1. Quick Review
2. Vorticity
3. Kelvin's Theorem
4. Examples

How to solve fluid problems?
(Like those in textbook)

STEP #1 - DRAW A PICTURE
- WRITE DOWN ALL KNOWN GEOMETRY
  AND FORCINGS

STEP #2 - IS THIS A DYNAMICAL OR STATIC PROBLEM?
- MOST DYNAMICS ARE SOLVED BY LINEARIZATION
  (UNLESS OTHERWISE REQUIRED) BECAUSE NONLINEAR
  PROBLEMS ARE MUCH MORE DIFFICULT.

STEP #3 - SELECT THE MOST CONVENIENT COORDINATE
  SYSTEM. (NOTE: CURVILINEAR COORDINATES
  IN APPENDIX B.)

STEP #4 - CAN YOU APPLY CONSERVATION PRINCIPLES
  (MOMENTUM, ANGULAR MOMENTUM, ENERGY)
  AND STEVEN'S PRINCIPLES TO CONSTRAINT
  YOUR SOLUTION?

STEP #5 - APPLY YOUR INTUITION!
Hints with HW #2

Vorticity Dynamics

\[ \bar{\omega} = \text{Vorticity, a vector field} \]

\[ = \nabla \times \bar{u} \quad \text{(Thus, } \nabla \cdot \bar{\omega} = 0 \quad \bar{\omega} \text{ is solenoidal.)} \]

\[ \nabla \times \left[ \frac{2}{\varepsilon} \bar{u} + \bar{\omega} \times \bar{u} \right] = \bar{\omega} - \frac{1}{\rho} \nabla \rho - \nabla (\frac{1}{2} u^2) + \mu [\nabla^2 \bar{u} + \frac{1}{3} \nabla (\nabla \cdot \bar{u})] \]

\[ \frac{2\bar{\omega}}{2\varepsilon} + \nabla \times (\bar{\omega} \times \bar{u}) = \mu \nabla^2 \bar{\omega} \]

Vorticity Equation!

(Remember the importance of viscosity.)
Simple (2D) Fluid Rotation

Cylindrical coordinates:

Rigid Rotation

\[ \vec{U}_0 = \omega \hat{r} \]

\[ \nabla \times \vec{U} = 0 \Rightarrow \text{No Viscous Dissipation} \]

Vorticity is constant everywhere.

\[ \nabla \times (\nabla \times \vec{U}) = \frac{1}{\rho} \nabla \rho \]

Line Vortex

\[ \Gamma = \text{Circulation} \]

\[ \vec{U}_0 = \frac{\Gamma}{2\pi r} \]

3D Fluid Flow

CFD calculation of flow in mixing tank

CFD calculation of atmosphere (Iso-Vorticity contours)
Kelvin's Theorem

\[ \frac{d}{dt} \int_S \mathbf{J} \cdot d\mathbf{A} = 0 \quad \text{(as } \mu \to 0) \]

\[ \int_S \mathbf{J} \cdot d\mathbf{A} = \left\{ \left( \mathbf{V} \times \mathbf{A} \right) \cdot d\mathbf{A} \right\} = \int_S \mathbf{U} \cdot d\mathbf{E} = \Gamma \]

\text{CIRCULATION \ OF \ FLUID \ \ IS \ \ CONSTANT \ \ AS } \mu \ \to \ 0

\text{VORTEX TUBE IS A TUBE OF CONSTANT VORTICITY FLUX.}

\text{CIRCULATION WITHIN TUBE IS CONSTANT AS IT MOVES WITH FLUID (AS } \mu \to 0)\]

A (General) Vector Theorem for a Moving Surface Flux

\[ \frac{d}{dt} \left[ \int_S \mathbf{a} \cdot d\mathbf{A} \right] = \lim_{\delta t \to 0} \frac{1}{\delta t} \left[ \int_{S_2} \mathbf{a}(\mathbf{x} + \delta t) \cdot d\mathbf{A} - \int_{S_1} \mathbf{a}(\mathbf{x}) \cdot d\mathbf{A} \right] \]

\[ \mathbf{a}(\mathbf{x}, t) = \text{VECTOR FIELD} \]

Proof:

\[ \int_V \mathbf{V} \cdot \mathbf{a} \, dV = \int_S (\mathbf{V} \cdot \mathbf{a})(\mathbf{U} \, dt) \cdot d\mathbf{A} \quad \text{as } \delta t \to 0 \]

\[ = \int_{S_2} \mathbf{a} \cdot d\mathbf{A} - \int_{S_1} \mathbf{a} \cdot d\mathbf{A} - \int_{S_0} \mathbf{a} \cdot (\mathbf{U} \, dt) \cdot d\mathbf{A} \]

\text{S}_{2}\quad \text{S}_{1}\quad \text{S}_{0}\text{ TOP SURFACE}\quad \text{BOTTOM SURFACE}\quad \text{"SIDE" SURFACE}
Vector Flux Identity (cont)

\[ \frac{1}{a} \frac{\partial}{\partial t} \left[ \iint_{S} \vec{a} \cdot \vec{A} \, dA \right] = \frac{1}{\delta t} \left[ \iint_{S_2} (\vec{a} + \delta t \frac{\partial \vec{a}}{\partial t}) \cdot \vec{A} \, dA \right] - \iint_{S_1} \vec{a} \cdot \vec{A} \, dA \]

\[ = \iint_{S} \frac{2 \vec{a}}{2 \delta t} \cdot \vec{A} \, dA + \frac{1}{\delta t} \left[ \iint_{S_2} \vec{a} \cdot \vec{A} \, dA - \iint_{S_1} \vec{a} \cdot \vec{A} \, dA \right] \]

\[ = \iint_{S} \left( \frac{2 \vec{a}}{2 \delta t} + \frac{(\nabla \cdot \vec{a}) \vec{u}}{\delta t} + \nabla \times (\vec{a} \times \vec{u}) \right) \cdot \vec{A} \, dA \]

Therefore, if

\[ \frac{2 \vec{a}}{2 \delta t} + (\nabla \cdot \vec{a}) \vec{u} + \nabla \times (\vec{a} \times \vec{u}) = 0 \]

then flux through surface moves with fluid is constant!!

For vorticity, \( \nabla \cdot \vec{a} = 0 \)

\[ \frac{2 \delta}{\delta t} + \nabla \times (\vec{a} \times \vec{u}) = 0 \] Q.E.D.

Two Line Vorticies

Two vorticies move in the same direction orbit one another.

Rate of orbit \( = \frac{\pi}{\pi h^2} \)
Two Counter-Directed Line Vorticies

TWO VORTICIES POINTED IN OPPOSITE DIRECTIONS
TRAVEL TOGETHER

RATE OF TRANSLATION = \frac{\pi}{2m}

Smoke Rings

TOROIDAL RING OF VORTICITY

Smoke ring moves at speed \approx O\left(\frac{t}{4\pi R}\right)
Toroidal Vortex Ring

Colliding Vortex Rings

http://serve.me.nus.edu.sg/limtt/#Video_Gallery
(Prof. Lim, Division of Fluid Mechanics, Melbourne, AU)
Surface of Rotating Bucket

How to solve fluid problems?
(like those in textbook)

**Step #1** - Draw a picture
- Write down all known geometry and forcings

**Step #2** - Is this a dynamical or static problem?
- Most dynamics are solved by linearization (unless otherwise required) because nonlinear problems are much more difficult.

**Step #3** - Select the most convenient coordinate system. (Note: curvilinear coordinates in Appendix B.)

**Step #4** - Can you apply conservation principles (mass, momentum, angular momentum, energy) and Bernoulli's principle to constrain your solution?

**Step #5** - Apply your intuition!
Surface of Rotating Bucket

1. Draw a picture.
2. Dynamic or Static? ...Static
3. Coordinate system? ...co-rotating cylindrical
   (no flow in this frame)
4. Can you apply conservation principles? ...Bernoulli’s
5. Use your intuition.

Co-Rotating Frame

\[
\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = \mathbf{g} - \left(\frac{1}{\rho}\right) \nabla p + \nu \nabla^2 \mathbf{u}
\]

\[
\frac{\partial \mathbf{u}^R}{\partial t} + \mathbf{u}^R \cdot \nabla \mathbf{u}^R = \mathbf{g} - \left(\frac{1}{\rho}\right) \nabla p + \nu \nabla^2 \mathbf{u}^R + \mathbf{R}_\perp \mathbf{\Omega}^2 - 2\mathbf{\Omega} \times \mathbf{u}^R
\]

with Centrifugal and Coriolis “apparent” forces.

(Very convenient when \( \mathbf{u}^R = 0 \).)
Co-Rotating With Bucket

Co-Rotating Frame of Reference
\[ \mathbf{U} = \text{Co-Rotating Frame Velocity} \]

Equations of Motion...
\[ \mathbf{U} \cdot \nabla \mathbf{U} = 0 \approx -\frac{1}{\rho} \nabla p + \mathbf{f} + \omega \times \mathbf{r} \]

Bernoulli’s Law
\[ \bar{q} = -\nabla \mathbf{q} \quad (\mathbf{q} \propto \frac{1}{\sqrt{\rho}}) \]
\[ \bar{p} = \bar{\nabla} \left( \rho \frac{\omega^2}{2} \right) \]

So
\[ \nabla \left( \frac{\rho}{\rho} + \mathbf{q} \cdot \omega^2 \frac{r^2}{2} \right) = 0 \]

Constant Throughout

Applying Bernoulli

\[ p_{\text{atm}} + \rho g (h + \delta) - \rho \omega^2 r^2/2 = p_2 - \rho \omega^2 r^2/2 \]
\[ \rho_2 = p_{\text{atm}} + \rho g (h + \delta) \]

\[ p_{\text{atm}} + \rho g h = p_3 \]
\[ p_2 - \rho \omega^2 r^2/2 = p_3 \]
\[ p_2 = \rho \omega^2 r^2 + \rho gh + p_{\text{atm}} \]
\[ \delta = \frac{\omega^2}{2g} \]

Surface is Parabolic!

(Note: To Achieve Rigid Rotation, Fluid Requires Viscos. But Rigid Co-Rotation Has No Viscos Forces.)
Problem 5.1

\[ \nu = \text{rate of sec} \]

\[ \begin{align*}
\rho \gamma & = \frac{1}{2} \rho \omega^2 R_2^2 \\
\rho \gamma & = \frac{1}{2} \rho \omega^2 R_1^2 \\
\frac{1}{2} \rho \omega^2 (R_2^2 - R_1^2) & = \rho g h
\end{align*} \]

\[ \text{Volume of paraboloid} = \frac{4}{3} \pi R^2 \]

Summary

- Fluid motion can be described by vorticity dynamics.
- Vorticity is conserved in invicid flow.
- Vorticity is solenoidal. Vorticity must either close upon itself (like a torus) or close at material boundaries (like a twirling spoon.)