describes the trajectory of the so-called guiding center; the term in brackets in eq. 1-5 arises from the accelerated motion in the direction of the magnetic field; the term $\mathbf{b} \times \mathbf{a}/\Omega$ is designated the *drift velocity* perpendicular to the magnetic field.

Consider the case in which $\mathbf{a}$ and $\mathbf{\Omega}$ depend on space and time but do not change much in a distance w/Ω or a time $1/\Omega$, where w is the magnitude of the component of the velocity of the particle orthogonal to the magnetic field measured relative to the drift velocity, and Ω is the value of the gyration frequency that prevails at the point in question on the trajectory of the particle. It then seems plausible that the solution of the equation of motion will be very much like that given in eqs. 1-4 to 1-6. If this is so, one is led to seek a solution effectively in powers of the small parameter

$$\varepsilon = \left|\Omega^{-1}\left(\frac{\partial}{\partial t} + \dot{\mathbf{R}}\cdot\nabla\right)\ln aB\right| + |(w/\Omega)\nabla \ln aB| \tag{1-11}$$

This is the program adopted by Kruskal (2) and leads to an asymptotic representation.

An alternative method that is more expeditious for obtaining lowest significant order results consists in the introduction of an auxiliary variable

θ contrived to describe the rapid gyration indicated in eq. 1-6 for the constant field case and a suitable iteration scheme. We shall require periodicity in θ and choose the period to be unity, so that θ has the character of an angle variable in Hamilton Jacobi theory. For the case of constant fields one sees from eq. 1-6 that $\theta = \Omega t/2\pi$ is an appropriate choice. For the general case we write

$$\mathbf{r} = \mathbf{R}(t) + \boldsymbol{\rho}(\theta,t) \tag{1-12}$$

whence if we denote partial derivatives by subscripts and set $\dot{\theta} = \nu(t)$,

$$\dot{\mathbf{r}} = \dot{\mathbf{R}}(t) + \nu(t)\, \boldsymbol{\rho}_\theta(\theta,t) + \boldsymbol{\rho}_t(\theta,t) \tag{1-13}$$

Presumably,

$$\nu^2 \rho_\theta{}^2 \gg \rho_t{}^2$$

and we expect that in the order of magnitude $\nu \sim \Omega/2\pi$.

In dealing with the equation of motion, since we anticipate that $\rho^2 = w^2/\Omega^2$, we are led to expand $\boldsymbol{\Omega}(\mathbf{r},t)$ and $\mathbf{a}(\mathbf{r},t)$ in powers of $\boldsymbol{\rho}$ because this is effectively an expansion in powers of the small parameter ε of eq. 1-11. Thus one writes

$$\mathbf{a}(\mathbf{R} + \boldsymbol{\rho},t) = \mathbf{a}(\mathbf{R},t) + \boldsymbol{\rho} \cdot \nabla \mathbf{a}(\mathbf{R},t) + \tfrac{1}{2}\boldsymbol{\rho}\boldsymbol{\rho} : \nabla\nabla \mathbf{a}(\mathbf{R},t) + \cdots \tag{1-14}$$

and a parallel expansion for $\boldsymbol{\Omega}$. When these expansions and the time derivative of eq. 1-13 are employed in the equation of motion 1-1, we obtain the result,

$$\begin{aligned}
\ddot{\mathbf{R}} + \nu^2 \boldsymbol{\rho}_{\theta\theta} + 2\nu \boldsymbol{\rho}_{\theta t} + \dot{\nu}\boldsymbol{\rho}_\theta + \boldsymbol{\rho}_{tt} = {} & \mathbf{a} + \boldsymbol{\rho} \cdot \nabla \mathbf{a} + \tfrac{1}{2}\boldsymbol{\rho}\boldsymbol{\rho} : \nabla\nabla \mathbf{a} + \cdots \\
& - \boldsymbol{\Omega} \times \dot{\mathbf{R}} - \boldsymbol{\rho} \cdot (\nabla \boldsymbol{\Omega}) \times \dot{\mathbf{R}} - \tfrac{1}{2}\boldsymbol{\rho}\boldsymbol{\rho} : (\nabla\nabla\boldsymbol{\Omega}) \times \dot{\mathbf{R}} \\
& - \cdots \\
& - \nu\boldsymbol{\Omega} \times \boldsymbol{\rho}_\theta - \nu\boldsymbol{\rho} \cdot (\nabla\boldsymbol{\Omega}) \times \boldsymbol{\rho}_\theta - \tfrac{1}{2}\nu\boldsymbol{\rho}\boldsymbol{\rho} : (\nabla\nabla\boldsymbol{\Omega}) \times \boldsymbol{\rho}_\theta \\
& - \cdots \\
& - \boldsymbol{\Omega} \times \boldsymbol{\rho}_t - \boldsymbol{\rho} \cdot (\nabla\boldsymbol{\Omega}) \times \boldsymbol{\rho}_t - \tfrac{1}{2}\boldsymbol{\rho}\boldsymbol{\rho} : (\nabla\nabla\boldsymbol{\Omega}) \times \boldsymbol{\rho}_t \\
& - \cdots
\end{aligned} \tag{1-15}$$

Note that in eq. 1-15 θ occurs only in $\boldsymbol{\rho}$ and its derivative, $\mathbf{a} = \mathbf{a}(\mathbf{R},t)$, $\boldsymbol{\Omega} = \boldsymbol{\Omega}(\mathbf{R},t)$, and ∇ denotes the gradient with respect to $\mathbf{R}$. We shall require that $\boldsymbol{\rho}$ be periodic in θ with period unity and the average of $\boldsymbol{\rho}$ over one period in θ vanish. That is, following Kruskal, one can write the Fourier series:

$$\boldsymbol{\rho}(\theta,t) = \sum_{n=1}^{\infty} [\boldsymbol{\rho}^{(n)}(t)e^{i2\pi n\theta} + \boldsymbol{\rho}^{(n)*}(t)e^{-i2\pi n\theta}] \tag{1-16}$$

Thus, if we integrate eq. 1-15 over one period of θ, we obtain

$$\ddot{\mathbf{R}} = \mathbf{a} + \nu \int_0^1 d\theta\, \boldsymbol{\rho}_\theta \times [\boldsymbol{\rho} \cdot (\nabla\boldsymbol{\Omega})] - \boldsymbol{\Omega} \times \dot{\mathbf{R}} + \cdots \tag{1-17}$$

As will be shown later, the dependence of $\boldsymbol{\rho}$ on θ to the lowest significant order is given by eq. 1-6 with $2\pi\theta$ replacing Ωt. Thus, if one carries out integrations over one period in θ, the integral of any quantity cubic in $\boldsymbol{\rho}$ or its derivatives will vanish to the lowest significant order. In particular, it follows from this observation that the terms indicated by dots above are smaller by a factor of the order ε^2 than the largest term retained explicitly.

When eq. 1-17 is subtracted from eq. 1-15, on regrouping terms and recognizing that $\dot{\mathbf{R}}$ is not a function of $\mathbf{R}$ but depends only on t, we find that

$$\begin{aligned}
(\nu^2 \boldsymbol{\rho}_\theta + \nu\boldsymbol{\Omega} \times \boldsymbol{\rho})_\theta = {} & -2\nu\boldsymbol{\rho}_{\theta t} - \dot{\nu}\boldsymbol{\rho}_\theta + \boldsymbol{\rho} \cdot \nabla(\mathbf{a} + \dot{\mathbf{R}} \times \boldsymbol{\Omega}) + \boldsymbol{\rho}_t \times \boldsymbol{\Omega} \\
& + \nu\boldsymbol{\rho}_\theta \times (\boldsymbol{\rho} \cdot \nabla\boldsymbol{\Omega}) - \int_0^1 d\theta\, \nu\boldsymbol{\rho}_\theta \times (\boldsymbol{\rho} \cdot \nabla\boldsymbol{\Omega}) + \cdots
\end{aligned} \tag{1-18}$$

The terms indicated by dots on the right-hand side of eq. 1-18 are smaller by a factor of order ε than these indicated explicitly on the right. These latter in turn are smaller by a factor of the order ε than those written on the left-hand side of eq. 1-18. Thus, to the lowest significant order, we require that the left-hand side above vanish, whence on integration in θ

$$\nu^2 \boldsymbol{\rho}_\theta + \nu\boldsymbol{\Omega} \times \boldsymbol{\rho} = \boldsymbol{\xi}(t) \tag{1-19}$$

where $\xi(t)$ is the *constant of integration*. If one integrates eq. 1-19 over one period in θ, it follows that the left-hand side vanishes because of eq. 1-16. Thus $\xi = 0$, and if we resolve eq. 1-19 in the Cartesian coordinate system associated with the unit vectors introduced prior to eq. 1-19, we obtain the result:

$$\rho_{1\theta} - \frac{\Omega}{\nu}\rho_2 = 0 \tag{1-20}$$

$$\rho_{2\theta} + \frac{\Omega}{\nu}\rho_1 = 0 \tag{1-21}$$

$$\rho_{3\theta} = 0 \tag{1-22}$$

It follows from eq. 1-22 and the requirement that $\boldsymbol{\rho}$ have no part constant in θ that $\rho_3 = 0$. Moreover, if one adds i times eq. 1-21 to eq. 1-20,

$$(\rho_1 + i\rho_2)_\theta = -i\frac{\Omega}{\nu}(\rho_1 + i\rho_2) \tag{1-23}$$

whence

$$\rho_1 + i\rho_2 = i\rho(t)e\left[-i\left(\frac{\Omega}{\nu}\theta + \phi(t)\right)\right]^{1/2} \tag{1-24}$$

where we have introduced the real constants of integration, $\rho(t)$ and $\phi(t)$.

In order that $\boldsymbol{\rho}$, as determined from eq. 1-24, be periodic in θ with period unity, we must require that

$$\nu(t) = \Omega(\mathbf{R}(t),t)/2\pi \tag{1-25}$$

Thus, on rewriting these results in vector form, to the lowest significant order in ε, we obtain

$$\boldsymbol{\rho}(\theta,t) = \rho[\mathbf{e}_2 \cos(2\pi\theta + \phi) + \mathbf{e}_1 \sin(2\pi\theta + \phi)] \tag{1-26}$$

where ρ, ϕ, $\mathbf{e}_1$, and $\mathbf{e}_2$ all depend on time. Clearly eq. 1-26 reduces to eq. 1-6 for the case of constant fields. The time dependence of θ is now determined from

$$\theta(t) = \int dt\, \Omega(\mathbf{R}(t),t)/2\pi \tag{1-27}$$

Note, however, that to this order in ε, ρ, and ϕ are not yet determined as functions of time.

If one wishes to calculate to the next order in ε, it is adequate to drop the terms indicated by dots in eq. 1-18. Rather than solve the resulting equation completely, we shall be content to derive an approximate constant of motion, correct to that order in ε corresponding to dropping the dots in eq. 1-18. The derivation proceeds by forming the scalar product of eq. 1-18 with ρ_θ, after deleting the terms indicated by dots:

$$\left(\frac{1}{2}\, v^2 \boldsymbol{\rho}_\theta^{\,2}\right)_\theta + (v\rho_\theta^{\,2})_t = \boldsymbol{\rho}_\theta \boldsymbol{\rho} \colon \nabla(\mathbf{a} + \dot{\mathbf{R}} \times \boldsymbol{\Omega}) + \boldsymbol{\rho}_\theta \times \boldsymbol{\rho}_t \cdot \boldsymbol{\Omega} - \boldsymbol{\rho}_\theta \cdot \int_0^1 d\theta\, v\boldsymbol{\rho}_\theta \times (\boldsymbol{\rho}\cdot\nabla\boldsymbol{\Omega}) \tag{1-28}$$

If we integrate eq. 1-28 over one period in θ, we obtain

$$\left(\int_0^1 d\theta v \rho_\theta^{\,2}\right)_t = \int_0^1 d\theta \boldsymbol{\rho}_\theta \boldsymbol{\rho} \colon \nabla(\mathbf{a} + \dot{\mathbf{R}} \times \boldsymbol{\Omega}) + \int_0^1 d\theta \boldsymbol{\rho}_\theta \times \boldsymbol{\rho}_t \cdot \boldsymbol{\Omega} \tag{1-29}$$

The integration over θ has removed the nominally large terms in eq. 1-28, and it is adequate to use the lowest significant order approximation eq. 1-26 in eq. 1-29. Thus

$$\begin{aligned} \int_0^1 d\theta \rho_\theta^{\,2} &= 4\pi^2\rho^2 \int_0^1 d\theta[\cos^2(2\pi\theta + \phi) + \sin^2(2\pi\theta + \phi)] \\ &= 4\pi^2\rho^2 \end{aligned} \tag{1-30}$$

and

$$\begin{aligned} \int_0^1 d\theta \boldsymbol{\rho}_\theta \boldsymbol{\rho} &= 2\pi\rho^2 \int_0^1 d\theta[-\mathbf{e}_2 \sin(2\pi\theta + \phi) + \mathbf{e}_1 \cos(2\pi\theta + \phi)] \\ &\quad \cdot [\mathbf{e}_2 \cos(2\pi\theta + \phi) + \mathbf{e} \sin(2\pi\theta + \phi)] \\ &= 2\pi\rho^2 \int_0^1 d\theta\{(\mathbf{e}_1\mathbf{e}_1 - \mathbf{e}_2\mathbf{e}_2) \sin(2\pi\theta + \phi)\cos(2\pi\theta + \phi) \\ &\qquad + \mathbf{e}_1\mathbf{e}_2 \cos^2(2\pi\theta + \phi) - \mathbf{e}_2\mathbf{e}_1 \sin^2(2\pi\theta + \phi)\} \\ &= \pi\rho^2(\mathbf{e}_1\mathbf{e}_2 - \mathbf{e}_2\mathbf{e}_1) \end{aligned} \tag{1-31}$$

In order conveniently to reduce the remaining integral, we note that on combining eqs. 1-19 and 1-25 we can write

$$\boldsymbol{\rho}_\theta = -2\pi\, \mathbf{b} \times \boldsymbol{\rho} \tag{1-32}$$

But substituting $\mathbf{b}$ in eq. 1-32, and using the result eq. of 1-26 that $\mathbf{b} \cdot \boldsymbol{\rho} = 0$, we find that

$$\mathbf{b} \times \boldsymbol{\rho}_\theta = 2\pi\boldsymbol{\rho} \tag{1-33}$$

Thus

$$\begin{aligned} \int_0^1 d\theta \boldsymbol{\rho}_\theta \times \boldsymbol{\rho}_t \cdot \boldsymbol{\Omega} &= \Omega \int_0^1 d\theta \mathbf{b} \times \boldsymbol{\rho}_\theta \cdot \boldsymbol{\rho}_t \\ &= 2\pi\Omega \int_0^1 d\theta \boldsymbol{\rho} \cdot \boldsymbol{\rho}_t \\ &= 2\pi\Omega \frac{\partial}{\partial t} \int_0^1 d\theta \frac{1}{2} \boldsymbol{\rho}^2 \\ &= \pi\Omega \frac{\partial}{\partial t} \rho^2 \end{aligned} \tag{1-34}$$

since $\boldsymbol{\rho}^2 = \rho^2$ is independent of θ. These results permit one to write eq. 1-29 as

$$\begin{aligned} (2\pi\Omega\rho^2)_t &= \pi\rho^2(\mathbf{e}_1\mathbf{e}_2 \cdot \nabla - \mathbf{e}_2\mathbf{e}_1 \cdot \nabla) \cdot (\mathbf{a} + \dot{\mathbf{R}} \times \boldsymbol{\Omega}) + \pi\Omega(\rho^2)_t \\ &= -\pi\rho^2[(\mathbf{e}_1 \times \mathbf{e}_2) \times \nabla] \cdot (\mathbf{a} + \dot{\mathbf{R}} \times \boldsymbol{\Omega}) + (\pi\Omega\rho^2)_t - \pi\rho^2\Omega_t \end{aligned} \tag{1-35}$$

On using eqs. 1-2 and 1-3 on the right-hand side above, after transposing the term that is a multiple of that on the left-hand side, we obtain the result, since $\mathbf{e}_1 \times \mathbf{e}_2 = \mathbf{b}$,

$$\begin{aligned} (\pi\Omega\rho^2)_t &= -\pi\rho^2\{(\mathbf{b} \times \nabla) \cdot (\mathbf{a} + \dot{\mathbf{R}} \times \boldsymbol{\Omega}) + \Omega_t\} \\ &= -\pi\rho^2\{\mathbf{b} \cdot \nabla \times (\mathbf{a} + \dot{\mathbf{R}} \times \boldsymbol{\Omega}) + \Omega_t\} \\ &= -\pi\rho^2 \frac{q}{m}\left\{\mathbf{b} \cdot \nabla \times \left[\mathbf{E} + \frac{1}{c}\dot{\mathbf{R}} \times \mathbf{B}\right] + \frac{1}{c} B_t\right\} \end{aligned} \tag{1-36}$$

In eq. 1-36, B_t is to be interpreted as a time derivative holding θ fixed, i.e., a convective derivative following the guiding-center motion characterized by $\dot{\mathbf{R}}$, namely,

$$B_t = \frac{\partial B}{\partial t} + \dot{\mathbf{R}} \cdot \nabla B \tag{1-37}$$

But from the Maxwell equation,

$$c\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{1-38}$$

it follows, since $\nabla \cdot \mathbf{B} = 0$ and $\nabla \dot{\mathbf{R}} = 0$, that

$$\begin{aligned}
\mathbf{B}_t &= \frac{\partial \mathbf{B}}{\partial t} + \dot{\mathbf{R}} \cdot \nabla \mathbf{B} \\
&= -c\nabla \times \left[\mathbf{E} + \frac{1}{c}\dot{\mathbf{R}} \times \mathbf{B}\right] + \mathbf{B} \cdot \nabla \dot{\mathbf{R}} + \dot{\mathbf{R}} \nabla \cdot \mathbf{B} - \mathbf{B} \nabla \cdot \dot{\mathbf{R}} \\
&= -c\nabla \times \left[\mathbf{E} + \frac{1}{c}\dot{\mathbf{R}} \times \mathbf{B}\right] \qquad (1\text{-}39)
\end{aligned}$$

Thus

$$\begin{aligned}
\mathbf{b} \cdot \nabla \times \left[E + \frac{1}{c}\dot{\mathbf{R}} \times \mathbf{B}\right] + \frac{1}{c} B_t &= -\frac{1}{c}\mathbf{b} \cdot \mathbf{B}_t + \frac{1}{c} B_t \\
&= -\frac{1}{cB}\mathbf{B} \cdot \mathbf{B}_t + \frac{1}{c}\left(\frac{B^2}{2}\right)_t \\
&= 0 \qquad (1\text{-}40)
\end{aligned}$$

Note that $\mathbf{E} + 1/c\,(\dot{\mathbf{R}} \times \mathbf{B})$ is just the electric field seen by an observer moving with the guiding center.

We can now conclude from eq. 1-36 that

$$(\pi\Omega\rho^2)_t = 0 \qquad (1\text{-}41)$$

whence

$$\mu = \frac{q}{c}\frac{\Omega}{2\pi}\pi\rho^2 = \frac{q}{c}\nu\pi\rho^2 \qquad (1\text{-}42)$$

$\sim \frac{q}{c}\frac{\frac{1}{2}v_m^2}{\Omega} \sim \frac{q}{c}\frac{\frac{1}{2}v_m^2}{qB/mc} \sim \frac{m v_m^2/2}{B}$

the so-called magnetic moment, is an approximate constant of the motion. Such an approximate constant is conventionally termed an *adiabatic invariant*.

Let us now return to eq. 1-17. Note that the term therein involving $\nabla\Omega$, on using eq. 1-31, can be approximated by

$$\begin{aligned}
\nu \int_0^1 d\theta\, \boldsymbol{\rho}_\theta \times [\boldsymbol{\rho} \cdot (\nabla\Omega)] &= \nu\pi\rho^2(\mathbf{e}_1\mathbf{e}_2 \cdot \nabla - \mathbf{e}_2\mathbf{e}_1 \cdot \nabla) \times (q\mathbf{B}/mc) \\
&= -\frac{\mu}{m}[(\mathbf{e}_1 \times \mathbf{e}_2) \times \nabla] \times \mathbf{B} \\
&= -\frac{\mu}{m}(\mathbf{b} \times \nabla) \times \mathbf{B} \\
&= -\frac{\mu}{m}[(\nabla\mathbf{B}) \cdot \mathbf{b} - \mathbf{b}\nabla \cdot \mathbf{B}] \\
&= -\frac{\mu}{m}\nabla B \qquad (1\text{-}43)
\end{aligned}$$

Thus on dropping the terms indicated by dots, after dotting and substituting $\mathbf{b}$, eq. 1-17 yields

$$\mathbf{b} \cdot \ddot{\mathbf{R}} = \mathbf{b} \cdot \mathbf{a} - \frac{\mu}{m}\mathbf{b} \cdot \nabla B \qquad (1\text{-}44)$$

and

$$\begin{aligned}
\dot{\mathbf{R}}_\perp &\equiv \mathbf{b} \times (\dot{\mathbf{R}} \times \mathbf{b}) \\
&= \frac{1}{\Omega}\mathbf{a} \times \mathbf{b} + \frac{\mu}{m\Omega}\mathbf{b} \times \nabla B + \frac{1}{\Omega}\mathbf{b} \times \ddot{\mathbf{R}} \\
&= c\frac{\mathbf{E} \times \mathbf{B}}{B^2} + \frac{1}{\Omega}\mathbf{b} \times \nabla G + \frac{\mu}{m\Omega}\mathbf{b} \times \nabla B + \frac{1}{\Omega}\mathbf{b} \times \ddot{\mathbf{R}} \qquad (1\text{-}45)
\end{aligned}$$

If we define

$$u = \mathbf{b} \cdot \dot{\mathbf{R}} \qquad (1\text{-}46)$$

we can write eq. 1-44 in the form

$$m\dot{u} = \mathbf{b} \cdot [q\mathbf{E} - m\nabla G - \mu\nabla B] + m\dot{\mathbf{b}} \cdot \dot{\mathbf{R}} \qquad (1\text{-}47)$$

since $\mathbf{b} \cdot \dot{\mathbf{b}} = 0$. Clearly, the acceleration along the magnetic field should not be so large as to change the magnetic field in a time comparable with Ω^{-1}; otherwise, the theory here developed is invalid.

Equation 1-45 can be solved by iteration, assuming that the acceleration $\mathbf{a}$ dominates, namely, to the lowest order,

$$\dot{\mathbf{R}}_\perp = \frac{1}{\Omega}\mathbf{a} \times \mathbf{b} \qquad (1\text{-}48)$$

← See opposite

and to the next order

$$\dot{\mathbf{R}}_\perp = \frac{1}{\Omega}\mathbf{a} \times \mathbf{b} + \frac{\mu}{m\Omega}\mathbf{b} \times \nabla B + \frac{1}{\Omega}\mathbf{b} \times \left[\left(\frac{1}{\Omega}\mathbf{a} \times \mathbf{b}\right)^{\cdot} + (u\mathbf{b})^{\cdot}\right] \qquad (1\text{-}49)$$

In order to iterate once again and preserve accuracy, one would have to restore the terms indicated by dots in eq. 1-17 and also evaluate $\boldsymbol{\rho}$ to the next order in ε.

The details of the gyration can be fixed, e.g., by choosing

$$\mathbf{e}_2 = \dot{\mathbf{R}}_\perp / |\dot{\mathbf{R}}_\perp| \qquad (1\text{-}50)$$

in which event

$$\mathbf{e}_1 = \mathbf{e}_2 \times \mathbf{e}_3 = \mathbf{e}_2 \times \mathbf{b} \qquad (1\text{-}51)$$

The derivation of the equation governing the evolution in time of the slowly varying phase function $\phi(t)$ can be found by returning to eq. 1-18 and viewing it as an inhomogeneous equation for the second approximation. That is, one writes eq. 1-16 in the form

$$[\boldsymbol{\rho}_\theta + (\boldsymbol{\Omega}/v) \times \boldsymbol{\rho}]_\theta = \mathbf{f} \tag{1-52}$$

where $v^2\mathbf{f}$ is given by the right-hand side of eq. 1-18 with the terms indicated by dots deleted, and $\boldsymbol{\rho}$ given by eq. 1-26. It is readily seen that $\mathbf{f}$ involves only the first and second harmonics of $2\pi\theta$, namely,

$$\mathbf{f} = \sum_{n=1}^{2} (\mathbf{f}^{(n)} e^{2\pi i n\theta} + \mathbf{f}^{(n)*} e^{-2\pi i n\theta}) \tag{1-53}$$

In order that the solution of the homogeneous equation associated with eq. 1-52 reproduce eq. 1-26, we must as before select $v = 2\pi/\Omega$. If then, as before, we resolve eq. 1-52 in a Cartesian coordinate system defined by the orthornormal vectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3 = \mathbf{b}$, on adding i times the two-component to the one-component of eq. 1-52, we obtain

$$(\rho_1 + i\rho_2)_{\theta\theta} + 2\pi i(\rho_1 + i\rho_2)_\theta = f_1 + if_2 \tag{1-54}$$

We require that $\boldsymbol{\rho}$ be represented by eq. 1-16, namely, that it be periodic in θ with period one and have no part constant in θ. That is, if we write

$$\rho_1 + i\rho_2 = \sum_{n=-\infty}^{\infty} c_n e^{2\pi n i\theta} \tag{1-55}$$

$$f_1 + if_2 = \sum_{n=-2}^{2} d_n e^{2\pi n i\theta} \tag{1-56}$$

where c_0 and d_0 are both zero, then the insertion of these expressions in eq. 1-54 and the equating of the coefficients of like Fourier factors $e^{2\pi n i\theta}$ yields

$$4\pi^2 n(n+1)c_n = d_n \tag{1-57}$$

Clearly, when $n \neq -1$, one has

$$c_n = -d_n/4\pi^2 n(n+1) \tag{1-58}$$

and the c_n vanish for $n = \pm 3, \pm 4, \ldots$. In order that a solution exist for $n = -1$, one must have $d_{-1} = 0$, or equivalently as follows from eq. 1-56 on multiplication by $e^{2\pi i\theta}$ and integration over one period in θ,

$$\mathrm{d}_1 = \int_0^1 d\theta(f_1 + if_2)e^{2\pi i\theta} = 0 \tag{1-59}$$

Equation 1-59 is equivalent to two real conditions resulting from taking the real and imaginary parts. These are effective equations for $\dot\phi$ and $\dot\rho$. On judicious combination they yield eq. 1-41. We shall not develop them in detail.

If one is not interested in analyzing the details of the gyration, it suffices to consider the equations governing the guiding center $\mathbf{R}(t)$: eq. 1-47 which gives the time rate of change of the component of the guiding center velocity along the magnetic field at the location of the guiding center and eq. 1-49 which gives the velocity of the guiding center perpendicular to the magnetic field at the location of the guiding center. The advantages of these equations over eq. 1-1 are twofold: first, they exhibit no fast gyrations on the scale of the gyration frequency; second, they constitute a fourth-order system of ordinary differential equations as opposed to eq. 1-1 which is a sixth-order system. These features are useful both for purposes of numerical calculation, and also for analytic work and qualitative analysis.

Higher approximations can be found by iterating the results just found, but in general the results are so complicated that the virtues of the reduced description are lost.

It has been shown (6) that these lowest significant order results represent the leading term in an asymptotic expansion of the trajectory of the particle in powers of the small parameter ε of eq. 1-11. That is, if one writes the partial sum,

$$\mathbf{S}_N(t) = \mathbf{r}_0(t) + \varepsilon\mathbf{r}_1(t) + \varepsilon^2\mathbf{r}_2(t) + \cdots \varepsilon^N r_N(t)$$

then, for any fixed time t,

$$\lim_{\varepsilon\to 0} \frac{|\mathbf{r}(t) - \mathbf{S}_N(t)|}{\varepsilon^N} = 0$$

This is distinct from what would prevail were the procedure convergent, namely,

$$\lim_{N\to\infty} |\mathbf{r}(t) - \mathbf{S}_N(r)| = 0$$

II. The Second or Longitudinal Adiabatic Invariant

A further reduction of the preceding guiding-center description can be made when the motion along the lines of force is quasiperiodic and much more rapid than the motion associated with the drift. The demonstration is assisted by writing the magnetic field in terms of two scalar fields $\alpha(\mathbf{r},t)$ and $\beta(\mathbf{r},t)$ via

$$\mathbf{B} = (\nabla\alpha) \times (\nabla\beta) \tag{2-1}$$

which clearly satisfies $\nabla \cdot \mathbf{B} = 0$. To show that eq. 2-1 is possible, recall that one can define lines of force by the equation

$$d\mathbf{r} \times \mathbf{B}(\mathbf{r},t) = 0$$

Let S be some surface nowhere tangent to the lines of force. In this surface choose a family of lines. The set of all lines of force through one of the lines of this family defines a surface. Let $\alpha(r,t) = \text{const}$ be the equation of such "magnetic surfaces." Now choose a second family of lines in S nowhere tangent to the first, and in a parallel manner associate with them a family of magnetic surfaces $\gamma(\mathbf{r},t) = \text{const}$. By construction

$$\mathbf{B} \cdot \nabla\alpha = 0 \qquad \mathbf{B} \cdot \nabla\gamma = 0 \tag{2-2}$$

and as follows directly from the above, since $\nabla\alpha$, $\nabla\gamma$, and $(\nabla\alpha) \times (\nabla\gamma)$ are noncoplanar on writing $\mathbf{B} = (\nabla\alpha) \times (\nabla\gamma)/\lambda + (\nabla\alpha)\mu + (\nabla\gamma)\nu$,

$$(\nabla\alpha) \times (\nabla\gamma) = \lambda\mathbf{B}$$

If one takes the divergence of the above equation and uses $\nabla \cdot \mathbf{B} = 0$,

$$\mathbf{B} \cdot \nabla\lambda = 0$$

and λ must be a function of α and γ. We shall now introduce a new variable $\beta(\alpha,\gamma)$. Clearly, $\mathbf{B} \cdot \nabla\beta = 0$. If we view γ as a function of α and β, and denote partial derivatives by the subscripts,

$$\begin{aligned}(\nabla\alpha) \times (\nabla\gamma) &= (\nabla\alpha) \times [\gamma_\alpha \nabla\alpha + \gamma_\beta \nabla\beta] \\ &= \gamma_\beta(\nabla\alpha) \times \nabla\beta\end{aligned}$$

We choose $\gamma_\beta = \lambda$. This yields the desired result

$$\mathbf{B} = (\nabla\alpha) \times (\nabla\beta) = \nabla \times (\alpha\nabla\beta) \tag{2-3}$$

Therefore, the intersection of any two surfaces $\alpha = \text{const}$ and $\beta = \text{const}$ is a line of force, and one can interpret the associated pair of values α, β as the coordinates of the line of force. Even though the pattern of lines of force may change in time, we shall identify that line labeled by a given pair α, β as the same line of force. The functions α and β need not be single-valued. See Figures 1, 2, and 3.

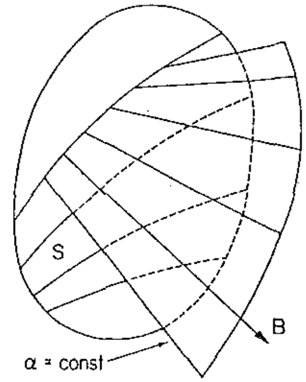

Fig. 1. Diagram illustrating the construction of surfaces. $\alpha = \text{const}$.

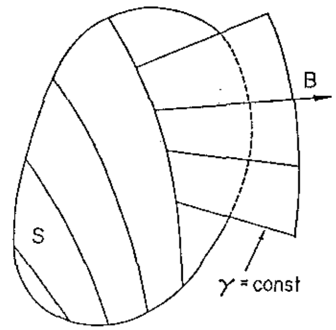

Fig. 2. Diagram illustrating the construction of surfaces. $\gamma = \text{const}$.

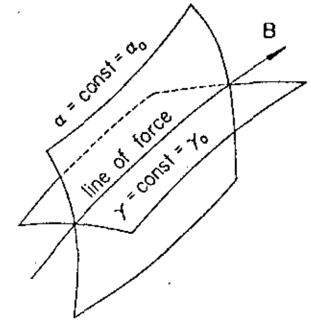

Fig. 3. Diagram illustrating the use of α, γ coordinates to label a line of force.

Now one can write the Maxwell equation

$$\begin{aligned}0 &= \nabla \times \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} \\ &= \nabla \times \left[\mathbf{E} + \frac{1}{c}\frac{\partial}{\partial t}(\alpha\nabla\beta)\right] = \nabla \times \Big[E + \frac{1}{c}\frac{\partial\alpha}{\partial t}\nabla\beta + \frac{1}{c}\alpha\nabla\end{aligned}$$

whence

$$\mathbf{E} = -\nabla\phi - \frac{1}{c}\frac{\partial\alpha}{\partial t}\nabla\beta - \frac{1}{c}\alpha\nabla\frac{\partial\beta}{\partial t} \tag{2-4}$$

and

$$\begin{aligned}\mathbf{b} \cdot \mathbf{E} &= -\mathbf{b} \cdot \nabla\phi - \frac{1}{c}\alpha\mathbf{b} \cdot \nabla\frac{\partial\beta}{\partial t} \\ &= -\mathbf{b} \cdot \nabla\left(\phi + \frac{\alpha}{c}\frac{\partial\beta}{\partial t}\right)\end{aligned} \tag{2-5}$$

since both $\mathbf{b} \cdot \nabla\alpha = 0$ and $\mathbf{b} \cdot \nabla\beta = 0$. Thus, if one defines the potential

$$V(\mathbf{r},t) = q\phi + \frac{q\alpha}{c}\frac{\partial\beta}{\partial t} + \mu B + mG \tag{2-6}$$

the equation of motion for the parallel velocity, assuming that $m\mathbf{a} \times \mathbf{b}$ and $\mu\mathbf{b} \times \nabla B$ are of the same order of magnitude can be written to the lowest consistent order

$$m\dot{u} = -\mathbf{b} \cdot \nabla V + mu\mathbf{b} \cdot (\nabla\mathbf{b}) \cdot \dot{\mathbf{R}}_\perp \tag{2-7}$$

since to this order we make the parallel assumption that

$$\dot{\mathbf{b}} = \partial\mathbf{b}/\partial t + (u\mathbf{b} + \dot{\mathbf{R}}_\perp) \cdot \nabla\mathbf{b} \sim u\mathbf{b} \cdot \nabla\mathbf{b} \quad \sim \; u K$$

The associated expression for $\dot{\mathbf{R}}_\perp$ can be expressed as

$$\dot{\mathbf{R}}_\perp = \frac{\mathbf{b}}{m\Omega} \times \left[\nabla V + \frac{q}{c}\left(\frac{\partial\alpha}{\partial t}\nabla\beta - \frac{\partial\beta}{\partial t}\nabla\alpha\right) + mu^2\mathbf{b} \cdot \nabla\mathbf{b}\right] \tag{2-8}$$

Observe that, in the expression for $m\dot{u}$, the term involving $\mathbf{R}_\perp$ is ostensibly small compared with $-\mathbf{b} \cdot \nabla V$. We assume, moreover, that V_t is small in a sense that we shall make precise later.

Let s be the arc length along a line of force and suppose that V vs. s has the character of a potential well, as indicated schematically in Figure 4. When

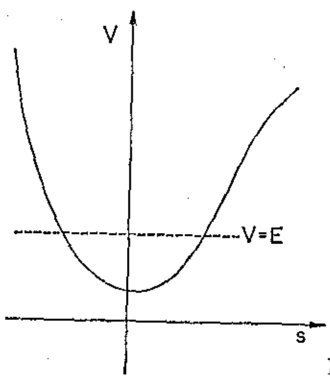

Fig. 4. A typical effective potential energy curve V vs. s.

both $\dot{\mathbf{R}}_\perp$ and V_t are zero, since

$$u = \dot{s}$$

there is a first integral of the equation of motion,

$$\tfrac{1}{2}m\dot{s}^2 + V = \text{const} = E$$

If one solves this for $\dot{s}$, it is easy to show that

$$t = \int^s ds\{2[E - V]/m\}^{-1/2}$$

Clearly, the motion is periodic with a period

$$\tau(E) = \oint ds\ \{2[E - V]/m\}^{-1/2}$$

The orbit of the particle in the s, $\dot{s}$ phase plane is the closed curve $E = \text{const}$. See Figure 5.

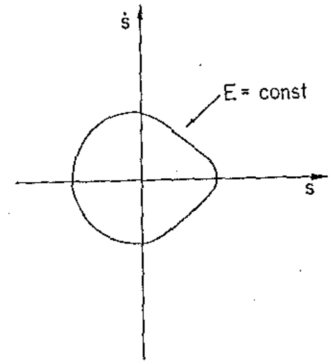

Fig. 5. A representative $\dot{s}$, s phase plane diagram for the case of constant E.

When $\dot{\mathbf{R}}$ and V_t do not vanish, the energy E will be a function of t. Suppose, however, that we extend the definition of the period $\tau(E)$ by means of the integral above to this case and assume that

$$\varepsilon = \tau|\partial \ln E/\partial t| + \dot{R}_\perp(2E/m)^{-1/2} \ll 1 \tag{2-9}$$

It seems plausible in this circumstance that the motion should be almost periodic. Let us assume so and seek a solution of the equation of motion via the introduction of an auxiliary variable $\theta(t)$ such that θ accounts for the rapid oscillation of period of the order τ, and any explicit dependence on t is associated with the slow time variation. That is, we write*

$$s = s(\theta,t) \tag{2-10}$$

whence if we define $\nu(t) \equiv \dot{\theta}$

$$\dot{s} = \nu s_\theta + s_t \tag{2-11}$$

* The symbols θ and ν are distinct from the quantities so labeled in the Introduction. We use the same symbols to illustrate the parallelism of the development.

where the subscripts denote partial derivatives. The equation of motion reads

$$v^2 s_{\theta\theta} + V_s = -2 v s_{\theta t} - \dot{v}_t s_\theta - (v s_\theta + s_t)\mathbf{b}\cdot(\nabla\dot{\mathbf{R}}_\perp)\cdot\mathbf{b} \tag{2-12}$$

where we have used the fact that $\mathbf{b}\cdot\dot{\mathbf{R}} = 0$ to write

$$(\nabla\mathbf{b})\cdot\dot{\mathbf{R}}_\perp = \nabla(\mathbf{b}\cdot\dot{\mathbf{R}}_\perp) - (\nabla\dot{\mathbf{R}}_\perp)\cdot\mathbf{b} = -(\nabla\dot{\mathbf{R}}_\perp)\cdot\mathbf{b}$$

The terms on the left-hand side above are presumably larger by a factor $1/\varepsilon$ than those on the right-hand side. Thus to the lowest order we require that the left-hand side above vanish. This requirement on multiplication by s_θ leads to

$$\left(\frac{m}{2} v^2 s_\theta^2 + V\right)_\theta = 0$$

whence on integration

$$\frac{m}{2} v^2 s_\theta^2 + V = E(t) \tag{2-13}$$

The constant of integration $E(t)$ is as yet unknown as a function of t. When one solves for s_θ from the above a further integration is possible, namely

$$\frac{\theta}{v} = \int^s ds\{2[E(t) - V(s,\alpha,\beta,t)]/m\}^{-1/2} \tag{2-14}$$

In the integrand we have indicated explicitly that the potential V depends on the point s on the line of force labeled by α and β, and by the time t. We have not indicated explicitly that it also depends on μ.

Let us pick $v = v(E,\alpha,\beta,t)$, so that θ is an angle variable; i.e., when s goes through one period of its motion for fixed α,β,t, we require that θ change by unity. Therefore,

$$\frac{1}{v} = \oint ds\{2[E - V]/m\}^{-1/2} \equiv \tau(E,\alpha,\beta,t) \tag{2-15}$$

and

$$\theta(t) = \int dt\, v \tag{2-16}$$

In order to determine $E(t)$, we revert to the equation of motion 2-13 and note that, if we retain terms to the next order in ε beyond that part which led to eq. 2-13, we find that

$$v^2 s_{\theta\theta} + V_s + 2 v s_{\theta t} + v_t s_\theta + v s_\theta \mathbf{b}\cdot(\nabla\dot{\mathbf{R}}_\perp)\cdot\mathbf{b} = 0$$

If we multiply this equation by s_θ, the result can be written

$$(\tfrac{1}{2} v^2 s_\theta^2 + V)_\theta + (v s_\theta^2)_\theta + v s_\theta^2 \mathbf{b}\cdot(\nabla\dot{\mathbf{R}}_\perp)\cdot\mathbf{b} = 0$$

If we integrate this equation with respect to θ from zero to one, and recall that $s(\theta,t)$ is presumably periodic in θ with period one, we obtain

$$\frac{\partial}{\partial t}\int_0^1 d\theta\, v s_\theta^2 + \int_0^1 d\theta\, \mathbf{b}\cdot(\nabla\dot{\mathbf{R}}_\perp)\cdot\mathbf{b}\, v s_\theta^2 = 0 \tag{2-17}$$

Let us in the above equation use s as the variable of integration and recognize that to the lowest significant order we may use eq. 2-13 to express s_θ in terms of E and V. The equation then reads on multiplication by m:

$$\frac{\partial}{\partial t}\oint ds\,\{2m[E(t) - V(s,\alpha,\beta,t]\}^{1/2}$$
$$+ \oint ds\,\mathbf{b}\cdot(\nabla\dot{\mathbf{R}}_\perp)\cdot\mathbf{b}\{2m[E(t) - V(s,\alpha,\beta,t)]\}^{1/2} = 0 \tag{2-18}$$

Note that $\partial/\partial t$ acting on the first integral above means a time derivative holding the line of force fixed. We shall now show that $ds\mathbf{b}\cdot(\nabla\dot{\mathbf{R}}_\perp)\cdot\mathbf{b}$ is just the time rate of change of the element of arc length ds due to the velocity $\dot{\mathbf{R}}_\perp$. See Figure 6.

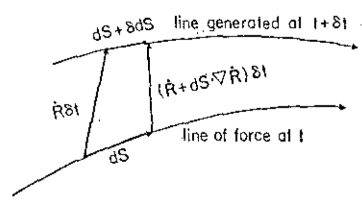

Fig. 6. Schematic diagram illustrating the calculation of the time rate of change of arc length along a line of force due to $\dot{\mathbf{R}}_\perp$.

Let us consider a vector

$$d\mathbf{s} = ds\mathbf{b}$$

In an infinitesimal time δt the end of $d\mathbf{s}$, as indicated in Figure 6, is carried a distance $\dot{\mathbf{R}}_\perp(\mathbf{R},t)\,\delta t$ by the guiding-center motion. The tip of $d\mathbf{s}$ is carried into

$$\dot{\mathbf{R}}_\perp(\mathbf{R} + d\mathbf{s},t)\,\delta t = [\dot{\mathbf{R}}_\perp(\mathbf{R},t) + d\mathbf{s}\cdot\nabla\dot{\mathbf{R}}_\perp(\mathbf{R},t) + \cdots]\,\delta t$$

The net change in $d\mathbf{s}$ is to lowest order

$$\delta d\mathbf{s} = d\mathbf{s}\cdot(\nabla\dot{\mathbf{R}}_\perp)\,\delta t$$

whence the square of the element of arc length is carried into

$$\begin{aligned}(ds + \delta ds)^2 &= (ds)^2 + 2ds \cdot (\delta ds) + \cdots \\ &= (ds)^2 + 2ds\mathbf{b} \cdot [ds\mathbf{b} \cdot (\nabla \dot{\mathbf{R}}_\perp)\, \delta t] + \cdots \\ &= (ds)^2\, [1 + 2\mathbf{b} \cdot (\nabla \dot{\mathbf{R}}_\perp) \cdot \mathbf{b}\, \delta t + \cdots]\end{aligned}$$

Thus

$$ds + \delta ds = ds\, [1 + \mathbf{b} \cdot (\nabla \dot{\mathbf{R}}_\perp) \cdot \mathbf{b}\, \delta t + \cdots]$$

and in the limit $\delta t \to 0$,

$$\frac{\delta ds}{\delta t} = \mathbf{b} \cdot (\nabla \dot{\mathbf{R}}_\perp) \cdot \mathbf{b}\, ds$$

Equation 2-18 is then to be interpreted as a time derivative of the integral following the guiding-center motion, and

$$J = \oint ds\{2m[E(t) - V(s,\alpha,\beta,t)]\}^{1/2} \tag{2-19}$$

is an approximate constant of the motion. For a given value J and known potential V this expression is an implicit equation for E. The constant J is conventionally termed the second or longitudinal adiabatic invariant.

To recapitulate then, the motion along the line is determined by eq. 2-14 with E given by eq. 2-19. The motion perpendicular to the line is then given by $\dot{\mathbf{R}}_\perp$ (see eq. 2-8), where we may replace mu^2 by $2(E - V)$. To find the trajectory associated with $\dot{\mathbf{R}}_\perp$ requires only the solution of a second-order system of ordinary differential equations.

It is interesting to note that, if the technique of this section is applied to the equation,

$$\ddot{x} + \omega(t)^2 x = 0$$

corresponding to

$$V = \tfrac{1}{2}\omega^2 x^2$$

and

$$(\dot{\omega}) \ll \omega^2$$

then it yields the well-known, lowest order WKB results.

III. The Third or Flux Invariant

When the fields involved in $\dot{\mathbf{R}}_\perp$ are changing sufficiently slowly, a notion that will be made more precise later, a further reduction in the description is possible. To demonstrate this, it is convenient to write equations for $\dot{\alpha}$ and $\dot{\beta}$, instead of dealing with $\dot{\mathbf{R}}$. To this end we view $\dot{\mathbf{R}}_\perp$ as a function of s,α,β, and t and write

$$\dot{\mathbf{R}} = \dot{s}\mathbf{R}_s + \dot{\alpha}\mathbf{R}_\alpha + \dot{\beta}\mathbf{R}_\beta + \mathbf{R}_t \tag{3-1}$$

where the subscripts indicate partial derivatives. Moreover, by the chain rule for differentiation, if $\mathscr{I}$ denotes the unit dyadic,

$$\nabla \mathbf{R} = \mathscr{I} = \nabla s \mathbf{R}_s + \nabla \alpha \mathbf{R}_\alpha + \nabla \beta \mathbf{R}_\beta \tag{3-2}$$

whence on taking the dot product on the left-hand side with $\mathbf{b}$, one has

$$\mathbf{b} = (\mathbf{b} \cdot \nabla s)\mathbf{R}_s = \mathbf{R}_s \tag{3-3}$$

since $\mathbf{b} \cdot \nabla s = s_s = 1$. If one takes the dot product of eq. 3-2 on the left-hand side with $\mathbf{b} \times \mathbf{R}_\alpha$ and $\mathbf{b} \times \mathbf{R}_\beta$,

$$\mathbf{b} \times \mathbf{R}_\alpha = \nabla \beta \mathbf{R}_\beta \cdot \mathbf{b} \times R_\alpha \tag{3-4}$$

$$\mathbf{b} \times \mathbf{R}_\beta = \nabla \alpha \mathbf{R}_\alpha \cdot \mathbf{b} \times \mathbf{R}_\beta \tag{3-5}$$

The cross product of these two equations yields

$$\begin{aligned}-\nabla \alpha \times \nabla \beta (\mathbf{b} \cdot \mathbf{R}_\alpha \times \mathbf{R}_\beta)^2 &= (\mathbf{b} \times \mathbf{R}_\beta) \times (\mathbf{b} \times \mathbf{R}_\alpha) \\ &= \mathbf{b}\mathbf{R}_\alpha \cdot \mathbf{b} \times \mathbf{R}_\beta\end{aligned}$$

But, since $\nabla \alpha \times \nabla \beta = \mathbf{B} = B\mathbf{b}$, one has

$$\mathbf{b} \cdot \mathbf{R}_\alpha \times \mathbf{R}_\beta = 1/B \tag{3-6}$$

Now the dot product of eq. 3-1 with $\mathbf{R}_s \times \mathbf{R}_\beta = \mathbf{b} \times \mathbf{R}_\beta$ yields

$$\dot{\alpha}\mathbf{R}_\alpha \cdot \mathbf{b} \times \mathbf{R}_\beta = \dot{\mathbf{R}} \cdot \mathbf{b} \times \mathbf{R}_\beta - \mathbf{R}_t \cdot \mathbf{b} \times \mathbf{R}_\beta$$

or on using eqs. 2-8, 3-5, and 3-6,

$$\begin{aligned}\dot{\alpha}/B = &-\mathbf{b} \times \mathbf{R}_\beta \cdot (\mathbf{b}/m\Omega) \\ &\times [\nabla V + (q/c)(a_t \nabla \beta - \beta_t \nabla \alpha) + 2(E - V)\mathbf{b}_s] + \mathbf{R}_t \cdot \nabla a/B\end{aligned}$$

Since, by the chain rule for differentiation, $\mathbf{R}_t \cdot \nabla \alpha = \alpha_t$, $\mathbf{R}_\beta \cdot \nabla V = V_\beta$, $\mathbf{b} \cdot \nabla V = \mathbf{R}_s \cdot \nabla V = V_s$, $\mathbf{R}_\beta \cdot \nabla \beta = \beta_\beta = 1$, $\mathbf{R}_\beta \cdot \nabla \alpha = \alpha_\beta = 0$, while $\mathbf{b} \cdot \nabla \alpha = 0$, $\mathbf{b} \cdot \nabla \beta = 0$, and since $\mathbf{b}$ is a unit vector $\mathbf{b} \cdot \mathbf{b}_s = 0$, the above reduces to

$$\begin{aligned}\dot{\alpha} &= \alpha_t(\mathbf{R}_\beta - \mathbf{b}\mathbf{b} \cdot \mathbf{R}_\beta) \cdot (c/q)[\nabla V + (q/c)(\alpha_t \nabla \beta - \beta_t \nabla \alpha) + 2(E - V)b_s] \\ &= \alpha_t - (c/q)[\mathbf{R}_\beta \cdot \nabla V + (q/c)\alpha_t \mathbf{R}_\beta \cdot \nabla \beta - (q/c)\beta_t \mathbf{R}_\beta \cdot \nabla \alpha \\ &\qquad + 2(E - V)\mathbf{R}_\beta \cdot \mathbf{b}_s - \mathbf{b} \cdot \nabla V \mathbf{b} \cdot \mathbf{R}_s] \\ &= -(c/q)[V_\beta - V_s \mathbf{b} \cdot \mathbf{R}_s + 2(E - V)\mathbf{b}_s \cdot \mathbf{R}_\beta]\end{aligned}$$

But, on recognizing that $\mathbf{R}_{\beta s} = (\mathbf{R}_s)_\beta = \mathbf{b}_\beta$, and $\mathbf{b} \cdot \mathbf{b}_\beta = 0$, this can be written

$$\dot{\alpha} = -(c/q)\{V_\beta + [2(E - V)/m]^{1/2}([2m(E - V)]^{1/2}\, \mathbf{b} \cdot \mathbf{R}_\beta)_s\}$$

Finally, if we introduce the angle variable θ in place of s, since

$$ds = [2(E - V)/m]^{1/2}\,\tau\,d\theta,$$
$$\dot\alpha = -(c/q)[V_\beta + \{\tau^{-1}\,[2m(E - V)]^{1/2}\,\mathbf{b}\cdot\mathbf{R}_\beta\}_\theta]$$

If one integrates this expression over one period in θ,

$$\int_0^1 d\theta\dot\alpha = -(c/q)\int_0^1 d\theta V_\beta$$
$$= -(c/q)\tau^{-1}\oint ds\{2(E - V)/m\}^{-1/2}V_\beta \qquad (3\text{-}7)$$

The right-hand side of eq. 3-7 can be related to the energy E, as defined implicitly by eq. 2-19, and considered to be a function of α,β,t, and of course the constants of the motion J and μ. If we take the partial derivative of eq. 2-19 with respect to β, we find that since J is an independent parameter,

$$0 = \oint ds\{2(E - V)/m\}^{-1/2}\,[E_\beta - V_\beta]$$

or on using eq. 2-15,

$$\oint ds\{2(E - V)/m\}^{-1/2}\,V_\beta = \tau E_\beta$$

Thus, if we interpret $\int_0^1 d\theta\dot\alpha$ as the time derivative of the average value of α associated with a particle over a period τ, we can cast eq. 3-7 in the form

$$\dot\alpha = -(c/q)E_\beta \qquad (3\text{-}8)$$

In similar fashion we can show that

$$\dot\beta = (c/q)E_\alpha \qquad (3\text{-}9)$$

where it is to be emphasized that α and β are the coordinates of the mean line of force on which the particle is gyrating and oscillating.

The equations of motion for α and β are in Hamiltonian form with $E(\alpha,\beta,t)$ playing the role of a time-dependent Hamiltonian. When $E_t = 0$, E is a constant of the motion, and the orbit in the α,β phase plane is the curve E = const. Suppose that this orbit is a closed curve, as shown schematically in Figure 7. Then the motion is periodic with period

$$T = (q/c)\oint d\beta/E_\alpha = (q/c)\oint d\alpha/E_\beta \qquad (3\text{-}10)$$

Suppose that

$$T|\partial \ln E/\partial t| \ll 1$$

$\dot\beta \sim \frac{1}{\sqrt{\ell m}}\,\frac{m\,L^2/t^2}{\alpha}$

UNITS $\frac{c}{q} \sim \frac{L/t}{\frac{\sqrt{m}\,\ell^{3/2}}{t}} \sim \frac{1}{\sqrt{Lm}}$ $\qquad \dot\alpha \sim \frac{1}{\sqrt{\ell m}}\,\frac{mL^2/t^2}{\beta}$

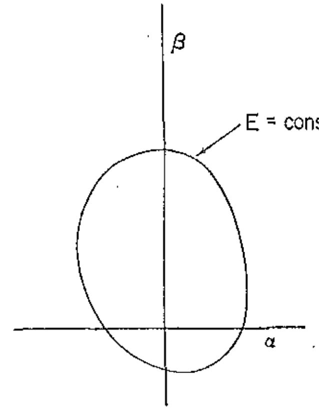

Fig. 7. Schematic diagram of constant flux surface illustrating a particle trajectory therein.

We anticipate as in the former cases that the motion will be almost periodic; we introduce an auxiliary variable $\chi(t)$ and write

$$\alpha = \alpha(\chi,t) \qquad \beta = \beta(\chi,t) \qquad (3\text{-}11)$$

If we define $\omega(t) = \dot\chi$, we can write the equations of motion as

$$\omega\alpha_\chi + \alpha_t = -(c/q)E_\beta \qquad (3\text{-}12)$$
$$\omega\beta_\chi + \beta_t = (c/q)E_\alpha \qquad (3\text{-}13)$$

To the lowest order we delete the ostensibly small term α_t and β_t and note that then

$$(c/q)E_\chi = (c/q)[\alpha_\chi E_\alpha + \beta_\chi E_\beta]$$
$$= \alpha_\chi\,\omega\beta_\chi + \beta_\chi(-\omega\alpha_\chi)$$
$$= 0$$

Thus to this order

$$E = H(t) \qquad (3\text{-}14)$$

where the constant of integration $H(t)$ is as yet undermined. Let us choose $\omega = 1/T$, where T is defined by eq. 3-10 but with the integrals extended over the closed curve $E = H$. This makes χ an angle variable, and one can formally integrate the approximate equations of motion, eqs. 3-12 and 3-13 with α_t and β_t deleted, to obtain

$$\chi T = (q/c)\int^\alpha d\alpha/E_\beta[\alpha,\beta(\alpha,t),t] = -(q/c)\int^\beta d\beta/E_\alpha[\alpha(\beta,t),\beta,t]$$

where $\beta(\alpha,t)$ is determined from $E(\alpha,\beta,t) = H(t)$, etc.

So $\frac{\dot\alpha\dot\beta}{\alpha\beta} \sim \frac{1}{\ell m}\,\frac{(mL^2/t^2)^2}{(\alpha\beta)^2}$ $\qquad (\alpha\beta)^2 \sim (BL^2)^2 \sim (E/L^3)\,L^4$

$\sim \frac{1}{\ell m}\,\frac{E}{L} \sim \frac{1}{t^2}$ Right!

In order to determine $H(t)$, we note that, without approximation of the equations of motion,

$$\begin{aligned}(c/q)E_\chi &= \alpha_\chi(c/q)E_\alpha + \beta_\chi(c/q)E_\beta \\ &= \alpha_\chi(\omega\beta_\chi + \beta_t) - \beta_\chi(\omega\alpha_\chi + \alpha_t) \\ &= \alpha_\chi\beta_t - \beta_\chi\alpha_t \\ &= -(\alpha\beta_\chi)_t + (\alpha\beta_t)_\chi\end{aligned}$$

Thus, if we integrate this result with respect to χ from zero to unity, we obtain

$$\left(\int_0^1 d\chi\, \alpha\beta_\chi\right)_t = 0$$

and to the lowest significant order

$$\psi = \oint d\beta\, \alpha \tag{3-15}$$

is an approximate constant of the motion, where the integral is extended over the closed curve $E = H$.

We shall now show that ψ is a magnetic flux. To demonstrate this, we note that the flux crossing any surface in x,y,z space is, on using Stokes theorem and $\mathbf{B} = (\nabla\alpha) \times (\nabla\beta) = \nabla \times (\alpha\nabla\beta)$,

$$\begin{aligned}\int d^2\mathbf{r} \cdot \mathbf{B} &= \int d^2\mathbf{r} \cdot \nabla \times (\alpha\nabla\beta) \\ &= \int d\mathbf{r} \cdot \alpha\nabla\beta \\ &= \int d\beta\, \alpha\end{aligned} \tag{3-16}$$

The line integral above is extended over any closed curve resulting from slicing the magnetic surface defined in x,y,z space by the equation $E(\alpha,\beta,t) = H(t)$, as shown schematically in Figure 8.

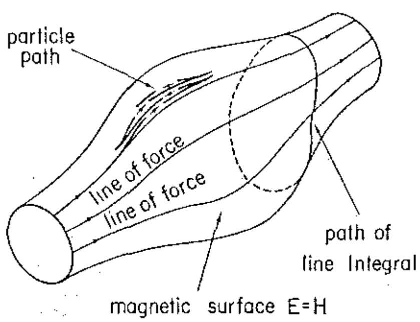

Fig. 8. Schematic diagram indicating particle path and line integral path in magnetic surface $E = H$.

Clearly, ψ is independent of the choice of line as long as it is topologically equivalent to that shown above. Equation 3-15 is then to be viewed as determining $H(t)$ implicitly, given ψ. The approximate constant ψ is conventionally termed the third adiabatic invariant, or alternatively the flux invariant.

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References

1. H. Alfvén, *Cosmical Electrodynamics*, Oxford University Press, New York, 1950.
2. M. D. Kruskal, *J. Math. Phys.*, 3, 806 (1962).
3. T. G. Northrup, *The Adiabatic Motion of Charged Particles*, Interscience, New York, 1963.
4. B. Lehnert, *Dynamics of Charged Particles*, North Holland, Amsterdam, 1964, pp. 26 ff.
5. J. Berkowitz and C. Gardner, *Communs. Pure and Appl. Math.*, **12**, 501 (1959).