Equilibrium Reconstruction with Grad-Shafranov Equation

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April 14, 2011

1 Grad-Shafranov equation

[1]For axial symmetric case, the toroidal B field is $\vec{B}_{tor} = \bar{F}\nabla\phi$. Where $2\pi\bar{F}(r,\theta)$ is the total poloidal current through the circle with radius at(r,θ). The total B field is given by

$$\vec{B} = \bar{F}\nabla\phi - \nabla\phi \times \nabla\bar{\psi}_p$$

notice that here $2\pi\bar{\psi}_p$ is the polodial flux, but defined slightly different from that in Plasma II. $\bar{\psi}_p$ is 0 at magnetic axis and is the integration of counter-clockwise B field from magnetic axis to the surface.

Then we have

$$\begin{aligned}
\mu_0 j &= \nabla \times B \\
&= \frac{d\bar{F}}{d\bar{\psi}_p}\nabla\bar{\psi}_p \times \nabla\phi - (R\frac{\partial}{\partial R}(\frac{1}{R}\frac{\partial}{\partial R}) + \frac{\partial^2}{\partial^2 Z})\bar{\psi}_p\nabla\phi \\
&= \frac{d\bar{F}}{d\bar{\psi}_p}\nabla\bar{\psi}_p \times \nabla\phi - \triangle^*\bar{\psi}_p\nabla\phi
\end{aligned}$$

thus

$$j_{torodial} = -\frac{\triangle^*\bar{\psi}_p}{\mu_0 r}$$

It should be pointed out that currently $\triangle^*$ is defined in cylindrical coordinates while other functions are defined in flux coordinates. For MHD model:

$$\begin{aligned}
\mu_0 j \times B &= \mu_0\nabla p \\
\Rightarrow \quad (-\bar{F}\frac{d\bar{F}}{d\bar{\psi}_p} - \triangle^*\bar{\psi}_p)(\nabla\phi)^2 &= \frac{d(\mu_0 p)}{d\bar{\psi}_p} \\
\Rightarrow \quad \triangle^*\bar{\psi}_p &= -r^2\frac{d(\mu_0 p)}{d\bar{\psi}_p} - \bar{F}\frac{d\bar{F}}{d\bar{\psi}_p} \\
&= -r^2 P - T
\end{aligned}$$

Grad-Shafronov equation is

$$\triangle^* \bar{\psi}_p = -r^2 P(\bar{\psi}_p) - T(\bar{\psi}_p) \tag{1}$$

and we have

$$\mu_0 j_{torodial} = rP + \frac{T}{r}$$

1.1 Green's function

Search for Greens function $G(r, z, r', z')$ such that[2]

$$\begin{aligned} \frac{\triangle^*_{r,z}}{r} G(r, z, r', z') &= \delta(r - r', z - z') \\ G|_{r,z\to\infty} &\to 0 \end{aligned} \tag{2}$$

In fact the poloidal component of B field:

$$-\nabla\phi \times \nabla\bar{\psi}_p = \nabla \times (\bar{\psi}_p \nabla\phi)$$

and $\bar{\psi}_p$ is thus the ϕ component of magnetic potential A_ϕ corresponding to the B field. The physical meaning of the RHS of (2) is a ring current $(1/\mu_0)$ locating at (r', z') and its magnectic potential is

$$\begin{aligned} \vec{A}(\vec{r}) &= \frac{\mu_0}{4\pi} \cdot \int \frac{j(\vec{r'})}{|\vec{r} - \vec{r'}|} d^3 r' \\ &= (\vec{\nabla}\phi) \cdot \frac{rr'}{4\pi} \cdot \int_{-\pi}^{\pi} \frac{cos(\theta)}{\sqrt{r^2 + r'^2 - 2rr'cos(\theta)}} d\theta \end{aligned}$$

thus

$$G(r, z, r', z') = \frac{rr'}{4\pi} \cdot \int_{-\pi}^{\pi} \frac{cos(\theta)}{\sqrt{r^2 + r'^2 - 2rr'cos(\theta)}} d\theta \tag{3}$$

1.2 Solution of GSh equation

We only want to solve GSh equation within a domain:

$$\frac{\triangle^*}{r} \bar{\psi}_p = f(r, z) \quad \forall (r, z) \in \Omega \tag{4}$$

we can extend Ω to the whole space to include the external coils and have

$$\begin{aligned} \bar{\psi}_p(r, z) &= \iint f(r', z') G(r, z, r', z') dr' dz' \\ &\quad + \sum_i I_i G(r, z, r'_i, z'_i) \end{aligned} \tag{5}$$

where I_i denote external coils. This is not very numerically efficient because if Ω is a $N - by - N$ space and $f(r, z)$ is non-zero almost every where in the

domain then this would require $O(N^4)$ operations to obtain the value of $\bar{\psi}_p$ at every point.

Another way to solve this is to first construct the boundary value $\bar{\psi}_p(r,z)|_{(r,z)\in\partial\Omega}$ using (5) (Which will take $O(N^3)$ operations) and then discretize (4) and solve it numerically (Which is expected to take $O(N^3)$ operations). [4]

To construct $\bar{\psi}_p$ on the boundary, we still need to calculate Green's function $O(N^3)$ times beforehand and store them in the memory. We can improve it further more.[3]

$$\begin{aligned}\frac{\triangle^* u}{r} &= (\frac{\partial}{\partial r}(\frac{1}{r}\frac{\partial}{\partial r}) + \frac{\partial^2}{\partial^2 z})u \\ &= \vec{\nabla}\cdot(\frac{1}{r}\vec{\nabla}u)\end{aligned}$$

using Green's identity

$$\begin{aligned}&\int G(r,z,r',z')\frac{\triangle^*}{r}\bar{\psi}_p(r,z) - \bar{\psi}_p(r,z)\frac{\triangle^*}{r}G(r,z,r',z')drdz \\ =& \int G(r,z,r',z')f(r,z) - \bar{\psi}_p(r,z)\delta(r-r',z-z')drdz \\ =& \int G(r,z,r',z')f(r,z)drdz - \bar{\psi}_p(r',z')\end{aligned}$$

since

$$\begin{aligned}&\int_\Omega G(r,z,r',z')\frac{\triangle^*}{r}\bar{\psi}_p(r,z) - \bar{\psi}_p(r,z)\frac{\triangle^*}{r}G(r,z,r',z')drdz \\ =& \int_\Omega G(r,z,r',z')\vec{\nabla}\cdot(\frac{1}{r}\vec{\nabla}\bar{\psi}_p(r,z)) - \bar{\psi}_p(r,z)\vec{\nabla}\cdot(\frac{1}{r}\vec{\nabla}G(r,z,r',z'))drdz \\ =& \int_{\partial\Omega}\frac{G(r,z,r',z')}{r}\frac{\partial}{\partial n}\bar{\psi}_p(r,z) - \frac{\bar{\psi}_p(r,z)}{r}\frac{\partial}{\partial n}G(r,z,r',z')\ dS_{r,z}\end{aligned}$$

we have

$$\begin{aligned}\bar{\psi}_p(r',z') &= \int_\Omega G(r,z,r',z')f(r,z)drdz \\ &+ \int_{\partial\Omega}\frac{\bar{\psi}_p(r,z)}{r}\frac{\partial}{\partial n}G(r,z,r',z') - \frac{G(r,z,r',z')}{r}\frac{\partial}{\partial n}\bar{\psi}_p(r,z)\ dS_{r,z} \quad (6)\end{aligned}$$

Notice that the first term of RHS of (6) is what we need in (5), we can obtain the value of this term by the following procedure:

Solve the equation numerically by discretizing the operator

$$\begin{aligned}\frac{\triangle^*}{r}\tilde{\psi}_p &= f(r,z) \quad \forall(r,z)\in\Omega' \\ \tilde{\psi}_p|_{\partial\Omega'} &= 0\end{aligned}$$

where the domain Ω' is larger than Ω. Then calculate $\frac{\partial}{\partial n}\tilde{\psi}_p$ on the boundary and since $\tilde{\psi}_p$ is zero on the boundary, we have

$$\begin{aligned}\int_{\Omega} G(r,z,r',z')f(r,z)drdz &= \int_{\Omega'} G(r,z,r',z')f(r,z)drdz \\ &= \tilde{\psi}_p(r',z') + \int_{\partial\Omega'} \frac{G(r,z,r',z')}{r}\frac{\partial}{\partial n}\tilde{\psi}_p(r,z)\ dS_{r,z}\end{aligned}$$

and we only need to know the value of $G(r,z,r',z')$with $(r,z) \in \partial\Omega,\ (r',z') \in \partial\Omega'$. Substitute this into (5) we have

$$\begin{aligned}\bar{\psi}_p(r,z) &= \tilde{\psi}_p(r,z) + \int_{\partial\Omega'} \frac{G(r',z',r,z)}{r'}\frac{\partial}{\partial n}\tilde{\psi}_p(r',z')\ dS_{r',z'} \\ &\quad + \sum_i I_i G(r,z,r_i',z_i')\end{aligned} \tag{7}$$

Use (7) to calculate $\bar{\psi}_p(r,z)$ on the boundary and then solve it numerically.

1.3 Non-linearity of GSh equation.

In the previous section we are assuming the RHS of GSh equation is $f(r,z)$ while in reality the RHS of (1) is $f(\bar{\psi}_p)$ which depends on the function of $p(\bar{\psi}_p)$ and $\bar{F}(\bar{\psi}_p)$. Even assuming that p and $\bar{F}$ is given. f is a function of (r,z) in a sense that we already know the function $\bar{\psi}_p(r,z)$ and $f(r,z) = f(\bar{\psi}_p(r,z))$. But $\bar{\psi}_p(r,z)$ is the function that we are solving for!

One way to treat this problem is to solve this equation iteratively. Start with an initial guess of $\bar{\psi}_{p,0}(r,z)$ and use this to calculate $f_{n+1}(r,z) = f(\bar{\psi}_{p,n}(r,z))$ and then solve the equation to obtain $\bar{\psi}_{p,n+1}$, iterate until $\bar{\psi}_p$ converges.

In fact, the plasma boundary can also change during each iteration. Plasma boundary is determined by a set of limiters located at (r_s, z_s) in this case and the limiter with lowest $\bar{\psi}_{p,n,min} = \bar{\psi}_{p,n}(r_s,z_s)$ determines the boundary of plasma. The function of $\bar{p}$ and $\bar{F}$ are actually functions of normalized poloidal flux $\bar{p}(\bar{\psi}_p(r,z)/\bar{\psi}_{p,min}(r,z))$, $\bar{F}(\bar{\psi}_p(r,z)/\bar{\psi}_{p,min}(r,z))$ because they are defined on the plasma region.

1.4 GSh equation on flux coordinates

[1]Solving GSh equation on (r,z) coordinates gives simplicity in discretizing the operator $\triangle^*$, but it it may not be the most efficient way. Use general coordinates (a,θ) to describe the (r,z) plane, where a can be treated as a surface label and θ the poloidal angle. Let

$$D = \frac{D(r,z)}{D(a,\theta)}$$

then

$$\begin{aligned}
\frac{\triangle^*}{r} f &= \vec{\nabla} \cdot (\frac{1}{r} \vec{\nabla} f) \\
&= \frac{1}{D} \frac{\partial}{\partial a} (\frac{DG_{a,a}}{r} \frac{\partial}{\partial a} f) + \frac{1}{D} \frac{\partial}{\partial a} (\frac{DG_{a,\theta}}{r} \frac{\partial}{\partial \theta} f) \\
&\quad + \frac{1}{D} \frac{\partial}{\partial \theta} (\frac{DG_{\theta,a}}{r} \frac{\partial}{\partial a} f) + \frac{1}{D} \frac{\partial}{\partial \theta} (\frac{DG_{\theta,\theta}}{r} \frac{\partial}{\partial \theta} f)
\end{aligned}$$

where

$$G_{a,a} = \frac{\partial \vec{x}}{\partial a} \cdot \frac{\partial \vec{x}}{\partial a} \qquad G_{a,\theta} = G_{\theta,a} = \frac{\partial \vec{x}}{\partial a} \cdot \frac{\partial \vec{x}}{\partial \theta} \qquad G_{\theta,\theta} = \frac{\partial \vec{x}}{\partial \theta} \cdot \frac{\partial \vec{x}}{\partial \theta}$$

The GSh equation is thus

$$\begin{aligned}
\frac{D\triangle^*}{r} \bar{\psi}_p &= -rDP - \frac{D}{r} T \\
&= -VP - LT \quad where\ V \equiv rD,\ L = \frac{D}{r}
\end{aligned}$$

If (a, θ) is already the flux coordinate (a^*, θ) that

$$\bar{\psi}_p = \bar{\psi}_p(a^*), \quad \bar{p} = \bar{p}(a^*), \quad \bar{F} = \bar{F}(a^*)$$

then we shall have

$$\frac{\partial}{\partial a^*} (\frac{DG_{a,a}}{r} \bar{\psi}_p') + \frac{\partial}{\partial \theta} (\frac{DG_{\theta,a}}{r} \bar{\psi}_p') = -VP - LT \tag{8}$$

averaging over θ we have

$$((\frac{DG_{a,a}}{r})_0 \bar{\psi}_p')' = -TL_0 - PV_0, \quad where\ (f)_0 = \frac{1}{2\pi} \int f d\theta \tag{9}$$

Define the total cross-section current

$$\begin{aligned}
\bar{J}(a^*) &= \frac{1}{2\pi} \int_0^{2\pi} d\theta \int_0^{a^*} \mu_0 j_{toroidal} \frac{D(r,z)}{D(a,\theta)} da \\
&= \frac{1}{2\pi} \int_0^{2\pi} d\theta \int_0^{a^*} rDP + \frac{DT}{r} da \\
&= \int_0^{a^*} V_0 P + L_0 T da
\end{aligned}$$

Compare it with (9) we have:

$$((\frac{DG_{a,a}}{r})_0 \bar{\psi}_p')|_0^{a^*} = -\bar{J}(a^*)$$

and since

$$\begin{aligned}
B_\theta(a = 0) &= 0 \\
\Rightarrow \bar{\psi}_p'|_{a=0} &= 0
\end{aligned}$$

we have

$$(\frac{DG_{a,a}}{r})_0 \bar{\psi}_p' = -\bar{J} \tag{10}$$

Which is an 1D equation.

1.5 Perturbed GSh equation

[1]When solving the GSh equation, we don't really have the flux coordinate (a^*, θ) until we get the true solution. However, we can use the coordinate (a, θ) generated from each approximation of $\bar{\psi}_p$ and assume that (a, θ) is close to (a^*, θ) in the sense that

$$a^* = a - \xi(a, \theta)$$

then (r, z) which is initially a function of (a, θ) can be expressed as

$$\begin{aligned} r &= r(a, \theta) = r(a^* + \xi, \theta) \approxeq r(a^*, \theta) + r'_a \xi \\ z &= z(a, \theta) = z(a^* + \xi, \theta) \approxeq z(a^*, \theta) + z'_a \xi \end{aligned}$$

the surface of (a, θ) has to satisfy that

$$\bar{\psi}_p(a, \theta) \equiv \bar{\psi}_p^0 + \psi(a, \theta) = const$$

where $\bar{\psi}_p^0$ is the main order approximation and can be obtained by solving the equation (10), in the coordinate (a, θ) but taken as (a^*, θ). Thus:

$$\bar{\psi'}_p^0 \xi + \psi = const$$

Notice that ξ have the part ξ_0 and oscillatory part $\tilde{\xi}$ thus

$$\begin{aligned} \tilde{\xi} &= -\frac{\tilde{\psi}}{\bar{\psi'}_p^0} \\ \xi_0 &= \frac{\bar{\psi}_p(a, \theta) - \bar{\psi}_p^0 - \psi_0}{\bar{\psi'}_p^0} \end{aligned}$$

chose $\bar{\psi}_p(a, \theta) = \bar{\psi}_p^0$ then

$$\xi = -\frac{\psi}{\bar{\psi'}_p^0}$$

Now, GSh equation can be linearized:

$$\begin{aligned} \triangle^* \bar{\psi}_p &= -r^2 P(a^*) - T(a^*) \\ with \quad P(a^*) &= P(a) - P'\xi + \delta P \\ T(a^*) &= T(a) - T'\xi + \delta T \end{aligned}$$

(δP and δT denote the variations of the given function profile which will be used later.) such that

$$\mathcal{L}\bar{\psi}_p = -VP - LT + VP'\xi + LT'\xi - V\delta P - L\delta T \tag{11}$$

For the previous algorithms, at each step, when $\bar{\psi}_p^n \to \bar{\psi}_p^{n+1}$, only the change to the LHS of the above equation is considered. If one can also calculated the related ξ due to the change in $\bar{\psi}_p$, and use both of them in the above equation, a better converging rate should be achieved.

2 Equilibrium reconstruction

Solving GSh equation is one major part of reconstructing the equilibrium profile. Yet, one also needs to have the profile of $\bar{P}(\bar{\psi}_p)$ and $\bar{T}(\bar{\psi}_p)$ to solve the equation. These profiles are the input parameters for GSh equation solver and are often determined by the measurement from probes.

We start with an initial guess of the profiles with parameters

$$\begin{aligned} \bar{P}(\bar{\psi}_p) &= \bar{P}(\alpha_1, \alpha_2, \ldots, \alpha_n, \bar{\psi}_p) \\ \bar{T}(\bar{\psi}_p) &= \bar{T}(\beta_1, \beta_2, \ldots, \beta_n, \bar{\psi}_p) \end{aligned}$$

and solve for $\bar{\psi}_p(r, z)$. Then we have $\bar{P}(\alpha_1, \alpha_2, \ldots, \alpha_n, r, z)$ and $\bar{T}(\beta_1, \beta_2, \ldots, \beta_n, r, z)$, from which we can get the expect reading from the ith probe

$$E_i(\alpha_1, \alpha_2, \ldots, \alpha_n, \beta_1, \beta_2, \ldots, \beta_n)$$

Since we also have the actually readings, the parameters can be adjusted to better fit the reading. Update the parameters and solve the GSh equation again until it converges. [4]

One can see that the convergence rate does not tend to be very fast, because the fitting procedure does not take into account that $\bar{\psi}_p(r, z)$ would be changed when the parameters change. A better way to do it might be using δP and δT in the equation (11) and obtain an estimated response of $\bar{\psi}_p$. Taking that into account when doing the fitting procesure may result in a better convergence rate.

2.1 Effects of measuring techniques on reconstruction.

It is a commen sense that the more accurate your measurements are, the better reconstruction you will get. But, it would be better if one can tell quantitatively how does the accuracy of the probe readings affects the credibility of one's reconstruction result. In L.E. Zakharov's paper[5], he gives the relations quantitatively.

The idea is to view errors as perturbations and see how large the magnitude of the perturbation could be as long as they are undetectable by the device. When we change the P and T profile by δP and δT, it will result in a change of $\delta\bar{\psi}_p$ and eventually a change of signals we read, δS. For smal pertubation, we can linearize the relationship and have

$$\delta S = A\delta X$$

where δS is a N dimensional vector of all the signals from probes and δX are a M dimensional discrete version of the change of δP and δT. Since all probes have certain degrees of error allowance, we use $\delta\epsilon$ to denote that. Normalize each row of $A(i, :)$ with $\delta\epsilon_i$ and we get $\tilde{A}$. Thus,

$$\delta\tilde{S} = \tilde{A}\delta X$$

and each entry of $\delta\tilde{S}$ has to have a magnitude less than 1 so that the perturbation is within the error range of probes. One can single value decompse

$$\tilde{A} = UWV^T$$

and for $\delta X_k = \gamma V_k$ we have $\delta\tilde{S}_k = \gamma w_k U_k$. Set certain constrains for the norm of δS and we'll get the error range δX for the reconstruction.

References

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