1. Quick Review
2. Vorticity
3. Kelvin’s Theorem
4. Examples
How to solve fluid problems?
(Like those in textbook)

**STEP #1** - DRAW A PICTURE
- WRITE DOWN ALL KNOWN GEOMETRY AND FORCINGS

**STEP #2** - IS THIS A DYNAMICAL OR STATIC PROBLEM?
- MOST DYNAMICS ARE SOLVED AT LINEARIZATION (UNLESS OTHERWISE REQUIRED) BECAUSE NONLINEAR PROBLEMS ARE MUCH MORE DIFFICULT.

**STEP #3** - SELECT THE MOST CONVENIENT COORDINATE SYSTEM. (NOTE: CURVILINEAR COORDINATES IN APPENDIX D.)

**STEP #4** - CAN YOU APPLY CONSERVATION PRINCIPLES (MASS, MOMENTUM, ANGULAR MOMENTUM, ENERGY) AND BERNOULLI'S PRINCIPLE TO CONSTRAINT YOUR SOLUTION?

**STEP #5** - APPLY YOUR INTUITION!
Mechanical Energy Density for a Stokes Fluid

\[ \rho \frac{D}{Dt} \left( \frac{1}{2} u^2 \right) = \rho \, u_i \, g_i + \frac{2}{3} x_j \left( u_i \, \tau_{ij} \right) - \tau_{ij} \epsilon_{ij} \]

with \[ \tau_{ij} = -\rho \delta_{ij} + 2\mu \epsilon_{ij} - \frac{2}{3} \mu (\nabla \cdot u) \delta_{ij} \]

gives \[ \tau_{ij} \epsilon_{ij} = -\rho \, \epsilon_{ii} + 2\mu \epsilon_{ij} \epsilon_{ij} - \frac{2}{3} \mu (\nabla \cdot u) \epsilon_{ii} \]

where \[ \epsilon_{ii} = \frac{1}{2} \epsilon_{ij} = \nabla \cdot \mathbf{u} \]

On

\[ \rho \frac{D}{Dt} \left( \frac{1}{2} u^2 \right) = \rho \, u_i \, g_i + \rho (\nabla \cdot u) + \frac{2}{3} x_j \left( u_i \, \tau_{ij} \right) - \mu \left[ 2 \epsilon_{ij} \epsilon_{ij} - \frac{2}{3} (\nabla \cdot u)^2 \right] \]

work due to expansion

viscous dissipation of mechanical work
Fluids Equations

**CONTINUITY**
\[
\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{U} = 0
\]

**NAVIER-STOKES**
\[
\rho \frac{\partial \mathbf{U}}{\partial t} = \rho \mathbf{g} - \nabla p + \mu \left( \nabla^2 \mathbf{U} + \frac{1}{3} \nabla (\nabla \cdot \mathbf{U}) \right)
\]

**EQUATION OF STATE**
\[
p = \rho R T \quad \text{(ideal fluid)}
\]
\[
\frac{\rho}{\rho_s} \text{ constant} \quad \text{(isentropic)}
\]

**CONSERVATION OF INTERNAL ENERGY**
\[
\rho \frac{\partial E}{\partial t} = \nabla \cdot \mathbf{k} \nabla T - \rho (\nabla \cdot \mathbf{U}) + \mu \left[ 2 \epsilon_{ij} \epsilon_{ij} - \frac{2}{3} (\nabla \cdot \mathbf{U})^2 \right]
\]

Simplifying Assumptions...

- **INCOMPRESSIBLE** \( \nabla \cdot \mathbf{U} = 0 \), so \( \frac{\partial \rho}{\partial t} = 0 \)
- **UNIFORM DENSITY** \( \rho = \text{constant} \)
- **IRROTATIONAL** \( \nabla \times \mathbf{U} = 0 \)
- **INVISCID** \( \mu \rightarrow 0 \) (Euler Eq.)
Vorticity Dynamics

(Remember the importance of viscosity.)

\[
\vec{\omega} = \text{VORTICITY, A VECTOR FIELD} = \nabla \times \vec{u} \quad (\text{THUS, } \nabla \cdot \vec{\omega} = 0 \quad \vec{\omega} \text{ IS SOLENOIDAL.})
\]

\[
\nabla \times \left[ \frac{2\vec{u}}{2t} + \vec{\omega} \times \vec{u} \right] = \bar{\mathbf{g}} - \frac{1}{\rho} \nabla p - \nabla (\frac{1}{2} \vec{u}^2) + \mu \left[ \nabla^2 \vec{u} + \frac{1}{3} \nabla (\nabla \cdot \vec{u}) \right]
\]

\[
\frac{2\vec{\omega}}{2t} + \nabla \times (\vec{\omega} \times \vec{u}) = \mu \nabla \times \nabla^2 \vec{u} = \mu \nabla^3 \vec{\omega}
\]

VORTICITY EQUATION!

\[
\nabla \times (\Omega \times \vec{U}) = \Omega \nabla \cdot \vec{U} + \vec{U} \cdot \nabla \Omega - \Omega \cdot \nabla \vec{U}
\]

\[
\nabla \times (\vec{A} \times \vec{B}) = \vec{A}(\nabla \cdot \vec{B}) - \vec{B}(\nabla \cdot \vec{A}) + (\vec{B} \cdot \nabla)\vec{A} - (\vec{A} \cdot \nabla)\vec{B}
\]
Simple (2D) Fluid Rotation

Cylindrical Coordinates:

Rigid Rotation

\[ \dot{\mathbf{r}} = \omega \times \mathbf{r} \]

\[ U_\theta = \omega R \]

\[ J^2 = (\nabla \times \mathbf{U})^2 = \frac{1}{2} \nabla (\nabla U_\theta) = 2\omega^2 \]

Vorticity = 2 x Rotation

Vorticity is constant for everywhere

\[ \nabla \cdot \mathbf{J} = 0 \quad \text{ie. no viscous dissipation} \]

Line Vortex

\[ \Gamma = \text{Circulation} \]

\[ U_\theta = \frac{\Gamma}{2\pi R} \]

\[ J^2 = (\nabla \times \mathbf{U})^2 = \phi = \text{irrotational} \]

No vorticity within fluid but "singular" line vortex at origin.

\[ \text{Circulation} = \oint \nabla \theta \cdot \mathbf{U}_\theta = \Gamma \]

Since \( J^2 = 0 \), no viscous dissipation within fluid.
3D Fluid Flow

CFD calculation of flow in mixing tank

CFD calculation of atmosphere (Iso-Vorticity contours)
Kelvin's Theorem

\[
\frac{d}{dt} \int_S \mathbf{r} \cdot d\mathbf{A} = 0 \quad (\text{as } \mu \to 0)
\]

\[
\int_S \mathbf{r} \cdot d\mathbf{A} = \int_S (\mathbf{\nabla} \times \mathbf{u}) \cdot d\mathbf{A}
\]

\[
= \oint \mathbf{u} \cdot d\mathbf{l} = \Gamma
\]

\underline{\text{CIRCULATION OF FLUID IS CONSTANT AS } \mu \to 0}

\underline{\text{VORTEX TUBE IS A TUBE OF CONSTANT VORTICITY FLUX.}}

\underline{\text{CIRCULATION WITHIN TUBE IS constant as it moves with fluid (as } \mu \to 0)}
A (General) Vector Theorem for a Moving Surface Flux

\[
\frac{d}{dt} \left[ \int_S \mathbf{a} \cdot \hat{n} \, dA \right] = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \left[ \int_{S_2} \mathbf{a}(t+\Delta t) \cdot \hat{n} \, dA - \int_{S_1} \mathbf{a}(t) \cdot \hat{n} \, dA \right]
\]

\[\mathbf{a}(\mathbf{x}, t) = \text{vector field}\]

Surface \(S_1\) moves to surface \(S_2\) in time \(\Delta t\)

Identity:

\[
\iiint_V \nabla \cdot \mathbf{a} \, dV = \int_S (\nabla \cdot \mathbf{a})(\mathbf{udt} \cdot \hat{n}) \, dA \quad \text{as } \Delta t \to 0
\]

\[
= \int_{S_2} \mathbf{a} \cdot \hat{n} \, dA - \int_{S_1} \mathbf{a} \cdot \hat{n} \, dA - \oint_{S_{side}} \mathbf{a} \cdot (\mathbf{udt} \times \hat{n}) \, d\ell
\]

Top Surface \(S_{top}\)

Bottom Surface \(S_{bottom}\)

'Side' Surface \(S_{side}\)
Vector Flux Identity (cont)

\[
\frac{d}{dt} \left[ \int_S \mathbf{\bar{a}} \cdot \mathbf{n} \, dA \right] = \lim_{\delta t \to 0} \frac{1}{\delta t} \left[ \int_S \mathbf{\bar{a}} (\mathbf{\bar{a}} + \delta t \frac{2\mathbf{\bar{a}}}{2t} + \ldots) \cdot \mathbf{n} \, dA \right. \\
\left. - \int_S \mathbf{\bar{a}} \cdot \mathbf{n} \, dA \right]
\]

\[
= \int_S \frac{2\mathbf{\bar{a}}}{2t} \cdot \mathbf{n} \, dA + \frac{1}{\delta t} \left[ \int_S \mathbf{\bar{a}} \cdot \mathbf{n} \, dA - \int_S \mathbf{\bar{a}} \cdot \mathbf{n} \, dA \right]
\]

\[
= \int_S \frac{2\mathbf{\bar{a}}}{2t} \cdot \mathbf{n} \, dA + \int_S (\nabla \cdot \mathbf{\bar{a}}) \mathbf{\bar{u}} \cdot \mathbf{n} \, dA + \oint \frac{\mathbf{\bar{a}} \cdot (\mathbf{\bar{u}} \times \mathbf{\bar{t}}) \, dl}{(\mathbf{\bar{a}} \times \mathbf{\bar{u}}) \cdot \mathbf{\bar{t}}}
\]

\[
= \int_S \left( \frac{2\mathbf{\bar{a}}}{2t} + (\nabla \cdot \mathbf{\bar{a}}) \mathbf{\bar{u}} + \nabla \times (\mathbf{\bar{a}} \times \mathbf{\bar{u}}) \right) \cdot \mathbf{n} \, dA
\]

Therefore, if

\[
\frac{2\mathbf{\bar{a}}}{2t} + (\nabla \cdot \mathbf{\bar{a}}) \mathbf{\bar{u}} + \nabla \times (\mathbf{\bar{a}} \times \mathbf{\bar{u}}) = 0
\]

then FLUX THROUGH SURFACE MOVING WITH FLUID IS \underline{CONSTANT}!!

For vorticity, \( \nabla \cdot \mathbf{\bar{w}} = 0 \)

\[
\frac{2\mathbf{\bar{w}}}{2t} + \nabla \times (\mathbf{\bar{w}} \times \mathbf{\bar{u}}) \equiv 0 \quad \text{Q.E.D.}
\]
Two Line Vorticies

Side View

Two Vorticies Pointed in the Same Direction Orbit One Another.

Rate of Orbit $= \frac{\pi}{\pi h^2}$
Two Counter-Directed Line Vorticies

Two vorticies pointed in opposite directions travel together.

Rate of translation = $\frac{\Gamma}{2\pi \, l}$
Ring of Point Vortices

http://www.student.math.uwaterloo.ca/~amat361/Fluid%20Mechanics/topics/vorticity.htm
Clouds of Opposite Vortices

http://www.student.math.uwaterloo.ca/~amat361/Fluid%20Mechanics/topics/vorticity.htm
Smoke Rings

Toroidal Ring of Vorticity

$\nabla \times \mathbf{A}$

Flow out of board

Flow due to opposite side

$2\pi R$

Net motion

Flow due to opposite side

Cross-section

Smoke ring moves at

$\text{speed } \leq O\left(\frac{\rho}{\eta R}\right)$
Toroidal Vortex Ring

Colliding Vortex Rings

CASE #1
OPPOSITE HEELOCITY

CASE #2
EQUAL HEEILITY
http://serve.me.nus.edu.sg/limtt/#Video_Gallery
(Prof. Lim, Division of Fluid Mechanics, Melbourne, AU)
http://serve.me.nus.edu.sg/limtt/#Video_Gallery
(Prof. Lim, Division of Fluid Mechanics, Melbourne, AU)
Surface of Rotating Bucket

Water rotates with the bucket, hence it is *at rest relative to the latter*.
How to solve fluid problems?

(Like those in textbook)

**STEP #1**
- Draw a picture
- Write down all known geometry and forcings

**STEP #2**
- Is this a dynamical or static problem?
- Most dynamics are solved at linearization (unless otherwise required) because nonlinear problems are much more difficult.

**STEP #3**
- Select the most convenient coordinate system. (Note: curvilinear coordinates in Appendix D.)

**STEP #4**
- Can you apply conservation principles (mass, momentum, angular momentum, energy) and Bernoulli's principle to constrain your solution?

**STEP #5**
- Apply your intuition!
Surface of Rotating Bucket

1. Draw a picture.
2. Dynamic or Static?
3. Coordinate system?
4. Can you apply conservation principles?
5. Use your intuition.

...Static
...co-rotating cylindrical (no flow in this frame)

...Bernoulli’s
Co-Rotating Frame

\[
\frac{\partial u}{\partial t} + u \cdot \nabla u = g - \left(\frac{1}{\rho}\right) \nabla p + \nu \nabla^2 u
\]

\[
\frac{\partial u^R}{\partial t} + u^R \cdot \nabla u^R = g - \left(\frac{1}{\rho}\right) \nabla p + \nu \nabla^2 u^R + \mathbf{R} \times \mathbf{u}^R
\]

with Centrifugal and Coriolis “apparent” forces.
(Very convenient when \( u^R = 0 \).)
Co-Rotating With Bucket

**Co-Rotating frame of reference**

\( \overline{U}^R = \text{co-rotating frame velocity} > 0 \)

**Equations of motion...**

\[ \overline{U} \cdot \nabla \overline{U} = 0 = -\frac{1}{\rho} \nabla p + \overline{\bar{g}} + \omega^2 \overline{r} \]

Force balance:

\[ \overline{\bar{g}} = -\nabla \varphi_g \quad (\varphi_g < \varphi_z) \]

\[ \overline{r} = \nabla (r^2/2) \]

So

\[ \nabla \left( \frac{p}{\rho} + \varphi_g - \omega^2 (r^2/2) \right) = 0 \]

\[ \text{\underline{Constant throughout}} \]
Rigid Rotation Statics

\[ P_{\text{ATM}} + \rho g (h + \delta) - \frac{1}{2} \rho \omega^2 R^2 = p_2 - \frac{1}{2} \rho \omega^2 R^2 \]

\[ p_2 = P_{\text{ATM}} + \rho g (h + \delta) \]

\[ P_{\text{ATM}} + \rho g h = p_3 \]

\[ p_2 - \frac{1}{2} \rho \omega^2 R^2 = p_3 \]

\[ p_2 = \rho \omega^2 R + \rho g h + P_{\text{ATM}} \]

\[ \delta = \frac{4 \omega^2}{2g} R^2 \]

Surface is parabolic!

(Note: to achieve rigid rotation, fluid requires viscosity. But rigid co-rotation has no viscous forces.)
Problem 5.1

Force balance:

\[
\begin{align*}
\rho g h - \frac{1}{2} \rho \omega^2 R_2^2 &= 0 \\
\rho g h &= \frac{1}{2} \rho \omega^2 R_1^2
\end{align*}
\]

\[
\frac{1}{2} \rho \omega^2 (R_2^2 - R_1^2) = \rho g h
\]

But Volume of Air (Water) does not change with rotation

\[
V_0(Air) = \pi R_0^2 h_1 = \frac{\pi}{2} R_2^2 (h + \delta) - \frac{\pi}{2} R_1^2 \delta
\]

where \( \delta = \frac{\omega^2}{2g} R_1^2 \)

Two equations for two unknowns \( (R_1, R_2) \)

\[
\begin{align*}
R_2^2 - R_1^2 &= \frac{2gh}{\omega^2} \\
R_2^2 (h + \delta) - R_1^2 \delta &= 2R_0^2 h_1
\end{align*}
\]

Solve for \( R_1 \)...

\[
\begin{align*}
(R_1^2 - \frac{2gh}{\omega^2}) \chi (h + \frac{\omega^2}{2g} R_1^2) - R_1 \frac{\omega^2}{2g} &= 2R_0^2 h_1 \\
on R_1 &= \frac{2R_0^2 h_1 + 2gh^2/\omega^2}{h - (\omega^2/2g)}
\end{align*}
\]
Summary

- Fluid motion can be described by vorticity dynamics.

- Vorticity is conserved in invicid flow.

- Vorticity is solenoidal, $\nabla \cdot \Omega = 0$. Vorticity must either close upon itself (like a torus) or close at material boundaries (like a twirling spoon.)