dissipation, and small masses (10^{15}–10^{17} kg), these devices are well suited to such explorations. Their dimensions not only make them susceptible to local forces, but also make it possible to integrate and tightly couple them to a variety of interesting electronic structures, such as solid-state two-level systems (quantum bits, or qubits), that exhibit quantum mechanical coherence. In fact, the most-studied systems, nanoresonators coupled to various superconducting qubits, are closely analogous to cavity quantum electrodynamics, although they are realized in a very different parameter space.

Quantized nanomechanical resonators

The classical and quantum descriptions of a mechanical resonator are very similar to those of the electromagnetic field in a dielectric cavity: The position- and time-dependent mechanical displacement $u(r,t)$ is the dynamical variable analogous to the vector potential $A(r,t)$. In each case, a wave equation constrained by boundary conditions gives rise to a spectrum of discrete modes. For sufficiently low excitation amplitudes, for which nonlinearities can be ignored, the energy of each mode is quadratic in both the displacement and momentum, and the system can be described as essentially independent simple harmonic oscillators.

Spatially extended mechanical devices, such as those in figure 1, possess a total of $3^A$ modes of oscillation, where $A$ is the number of atoms in the structure. Knowing the amplitude and phase of all the mechanical modes is equivalent to having complete knowledge of the position and momentum of every atom in the device. Continuum mechanics, with bulk parameters such as density and Young's modulus, provides an excellent description of the mode structure and the classical dynamics, because the wavelengths (100 nm–10 mm) of the lowest-lying vibrational modes are long compared to the interatomic spacing.

It is natural to make the distinction between nanomechanical modes and phonons: The former are low-frequency, long-wavelength modes strongly affected by the boundary conditions of the nanodevice, whereas the latter are vibrational modes with wavelengths much smaller than typical device dimensions. Phonons are relatively unaffected by the geometry of the resonator and, except in devices such as nanotubes that approach atomic dimensions, are essentially identical in nature to phonons in an infinite medium.

It is an assumption that quantum mechanics should even apply for such a large, distributed mechanical structure. Setting that concern aside for the moment, one can follow the standard quantum mechanical protocol to establish that the energy of each mode is quantized:

$$E \subset N + \frac{1}{2},$$

where $N \subset 0, 1, 2, \ldots$ is the occupation factor of the mechanical mode of angular frequency $\omega$. The http://www.physicstoday.org July 2005    Physics Today

Putting Mechanics into Quantum Mechanics

![Figure 1. Nanoelectromechanical devices.](a) A 200 MHz nanomechanical resonator capacitively coupled to a single-electron transistor (Keith Schwab, Laboratory for Physical Sciences).

(b) An ultrasensitive magnetic force detector that has been used to detect a single electron spin (Dan Rugar, IBM).

(c) A torsional resonator used to study Casimir forces and look for possible corrections to Newtonian gravitation at short length scales (Ricardo Decca, Indiana University–Purdue University Indianapolis).

(d) A parametric radio-frequency mechanical amplifier that provides a thousandfold boost of signal displacements at 17 MHz (Michael Roukes, Caltech).

(e) A 16 MHz nanomechanical resonator coupled to a single-electron transistor (Andrew Cleland, University of California, Santa Barbara).

(f) A unable carbon nanotube resonator operating at 3–300 MHz (Paul McEuen, Cornell University).
Nanoelectromechanical structures are starting to approach the ultimate quantum mechanical limits for detecting and exciting motion at the nanoscale. Nonclassical states of a mechanical resonator are also on the horizon.

Keith C. Schwab and Michael L. Roukes

Today, micro- and nanoelectromechanical systems (MEMS and NEMS) are widely employed in ways similar to those early force detectors, yet with vastly greater force and mass sensitivity—now pushing into the realm of zeptonewtons (10^{-21} N) and zeptograms (10^{-21} g). These ultraminiature sensors also can provide spatial resolution at the atomic scale and vibrate at frequencies in the gigahertz range. Among the breadth of applications that have become possible are measurements of forces between individual biomolecules, forces arising from magnetic resonance of single spins, and perturbations that arise from mass fluctuations involving single atoms and molecules. The patterning of mechanical structures with nanometer-scale features is now commonplace; figure 1 and the cover display examples of current devices.
What does it take to observe the quantum nature of an ordinary system?
Approaching the Quantum Limit of a Nanomechanical Resonator

M. D. LaHaye,¹,² O. Buu,¹,² B. Camarota,¹,² K. C. Schwab¹*

By coupling a single-electron transistor to a high-quality factor, 19.7-megahertz nanomechanical resonator, we demonstrate position detection approaching that set by the Heisenberg uncertainty principle limit. At millikelvin temperatures, position resolution a factor of 4.3 above the quantum limit is achieved and demonstrates the near-ideal performance of the single-electron transistor as a linear amplifier. We have observed the resonator’s thermal motion at temperatures as low as 56 millikelvin, with quantum occupation factors of $N_{\text{TH}} = 58$. The implications of this experiment reach from the ultimate limits of force microscopy to qubit readout for quantum information devices.
Outline

1. Mechanical resonator
2. Quantum modes and occupancy
3. SSET (superconducting single-electron transistor)
4. Measurements...
1. Mechanical resonator

2. Quantum modes and occupancy

3. SSET (superconducting single-electron transistor)

4. Measurements...

100 fm = 10^{-13} m \sim \text{size of 12 gold atoms}
19.7 MHz Nanomechanical Resonator

(C) Details of the 19.7-MHz nanomechanical resonator (200 nm wide, 8 µm long, coated with 20 nm of Au atop 100 nm SiN), defined by the regions in black where the SiN has been etched through. The SSET island (5 µm long and 50 nm wide) is positioned 600 nm away from the resonator. Tunnel junctions, marked “J,” are located at corners. A 70-nm-thick gold gate is positioned to the right of the resonator and is used both to drive the resonator and to control the bias point of the SSET.
Vibrating Beam

\[ 2\pi f = \sqrt{\frac{k}{m}} \]

\[ m = 9.7 \times 10^{-16} \text{ kg} \]
Quantum Harmonic Oscillator

\[ i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} + V(x)\psi \]

\[ E_n \psi = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi_n}{\partial x^2} + V(x)\psi_n \]

\[ \psi_n(x) \propto e^{-\beta^2 x^2/2} H_n(\beta x) \]

\[ E_n = \hbar \omega \left( \frac{1}{2} + n \right) = \hbar \sqrt{k/m} \left( \frac{1}{2} + n \right) \]
Quanta of Oscillations
(Phonons)

<table>
<thead>
<tr>
<th>$T$</th>
<th>$n_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.4 mK</td>
<td>1</td>
</tr>
<tr>
<td>2.3 mK</td>
<td>2</td>
</tr>
<tr>
<td>47.7 mK</td>
<td>50</td>
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</table>

$$n_i = \frac{1}{\exp(\frac{\hbar \omega}{kT}) - 1} \rightarrow \frac{kT}{\hbar \omega} \text{ for } kT \gg \hbar \omega$$
Vibrating Beam

\[ \rho \frac{\partial^2 \mathbf{u}}{\partial t^2} = \nabla \cdot \mathbf{\tau} \]

\[ \mathbf{\tau} = 2 \mu \varepsilon \]

\[ \rho \frac{\partial^2 u_x}{\partial t^2} = \frac{\partial}{\partial z} \left( \mu \frac{\partial u_x}{\partial z} \right) \]

\[ \rho A \frac{\partial^2 \delta x}{\partial t^2} = \int dx dy \mu \frac{\partial^2 \delta x}{\partial z^2} \]

\[ 2\pi f = \frac{\pi}{2L} \sqrt{\frac{E I \pi^2}{\rho A L^2}} \]
# Young's & Shear Modulus (E, μ)

<table>
<thead>
<tr>
<th>Material</th>
<th>Typical values for shear modulus (GPa)</th>
<th>Young's Modulus (GPa)</th>
<th>Mass Density (kg/m³)</th>
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<tbody>
<tr>
<td>Diamond[2]</td>
<td>478</td>
<td>1220</td>
<td>3,530</td>
</tr>
<tr>
<td>Steel[3]</td>
<td>79</td>
<td>200</td>
<td>~ 8,000</td>
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<tr>
<td>Copper[4]</td>
<td>45</td>
<td>117</td>
<td>8,960</td>
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<td>Glass[3]</td>
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<tr>
<td>Aluminium[3]</td>
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<td>69</td>
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<td>Rubber[5]</td>
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<td>~0.05</td>
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<tr>
<td>Gold</td>
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<td>79</td>
<td>19,300</td>
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<tr>
<td>Si₃N₄</td>
<td>~100</td>
<td>310</td>
<td>3,440</td>
</tr>
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</table>

\[
2\pi f = \frac{\pi}{2L} \sqrt{\frac{E I \pi^2}{\rho A L^2}}
\]
Fig. 1. (Bottom) The solid lines show the expected position resolution due to shot-noise (black), back-action noise (red), and the uncorrelated sum (blue) as a function of coupling voltage $V_{NR}$, assuming the device parameters realized in this experiment. The points are the observed sensitivity where the deviation from the blue curve is due to nonidealities in the RF-SSET readout circuit. The dashed line is the expected sensitivity calculated from the measured charge sensitivity. Error bars are on the quantity $\Delta X/\Delta X_{QL}$. (Top) The simplified schematic shows the RF-SSET capacitively coupled to a voltage-biased Au/SiN nanomechanical resonator on-chip LC resonator formed by the square spiral, $L_T$, and an interdigitated capacitor, $C_T$.

Fig. 2. Colorized scanning electron micrographs of the sample. (A) Metallizations (170 nm Al / 20 nm Ti / 20 nm Au) atop a [100] silicon wafer coated with 100 nm of SiN, which has been back-etched using KOH to form a 55-μm by 55-μm SiN membrane (shown as the black square in the center.) The Al/Ti/Au film is in contact with the silicon, which both provides electronic protection for the delicate device at room temperature and superconducts below 0.8 K. The inset on the left shows the 130-μm by 130-μm square coil used for the 1.35 GHz LC resonator. (B) SiN membrane (dark square), the Al leads to the SSET, and the Au leads to the nanomechanical resonator and electrostatic gates. (C) Details of the 19.7-MHz nanomechanical resonator (200 nm wide, 8 μm long, coated with 20 nm of Au atop 100 nm SiN), defined by the regions in black where the SiN has been etched through. The SSET island (5 μm long and 50 nm wide) is positioned 600 nm away from the resonator. Tunnel junctions, marked “J,” are located at corners. A 70-nm-thick gold gate is positioned to the right of the resonator and is used both to drive the resonator and to control the bias point of the SSET.
The SET is based on the tunnel effect through a metal-insulator-metal junction.

When two metallic electrodes are separated by an insulating barrier whose thickness is about 1 nm, electrons can traverse the insulator even though their energy is too low to overcome, in a classical motion, the large potential barrier of the insulating region. The tunnel effect manifests itself by a finite resistance $R_T$ of the insulating barrier.

For barriers with small tunneling, the charge $Q$ transferred through the barrier becomes quantized. $Q = Ne$, where $N$ is an integer. For $N$ not to be subject to quantum fluctuations, the resistance of the junction must be large compared with the resistance quantum $RT \gg h/e^2 = 25.8$ k Ohm.

If the dimensions of the island are sufficiently small, the charging energy, $e^2/(2C_i)$, of one extra electron in the island will become larger than the energy of thermal fluctuations, $kT$.

In practice, for devices fabricated by standard electron-beam lithography, $C_i$ is of the order of a FF and the charging energy is of order 1 K. Temperatures must be below 300 mK to satisfy the above charging energy criterion.

**Figure 3** The single-electron tunnelling transistor (SET). a. Simplified three-dimensional structure of the SET. The channel of the FET is replaced here by a sandwich consisting of a nanoscale metal electrode (island), which is connected to the drain and the source by tunnel junctions. As in the FET, a gate electrode influences the island electrostatically. b. Circuit diagram of the SET. The square box symbol represents a tunnel junction, and integers $N_i$ and $N_f$ denote the numbers of electrons having tunnelled through the two junctions. Each junction is characterized by its capacitance and its tunnel resistance.
Noise Power at Resonance

Fig. 3. Charge noise power around the mechanical resonance with $V_{NR} = 15$ V. Right peak is taken at 100 mK and is fit with a Lorentzian, shown as a red line. This noise power is used to scale the left peak taken with the refrigerator at 35 mK and corresponds to a resonator noise temperature of $T_{NR}^{\text{res}} = 73$ mK. This then scales the white-noise floor, which corresponds to a system-noise temperature of $T_{SSET}^{\text{res}} = 16$ mK = $18 T_{QL}$. Using the equipartition relation, the displacement resolution is 3.8 fm/√Hz. The inset shows the driven response, approximately 800 pm on resonance, with the data as circles and a Lorentzian fit as the solid lines. All SSET measurements are taken with the SSET biased near the double Josephson quasiparticle resonance peak.
Fig. 4. The integrated charge noise power, PNR, scaled by $V_{NR}$, versus refrigerator temperature for different $V_{NR}$. Right axis shows the quantum occupation factor, $N_{TH}$. Above 100 mK, we find excellent agreement with classical equipartition of energy, $P_{NR} \propto T$, shown as the solid line through the origin. Below 100 mK, we observe a deviation from this relationship, indicating a difficulty in thermalizing the nanomechanical mode. The arrow indicates the lowest observed noise temperature, $T_{NR}^{N} = 56$ mK and $N_{TH} = 58$. The upper plot shows both the quality factor, Q, and the resonant frequency shift, $\Delta F = F(T) - F(35 \text{ mK})$, versus temperature, which are extracted by fitting the thermal noise peaks at $V_{NR} = 6$ V.
Although our measurements at 20 MHz are essentially immune to nonintrinsic noise, which is ubiquitous at acoustic frequencies, it is interesting to compare our approach to the quantum limit with the current sensitivity of ultrasensitive gravitational wave detectors. The 4-km Laser Interferometer Gravitational-Wave Observatory (LIGO) interferometric detector has achieved $\Delta x = 1000 \cdot \Delta x_{QL}$ (22) at 100 Hz. A tabletop optical interferometer has achieved $\Delta x = 23 \cdot \Delta x_{QL}$ on the 2 MHz vibrational modes of a 100-g silica mirror at room temperature (23). The best performance on the readout of displacement transducers for cryogenic, acoustic gravitational wave detectors at 1 KHz is $\Delta x = 167 \cdot \Delta x_{QL}$ (24), with thermal occupation $N_{TH} \sim 10^9$. 

Figure 2 | View into the GEO 600 central building. In the front, the squeezing bench containing the squeezed-light source and the squeezing injection path is shown. The optical table is surrounded by several vacuum chambers containing suspended interferometer optics.