APPH 4200
Physics of Fluids

Problem Solving and Vorticity (Ch. 5)

1. Quick Review
2. Vorticity
3. Kelvin's Theorem
4. Examples

https://www.youtube.com/watch?v=-VL0M0jmu

Giant Smoke Rings - Cool Science Experiment

Sick Science!

Subscribe 381,314

2,212,933

Add to Share More 8,747 153

How to solve fluid problems?

(Like those in textbook)

STEP #1 – DRAW A PICTURE
– WRITE DOWN ALL KNOWN GEOMETRY AND FORCINGS

STEP #2 – IS THIS A DYNAMICAL OR STATIC PROBLEM?
– MOST DYNAMICS ARE SOLVED BY LINEARIZATION (UNLESS OTHERWISE REQUIRED) BECAUSE NONLINEAR PROBLEMS ARE MUCH MORE DIFFICULT.

STEP #3 – SELECT THE MOST CONVENIENT COORDINATE SYSTEM. (NOTE: CURVILINEAR COORDINATES IN APPENDIX B.)

STEP #4 – CAN YOU APPLY CONSERVATION PRINCIPLES (MASS, MOMENTUM, ANGULAR MOMENTUM, ENERGY) AND BERNOULLI'S PRINCIPLE TO CONSTRAIN YOUR SOLUTION?

STEP #5 – APPLY YOUR INTUITION!

Mechanical Energy Density for a Stokes Fluid

$$\rho \frac{D}{Dt}\left(\tfrac{1}{2}u^2\right) = \rho u_i g_i + \frac{\partial}{\partial x_j}(u_i \tau_{ij}) - \tau_{ij}\epsilon_{ij}$$

WITH $\tau_{ij} = -p\delta_{ij} + 2\mu\epsilon_{ij} - \frac{2}{3}\mu(\nabla\cdot u)\delta_{ij}$

GIVES $\tau_{ij}\epsilon_{ij} = -p\,\epsilon_{ii} + 2\mu\epsilon_{ij}\epsilon_{ij} - \frac{2}{3}\mu(\nabla\cdot u)\epsilon_{ii}$

WHERE $\epsilon_{ii} = \sum_i \epsilon_{ii} = \nabla\cdot\bar{u}$

OR

$$\rho \frac{D}{Dt}\left(\tfrac{1}{2}u^2\right) = \rho u_i g_i + \overbrace{p(\nabla\cdot u)}^{\text{WORK DUE TO EXPANSION}} + \frac{\partial}{\partial x_j}(u_i \tau_{ij}) \underbrace{- \mu\left[2\epsilon_{ij}\epsilon_{ij} - \frac{2}{3}(\nabla\cdot u)^2\right]}_{\text{VISCOUS DISSIPATION OF MECHANICAL WORK}}$$

Fluids Equations

CONTINUITY $\quad \frac{D\rho}{Dt} = -\rho \nabla \cdot \bar{u}$

NAVIER-STOKES $\quad \rho \frac{D\bar{u}}{Dt} = \rho \bar{g} \underbrace{- \nabla p + \mu \left[\nabla^2 \bar{u} + \frac{1}{3} \bar{\nabla} (\nabla \cdot \bar{u}) \right]}_{\nabla \cdot \bar{\bar{\tau}}}$

EQUATION OF STATE $\quad p = \rho R T \quad$ (IDEAL FLUID)

$p/\rho^{\gamma} \sim$ CONSTANT $\quad$ (ISENTROPIC)

CONSERVATION OF (internal) ENERGY $\quad \rho \frac{DE}{Dt} = \nabla \cdot k \nabla T - p(\nabla \cdot \bar{u}) + \boxed{\mu \left[2 \epsilon_{ij} \epsilon_{ij} - \frac{2}{3} (\nabla \cdot u)^2 \right]}$

SIMPLIFYING ASSUMPTIONS...

- INCOMPRESSIBLE $\quad \nabla \cdot \bar{u} = 0$, SO $\frac{D\rho}{Dt} = 0$
- UNIFORM DENSITY $\quad \rho =$ CONSTANT
- IRROTATIONAL: $\nabla \times \bar{u} = 0$
- INVISCID $\quad \mu \to 0 \quad$ (EULER EQ.)

Vorticity Dynamics

(Remember the importance of viscosity.)

$\bar{\Omega}$ = VORTICITY, A VECTOR FIELD

$= \nabla \times \bar{u} \quad$ (THUS, $\nabla \cdot \bar{\Omega} = 0 \quad \bar{\Omega}$ IS SOLENOIDAL.)

$$\nabla \times \left[\frac{\partial \bar{u}}{\partial t} + \bar{\Omega} \times \bar{u} = \bar{g} - \frac{1}{\rho} \nabla p - \nabla \left(\frac{1}{2} u^2 \right) + \mu \left[\nabla^2 \bar{u} + \frac{1}{3} \bar{\nabla} (\nabla \cdot \bar{u}) \right] \right]$$

?

$$\frac{\partial \bar{\Omega}}{\partial t} + \nabla \times (\bar{\Omega} \times \bar{u}) = \mu \bar{\nabla} \times \nabla^2 \bar{u} = \mu \nabla^2 \bar{\Omega}$$

$$\boxed{\frac{\partial \bar{\Omega}}{\partial t} + \nabla \times (\bar{\Omega} \times \bar{u}) = \mu \nabla^2 \bar{\Omega}}$$

VORTICITY EQUATION!

$$\nabla \times (\Omega \times \mathbf{U}) = \Omega \nabla \cdot \mathbf{U} + \mathbf{U} \cdot \nabla \Omega - \Omega \cdot \nabla \mathbf{U}$$

$$\nabla \times (\mathbf{A} \times \mathbf{B}) = \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A}) + (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B}$$

Momentum - Vorticity

$$\frac{D\mathbf{u}}{Dt} = \mathbf{g} - \frac{\nabla P}{\rho} + \nu \left[\nabla^2 \mathbf{u} + \frac{1}{3} \nabla (\nabla \cdot \mathbf{u}) \right]$$

$$\frac{D\mathbf{\Omega}}{Dt} = -\mathbf{\Omega} \nabla \cdot \mathbf{u} + (\mathbf{\Omega} \cdot \nabla)\mathbf{u} + \frac{\nabla \rho \times \nabla P}{\rho^2} + \nu \nabla^2 \mathbf{\Omega}$$

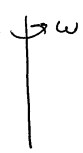
(Barotropic only $\Rightarrow$ Kelvin's Theorem)

$$\frac{\partial \mathbf{\Omega}}{\partial t} + \nabla \times (\mathbf{\Omega} \times \mathbf{u}) = \nu \nabla^2 \mathbf{\Omega}$$

Simple (2D) Fluid Rotation

CYLINDRICAL COORDINATES:

RIGID ROTATION



$u_\theta = \omega r$

$\Omega_z = (\nabla \times u)_z = \frac{1}{r}\frac{\partial}{\partial r}(r u_\theta) = 2\omega$

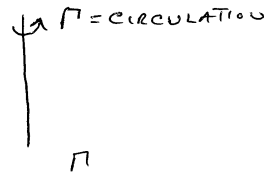
VORTICITY = 2 × ROTATION

VORTICITY IS CONSTANT FOR EVERYWHERE

RIGID ROTATION

$\nabla^2 \Omega = 0$ ∴ NO VISCOUS DISSIPATION

LINE VORTEX



$u_\theta = \frac{\Gamma}{2\pi r}$

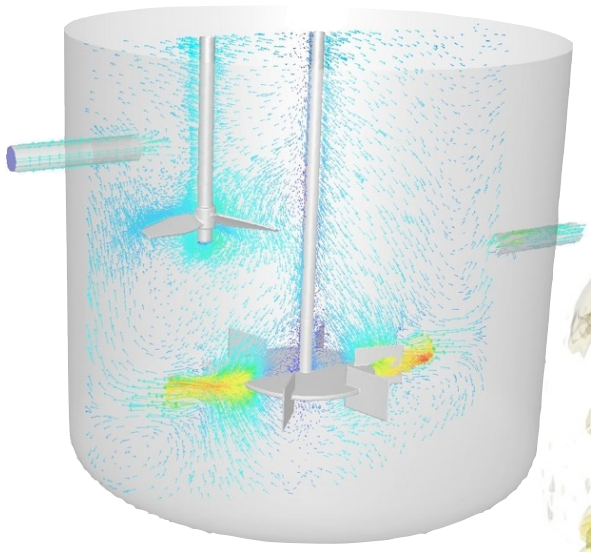
$\Omega_z = (\nabla \times u)_z = \phi$ ∴ IRROTATIONAL

NO VORTICITY WITHIN FLUID BUT 'SINGULAR' LINE VORTEX AT ORIGIN.

CIRCULATION $= \oint r\, d\theta\, u_\theta = \Gamma$

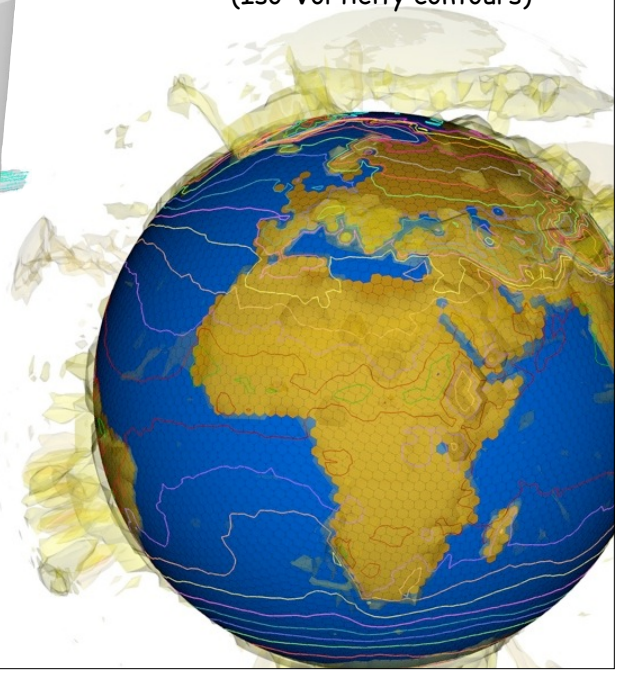
SINCE $\Omega = 0$, NO VISCOUS DISSIPATION WITHIN FLUID.

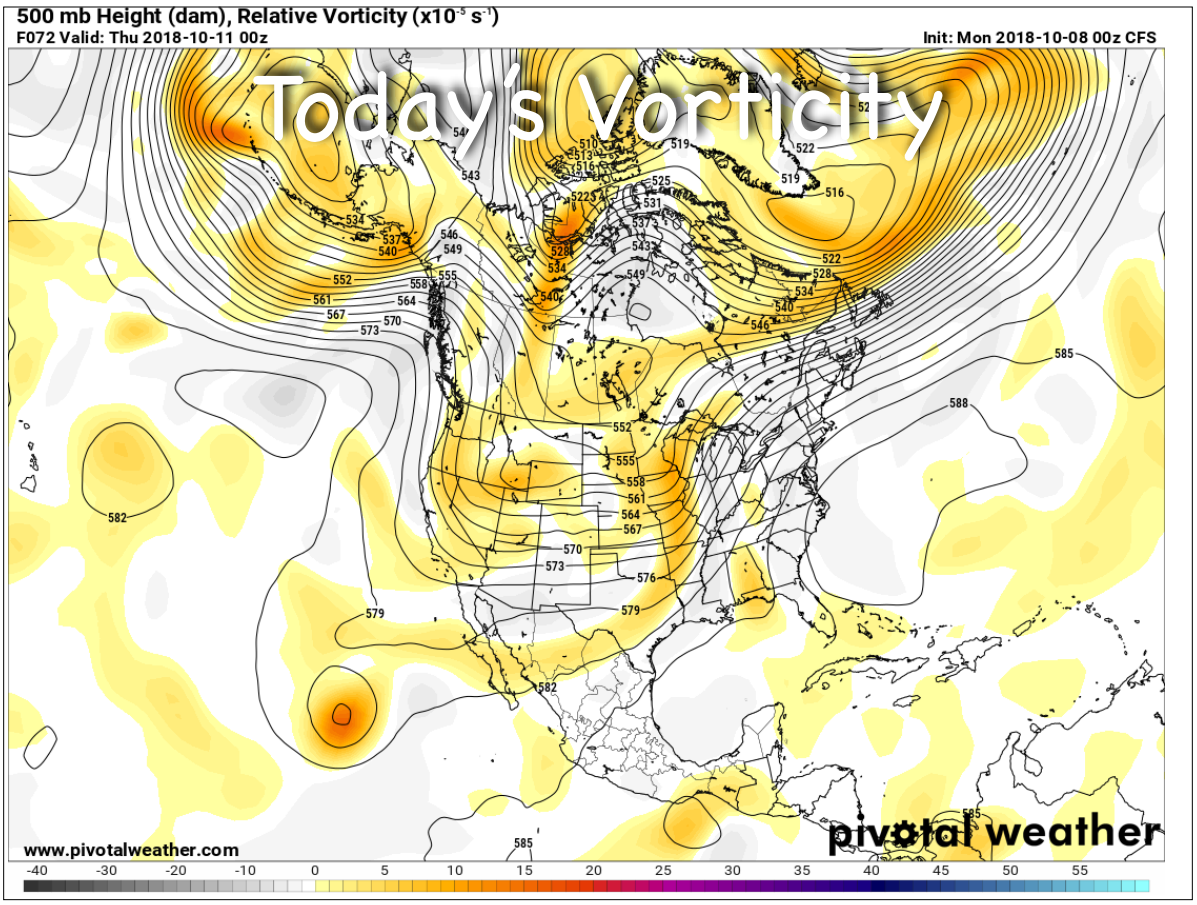
3D Fluid Flow



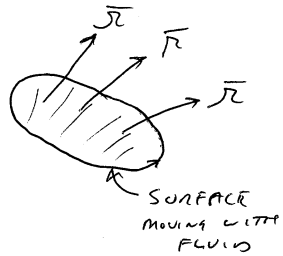
CFD calculation of flow in mixing tank

CFD calculation of atmosphere (Iso-Vorticity contours)





Kelvin's Theorem

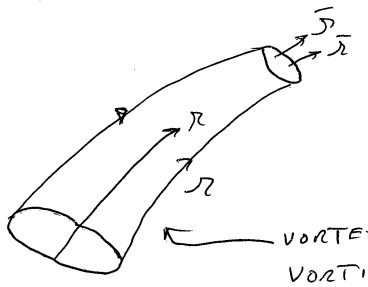


$$\frac{d}{dt}\iint_S \bar{\Omega}\cdot d\bar{A} = 0 \qquad (\text{AS } \mu \to 0)$$

$$\iint_S \bar{\Omega}\cdot d\bar{A} = \iint_S (\nabla\times\bar{u})\cdot d\bar{A}$$

$$= \oint \bar{u}\cdot d\bar{l} = \Gamma$$

CIRCULATION OF FLUID IS CONSTANT AS $\mu \to 0$



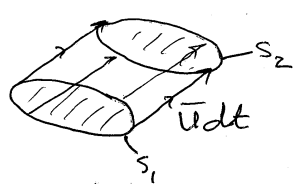
VORTEX TUBE IS A TUBE OF CONSTANT VORTICITY FLUX.

CIRCULATION WITHIN TUBE IS CONSTANT AS IT MOVES WITH FLUID (AS $\mu \to 0$)

A (General) Vector Theorem for a Moving Surface Flux

$$\frac{d}{dt}\left[\iint_S \bar{a}\cdot\hat{n}\,dA\right] = \lim_{\delta t\to 0}\frac{1}{\delta t}\left[\iint_{S_2}\bar{a}(t+\delta t)\cdot\hat{n}\,dA - \iint_{S_1}\bar{a}(t)\cdot\hat{n}\,dA\right]$$

$\bar{a}(\bar{x},t)$ = VECTOR FIELD



SURFACE S_1 MOVES TO SURFACE S_2 IN TIME δt

Identity:

$$\iiint_V \nabla\cdot\bar{a}\,dV = \iint_S (\nabla\cdot\bar{a})(\bar{u}dt\cdot\hat{n})\,dA \qquad \text{AS } \delta t\to 0$$

$$= \underbrace{\iint_{S_2}\bar{a}\cdot\hat{n}\,dA}_{\text{TOP SURFACE}} - \underbrace{\iint_{S_1}\bar{a}\cdot\hat{n}\,dA}_{\text{BOTTOM SURFACE}} - \underbrace{\oint \bar{a}\cdot(\bar{u}\times\hat{t})\,dt\,dl}_{\text{"SIDE" SURFACE}}$$

Vector Flux Identity (cont)

$$\frac{d}{dt}\left[\iint \bar{a}\cdot\hat{n}\,dA\right] = \lim_{\delta t\to 0}\frac{1}{\delta t}\left[\iint_{S_2}\left(\bar{a}+\delta t\frac{\partial\bar{a}}{\partial t}+\cdots\right)\cdot\hat{n}\,dA - \iint_{S_1}\bar{a}\cdot\hat{n}\,dA\right]$$

$$= \iint_S \frac{\partial\bar{a}}{\partial t}\cdot\hat{n}\,dA + \frac{1}{\delta t}\left[\iint_{S_2}\bar{a}\cdot\hat{n}\,dA - \iint_{S_1}\bar{a}\cdot\hat{n}\,dA\right]$$

$$= \iint_S \frac{\partial\bar{a}}{\partial t}\cdot\hat{n}\,dA + \iint_S(\nabla\cdot\bar{a})\bar{u}\cdot\hat{n}\,dA + \oint \underbrace{\bar{a}\cdot(\bar{u}\times\hat{t})}_{(\bar{a}\times\bar{u})\cdot\hat{t}}\,dl$$

$$= \iint_S\left(\frac{\partial\bar{a}}{\partial t} + (\nabla\cdot\bar{a})\bar{u} + \nabla\times(\bar{a}\times\bar{u})\right)\cdot\hat{n}\,dA$$

THEREFORE, IF

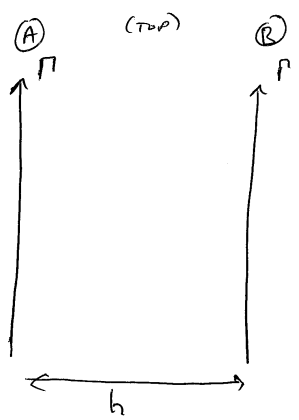
$$\frac{\partial\bar{a}}{\partial t} + (\nabla\cdot\bar{a})\bar{u} + \nabla\times(\bar{a}\times\bar{u}) = 0$$

THEN FLUX THROUGH SURFACE MOVING WITH FLUID IS <u>CONSTANT</u>!!

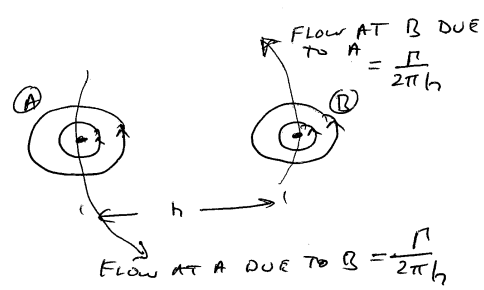
FOR VORTICITY, $\nabla\cdot\bar{\Omega} = 0$

$$\frac{\partial\bar{\Omega}}{\partial t} + \nabla\times(\bar{\Omega}\times\bar{u}) \stackrel{?}{=} 0 \qquad \text{Q.E.D.}$$

Two Line Vorticies



SIDE VIEW

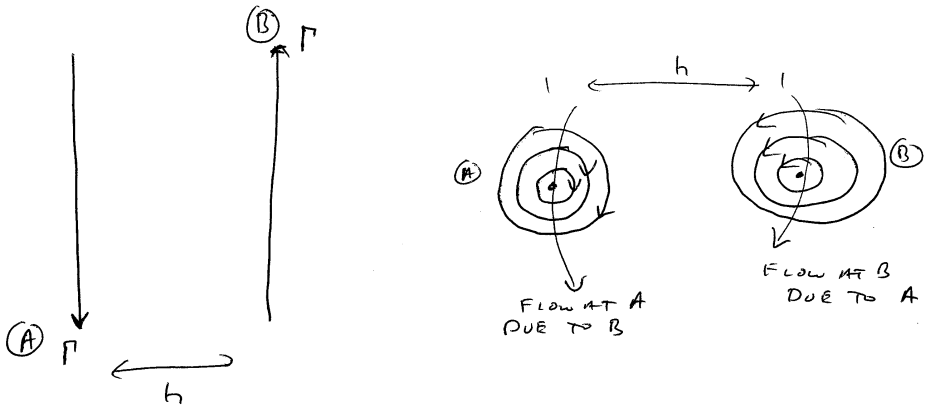


TOP VIEW

TWO VORTICIES POINTED IN THE <u>SAME</u> DIRECTION ORBIT ONE ANOTHER.

RATE OF ORBIT $= \dfrac{\Gamma}{\pi h^2}$

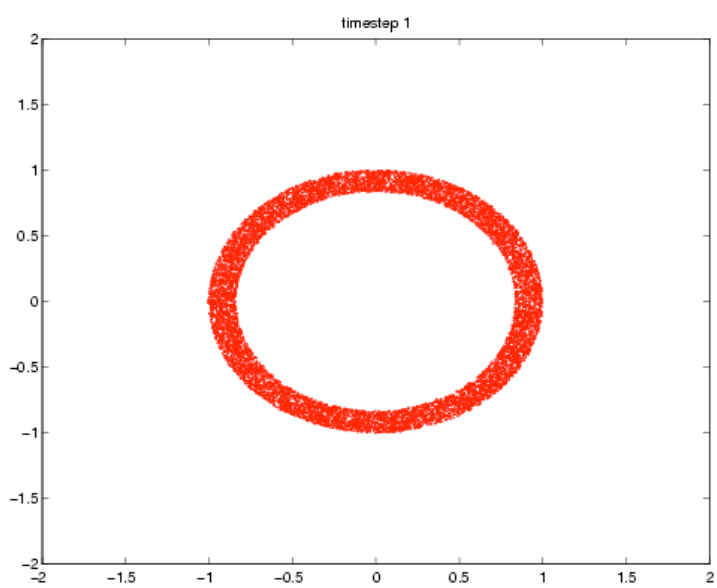
Two Counter-Directed Line Vorticies



TWO VORTICIES POINTED IN
OPPOSITE DIRECTIONS
TRAVEL TOGETHER

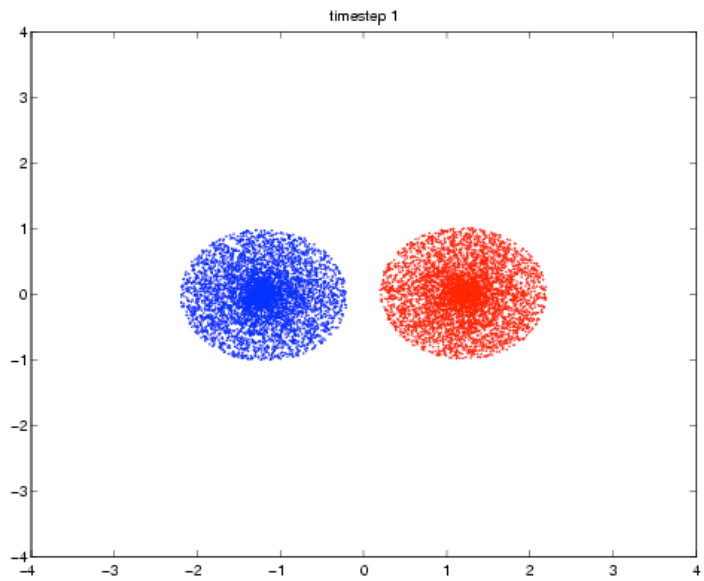
RATE OF TRANSLATION $= \frac{\Gamma}{2\pi h}$

Ring of Point Vortices



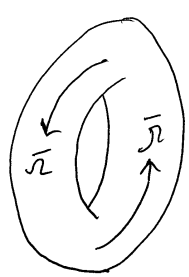
http://www.student.math.uwaterloo.ca/~amat361/Fluid%20Mechanics/topics/vorticity.htm

Clouds of Opposite Vortices

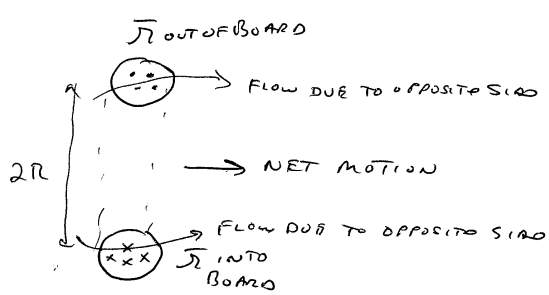


http://www.student.math.uwaterloo.ca/~amat361/Fluid%20Mechanics/topics/vorticity.htm

Smoke Rings



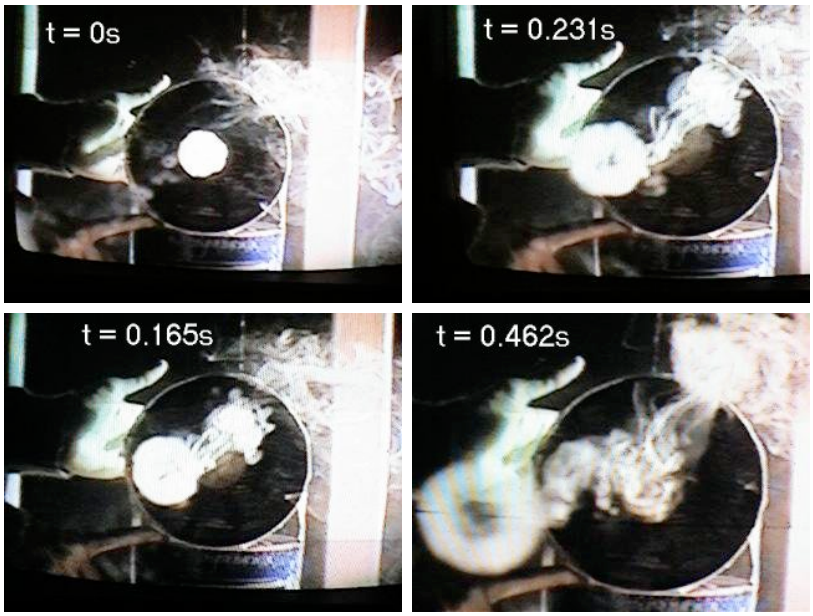
TOROIDAL
RING OF VORTICITY



CROSS-SECTION

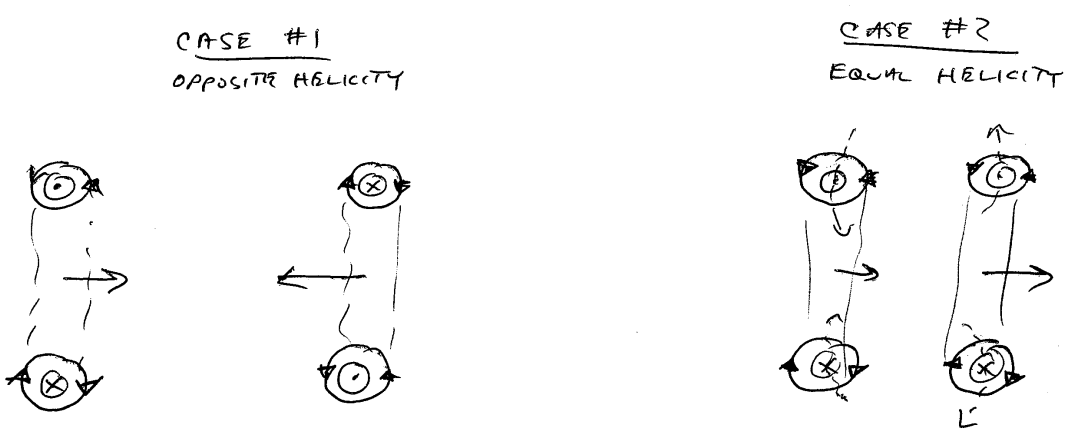
SMOKE RING MOVES AT
SPEED $\simeq \mathcal{O}\left(\frac{\Gamma}{4\pi R}\right)$

Toroidal Vortex Ring

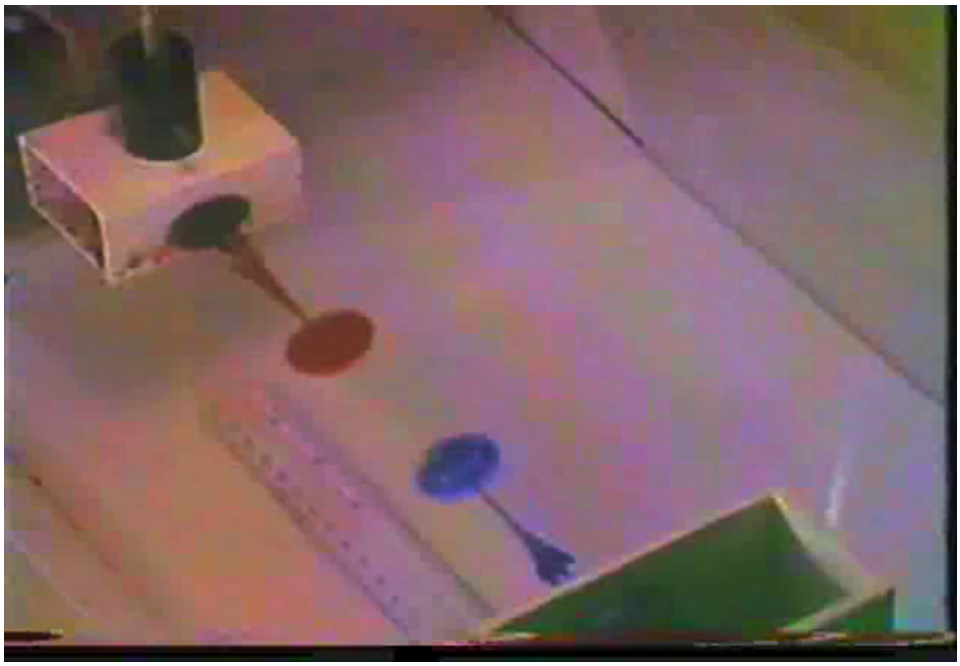


http://www.woodrow.org/teachers/esi/1999/princeton/projects/fluid_dynamics/vortex.html

Colliding Vortex Rings

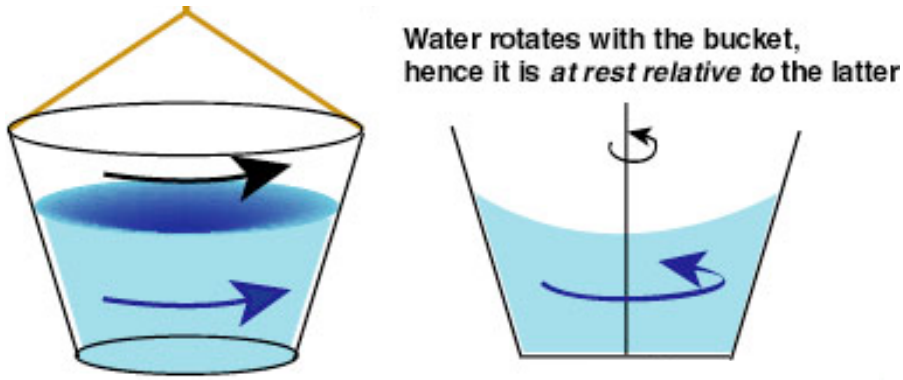


http://serve.me.nus.edu.sg/limtt/#Video_Gallery
(Prof. Lim, Division of Fluid Mechanics, Melbourne, AU)



http://serve.me.nus.edu.sg/limtt/#Video_Gallery
(Prof. Lim, Division of Fluid Mechanics, Melbourne, AU)

Surface of Rotating Bucket



How to solve fluid problems?

(Like those in textbook)

STEP #1 – DRAW A PICTURE
– WRITE DOWN ALL KNOWN GEOMETRY AND FORCINGS

STEP #2 – IS THIS A DYNAMICAL OR STATIC PROBLEM?
– MOST DYNAMICS ARE SOLVED BY LINEARIZATION (UNLESS OTHERWISE REQUIRED) BECAUSE NONLINEAR PROBLEMS ARE MUCH MORE DIFFICULT.

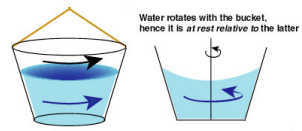
STEP #3 – SELECT THE MOST CONVENIENT COORDINATE SYSTEM. (NOTE: CURVILINEAR COORDINATES IN APPENDIX B.)

STEP #4 – CAN YOU APPLY CONSERVATION PRINCIPLES (MASS, MOMENTUM, ANGULAR MOMENTUM, ENERGY) AND BERNOULLI'S PRINCIPLE TO CONSTRAIN YOUR SOLUTION?

STEP #5 – APPLY YOUR INTUITION!

Surface of Rotating Bucket

1. Draw a picture.
2. Dynamic or Static?
3. Coordinate system?
4. Can you apply conservation principles?
5. Use your intuition.



...Static

...co-rotating cylindrical
(no flow in this frame)

...Bernoulli's

Co-Rotating Frame

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} = \mathbf{g} - (1/\rho)\nabla p + \nu\nabla^2\mathbf{u}$$

$$\frac{\partial \mathbf{u}^R}{\partial t} + \mathbf{u}^R\cdot\nabla\mathbf{u}^R = \mathbf{g} - (1/\rho)\nabla p + \nu\nabla^2\mathbf{u}^R + \mathbf{R}_\perp\Omega^2 - 2\Omega\times\mathbf{u}^R$$

with Centrifugal and Coriolis "apparent" forces.
(Very convenient when $u^R = 0$.)

Co-Rotating With Bucket

CO-ROTATING FRAME OF REFERENCE

$\bar{u}^R$ = CO-ROTATING FRAME VELOCITY ≈ 0

EQUATIONS OF MOTION...

$$\bar{u}\cdot\bar{\nabla}\bar{u} \approx 0 \approx -\frac{1}{\rho}\nabla p + \bar{g} + \omega^2 \bar{R}$$

Force balance:

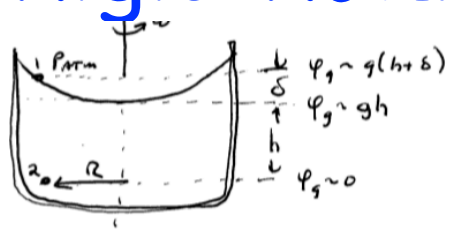
$$\bar{g} = -\nabla \varphi_g \qquad (\varphi_g \propto g z)$$

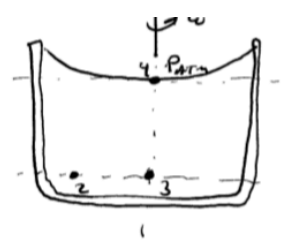
$$\bar{R} = \bar{\nabla}(R^2/2)$$

so

$$\bar{\nabla}\Big(\underbrace{\frac{p}{\rho} + \varphi_g - \omega^2 R^2/2}_{\text{CONSTANT THROUGHOUT}}\Big) = 0$$

Rigid Rotation Statics





$$\underbrace{P_{ATM} + \rho g(h+\delta) - \rho\omega^2 R^2/2 = P_2 - \rho\omega^2 R^2/2}$$

$$P_2 = P_{ATM} + \rho g(h+\delta)$$

$$\underbrace{P_{ATM} + \rho g h = P_3 \qquad P_2 - \rho\omega^2 R^2/2 = P_3}$$

$$P_2 = \rho\omega^2 R + \rho g h + P_{ATM}$$

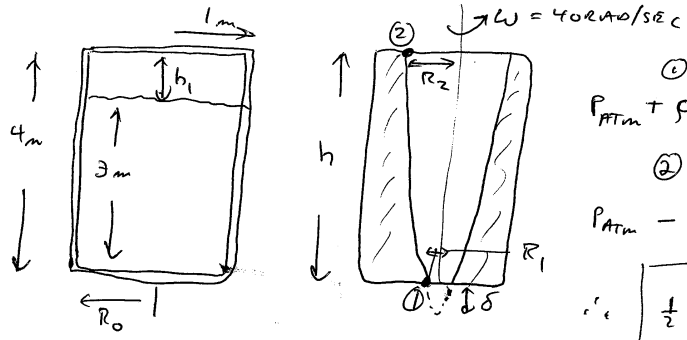
$$P_{ATM} + \rho g(h+\delta) = \rho\omega^2 R^2/2 + \rho g h + P_{ATM}$$

$$\delta = \frac{\omega^2}{2g} R^2$$

SURFACE IS PARABOLIC!

(NOTE: TO ACHIEVE RIGID ROTATION, FLUID REQUIRES VISCOSITY.
BUT RIGID CO-ROTATION HAS NO VISCOUS FORCES.)

Problem 5.1



Force balance:

① $P_{ATM} + \rho g h - \frac{1}{2}\rho\omega^2 R_2^2 =$

② $P_{ATM} - \frac{1}{2}\rho\omega^2 R_1^2$

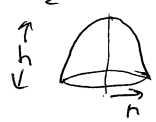
$$\therefore \boxed{\frac{1}{2}\rho\omega^2\left(R_2^2 - R_1^2\right) = \rho g h}$$

BUT VOLUME OF AIR (WATER) DOES NOT CHANG WITH ROTATION

$$Vol(AIR) = \pi R_0^2 h_1 = \frac{\pi}{2} R_2^2 (h+\delta) - \frac{\pi}{2} R_1^2 \delta$$

WHERE $\delta = \frac{\omega^2}{2g} R_1^2$

VOLUME OF PARABOLA $= \frac{1}{2}\pi R^2 h$



TWO EQUATIONS FOR TWO UNKNOWNS (R_1, R_2)

$$R_2^2 - R_1^2 = \frac{2gh}{\omega^2}$$

$$R_2^2(h+\delta) - R_1^2\delta = 2R_0^2 h_1$$

SOLVE FOR R_1 ....

$$\left(R_1^2 - \frac{2gh}{\omega^2}\right)\left(h + \frac{\omega^2}{2g}R_1^2\right) - R_1^4\frac{\omega^2}{2g} = 2R_0^2 h_1$$

OR $$R_1^4 = \frac{2R_0^2 h_1 + 2gh^2/\omega^2}{h - (\omega^2/2g)}$$

Summary

- Fluid motion can be described by vorticity dynamics.
- Vorticity is conserved in invicid flow.
- Vorticity is solenoidal, $\nabla \cdot \Omega = 0$. Vorticity must either close upon itself (like a torus) or close at material boundaries (like a twirling spoon.)