1. Quick Review
2. Vorticity
3. Kelvin’s Theorem
4. Examples

https://www.youtube.com/watch?v=-VL0M0jmu
How to solve fluid problems?
(like those in textbook)

**Step #1** - Draw a picture
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**Step #5** - Apply your intuition!

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**Mechanical Energy Density for a Stokes Fluid**

\[ \rho \frac{D}{Dt} \left( \frac{1}{2} u^2 \right) = \rho u_i \dot{u}_i + \frac{2}{3} \left( \mu \dot{e}_{ij} \right) - \tau_{ij} \epsilon_{ij} \]

where \( \epsilon_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \)

\[ \tau_{ij} = -\rho \delta_{ij} + 2\mu \epsilon_{ij} - \frac{2}{3} \mu (\sigma : \sigma) \delta_{ij} \]

**Or**

\[ \rho \frac{D}{Dt} \left( \frac{1}{2} u^2 \right) = \rho u_i \dot{u}_i + \rho (\sigma : \dot{u}) + \frac{2}{3} \left( \mu \dot{e}_{ij} \right) - \mu \left[ 2 \epsilon_{ij} \epsilon_{ij} - \frac{2}{3} (\sigma : \sigma) \right] \]

work due to expansion

viscous dissipation of mechanical work
**Fluids Equations**

**Continuity**
\[ \frac{\partial \rho}{\partial t} = - \rho \nabla \cdot \mathbf{U} \]

**Navier-Stokes**
\[ \frac{\partial \mathbf{U}}{\partial t} = \frac{\partial \mathbf{U}}{\partial t} + \mathbf{U} \nabla \cdot \mathbf{U} + \frac{1}{2} \nabla \mathbf{U} + \mathbf{U} \nabla \rho + \mathbf{U} \nabla \mathbf{U} \]

**Equation of State**
\[ \rho = \rho R T \quad \text{(ideal fluid)} \]
\[ \rho / \rho T \text{ constant} \quad \text{(isentropic)} \]

**Conservation of (internal) Energy**
\[ \frac{\partial Q}{\partial t} = \nabla \cdot \mathbf{Q} - \rho (\partial U / \partial t) + \mu \left[ \mathbf{D} \cdot \mathbf{D} - \frac{1}{2} \nabla \mathbf{Q} \right] \]

**Simplifying Assumptions...**
- **Incompressible** \( \nabla \cdot \mathbf{U} = 0 \), so \( \frac{\partial \rho}{\partial t} = 0 \)
- **Uniform Density** \( \rho = \text{constant} \)
- **Irrotational**: \( \nabla \times \mathbf{U} = 0 \)
- **Inviscid**: \( \mu = 0 \) (Euler Eq.)

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**Vorticity Dynamics**

*(Remember the importance of viscosity.)*

\[ \mathbf{J} = \text{VORTICITY, A VECTOR FIELD} \]
\[ = \nabla \times \mathbf{U} \quad \text{(thus, } \nabla \cdot \mathbf{J} = 0 \text{, } \mathbf{J} \text{ is solenoidal).} \]

\[ \nabla \times \left[ \frac{\partial \mathbf{U}}{\partial t} + \mathbf{J} \times \mathbf{U} \right] = \nabla - \mathbf{\hat{z}} \mathbf{\hat{z}} \mathbf{\hat{z}} - \nabla (\frac{1}{2} \mathbf{U} \cdot \mathbf{U}) + \mu \left[ \mathbf{D} \cdot \mathbf{D} - \frac{1}{2} \nabla \mathbf{Q} \right] \]

\[ \frac{\partial \mathbf{J}}{\partial t} + \nabla \left( \mathbf{J} \times \mathbf{U} \right) = \mu \nabla \times \nabla \mathbf{U} = \mu \nabla \mathbf{J} \]

\[ \frac{\partial \mathbf{J}}{\partial t} + \nabla \left( \mathbf{J} \times \mathbf{U} \right) = \mu \nabla \mathbf{J} \]

**Vorticity Equation!**
\[ \nabla \times (\mathbf{\Omega} \times \mathbf{U}) = \mathbf{\Omega} \nabla \cdot \mathbf{U} + \mathbf{U} \cdot \nabla \mathbf{\Omega} - \mathbf{\Omega} \cdot \nabla \mathbf{U} \]
\[ \nabla \times (\mathbf{A} \times \mathbf{B}) = \mathbf{A} (\nabla \cdot \mathbf{B}) - \mathbf{B} (\nabla \cdot \mathbf{A}) + \mathbf{B} \cdot (\nabla \mathbf{A}) - \mathbf{A} \cdot (\nabla \mathbf{B}) \]
Momentum - Vorticity

\[
\frac{Du}{Dt} = g - \frac{\nabla P}{\rho} + \nu \left[ \nabla^2 u + \frac{1}{3} \nabla (\nabla \cdot u) \right]
\]

\[
\frac{D\Omega}{Dt} = -\Omega \nabla \cdot u + (\Omega \cdot \nabla) u + \frac{\nabla \rho \times \nabla P}{\rho^2} + \nu \nabla^2 \Omega
\]

(Barotropic only ⇒ Kelvin’s Theorem)

\[
\frac{\partial \Omega}{\partial t} + \nabla \times (\Omega \times u) = \nu \nabla^2 \Omega
\]

Simple (2D) Fluid Rotation

Cylindrical Coordinates:

**Rigid Rotation**

\[
U_0 = \omega \hat{r}
\]

\[
J_2 = (\nabla \times u)^2 = \frac{1}{2} \frac{\partial}{\partial r} (\nabla U_0) = 2 \omega^2
\]

Vorticity = 2 x Rotation

Vorticity is constant for everywhere

Rigid Rotation

\[
\nabla^2 J_2 = 0 \quad \text{no viscous dissipation}
\]

**Line Vortex**

\[
U_0 = \frac{\Gamma}{\rho_0} \hat{\tau}
\]

\[
J_2 = (\nabla \times u)^2 = \phi \quad \text{irrotational}
\]

No vorticity within fluid

But "sink" of line vortex at origin.

Circulation = \oint \nabla \cdot u \, dr = \Gamma

Since \[ J_2 = 0 \], no viscous dissipation within fluid.
3D Fluid Flow

CFD calculation of flow in mixing tank

CFD calculation of atmosphere (Iso-Vorticity contours)

Today's Vorticity
Kelvin’s Theorem

\[ \frac{d}{dt} \int_S \mathbf{r} \cdot d\mathbf{A} = 0 \quad (\text{as } \mu \to 0) \]

\[ \int_S \mathbf{r} \cdot d\mathbf{A} = \int_S (\mathbf{v} \times \mathbf{u}) \cdot d\mathbf{A} \]

\[ = \oint \mathbf{u} \cdot d\mathbf{L} = \Gamma \]

\[ \text{Circulation of fluid is constant as } \mu \to 0 \]

Vortex tube is a tube of constant vorticity flux.

Circulation within tube is constant as it moves with fluid (as \( \mu \to 0 \))

A (General) Vector Theorem for a Moving Surface Flux

\[ \frac{d}{dt} \left[ \int_S \mathbf{a} \cdot d\mathbf{A} \right] = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \left[ \int_S \mathbf{a}(\mathbf{e}_t) \cdot d\mathbf{A} - \int_S \mathbf{a}(\mathbf{e}_0) \cdot d\mathbf{A} \right] \]

\[ \mathbf{a}(\mathbf{r}, t) = \text{vector field} \]

Surface \( S_1 \) moves to surface \( S_2 \) in time \( \Delta t \)

Identity:

\[ \int_S \mathbf{v} \cdot \mathbf{a} \, dV = \int_S (\mathbf{v} \cdot \mathbf{a}) (\mathbf{udt} \cdot \mathbf{n}) \, dA \quad \text{as } \Delta t \to 0 \]

\[ = \int_{S_2} \mathbf{a} \cdot dA - \int_{S_1} \mathbf{a} \cdot dA - \oint \mathbf{a} \cdot (\mathbf{u} \times \mathbf{e}_t) \, dL \]

\[ \text{Top surface} \quad \text{Bottom surface} \quad \text{Vortex surface} \]
Vector Flux Identity (cont)

\[
\frac{d}{dt} \left[ \int_S \tilde{\alpha} \cdot dA \right] = \lim_{\delta t \to 0} \frac{1}{\delta t} \left[ \int_S (\tilde{\alpha} + \delta \tilde{\alpha} + \cdots) \cdot dA - \int_S \tilde{\alpha} \cdot dA \right] \\
= \int_S \frac{2 \tilde{\alpha}}{\delta t} \cdot dA + \frac{1}{\delta t} \left[ \int_S \tilde{\alpha} \cdot dA - \int_S \tilde{\alpha} \cdot dA \right] \\
= \int_S \frac{2 \tilde{\alpha}}{\delta t} \cdot dA + \int_S (\nabla \times (\tilde{\alpha} \times \tilde{u})) \cdot dA \\
= \int_S \left( \frac{2 \tilde{\alpha}}{\delta t} + (\nabla \times (\tilde{\alpha} \times \tilde{u})) \right) \cdot dA
\]

There fore, if

\[
\frac{2 \tilde{\alpha}}{\delta t} + (\nabla \times (\tilde{\alpha} \times \tilde{u})) = 0
\]

then flux through surface moving with fluid is constant!!

For vorticity, \( \nabla \cdot \tilde{\alpha} = 0 \)
\[
\frac{2 \tilde{\alpha}}{\delta t} + (\nabla \times (\tilde{\alpha} \times \tilde{u})) = 0 \quad \text{Q.E.D.}
\]

Two Line Vorticies

Two vorticies move in the same direction orbit one another.

Rate of orbit = \( \frac{\pi}{\pi h^2} \)}
Two Counter-Directed Line Vortices

Two vortices pointed in opposite directions travel together.

Rate of translation = \( \frac{\pi}{2\pi h} \)

Ring of Point Vortices

http://www.student.math.uwaterloo.ca/~amat361/Fluid%20Mechanics/topics/vorticity.htm
Clouds of Opposite Vortices

http://www.student.math.uwaterloo.ca/~amat361/Fluid%20Mechanics/topics/vorticity.htm

Smoke Rings

TONOIDAL RING OF VORTICITY

\( \vec{F}_{\text{OUTERBOARD}} \rightarrow \text{FLOW DUE TO OPPOSITE SIGN} \)

\( \vec{F}_{\text{INNERBOARD}} \rightarrow \text{FLOW DUE TO OPPOSITE SIGN} \)

\( \text{CROSS-SECTION} \)

\( \text{SMOKE RING MOVES AT SPEED } \approx O(\frac{1}{\sqrt{n}}) \)
Toroidal Vortex Ring

Colliding Vortex Rings

http://serve.me.nus.edu.sg/limtt/#Video_Gallery
(Prof. Lim, Division of Fluid Mechanics, Melbourne, AU)
Surface of Rotating Bucket

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**Step 5** - Apply your intuition!
Surface of Rotating Bucket

1. Draw a picture.
2. Dynamic or Static?  ...Static
3. Coordinate system?  ...co-rotating cylindrical
   (no flow in this frame)
4. Can you apply conservation principles?  ...Bernoulli’s
5. Use your intuition.

Co-Rotating Frame

\[
\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = g - (1/\rho) \nabla p + \nu \nabla^2 \mathbf{u}
\]
\[
\frac{\partial \mathbf{u}^R}{\partial t} + \mathbf{u}^R \cdot \nabla \mathbf{u}^R = g - (1/\rho) \nabla p + \nu \nabla^2 \mathbf{u}^R + \mathbf{R} \perp \Omega^2 - 2\Omega \times \mathbf{u}^R
\]

with Centrifugal and Coriolis “apparent” forces.
(Very convenient when \(\mathbf{u}^R = 0\).)
Co-Rotating With Bucket

Co-Rotating Frame of Reference

\[ \bar{U}^R = \text{Co-Rotating Frame Velocity} = 0 \]

Equations of Motion...

\[ \bar{U} \cdot \bar{\nabla} \bar{U} = 0 \Rightarrow -\frac{1}{\rho} \bar{\nabla} p + \bar{f} + \bar{\omega} \times \bar{r} \]

Force balance:

\[ \bar{f} = -\bar{\nabla} \varphi_g \quad (\varphi_g < \varphi) \]

\[ \bar{r} = \bar{\nabla} \left( \bar{r}^2 / 2 \right) \]

So

\[ \bar{\nabla} \left( \frac{p}{\rho} + \varphi_g - \bar{\omega} \times \bar{r}^2 / 2 \right) = 0 \]

Constant Throughout

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Rigid Rotation Statics

\[ P_{atm} + \rho g (h + \delta) - \rho \bar{\omega}^2 R^2 / 2 \]

\[ = P_2 - \rho \bar{\omega}^2 R^2 / 2 \]

\[ P_2 = P_{atm} + \rho g (h + \delta) \]

\[ P_2 = P_{atm} + \rho g h \]

\[ P_2 - \rho \bar{\omega}^2 R^2 / 2 = P_3 \]

\[ P_3 = P_2 + \rho \bar{\omega}^2 R^2 + \rho g h + P_{atm} \]

Surface is Parabolic!

(Note: To achieve Rigid Rotation, fluid requires viscosity.
But Rigid Co-rotation has no viscous forces.)
**Summary**

- Fluid motion can be described by vorticity dynamics.

- Vorticity is conserved in invicid flow.

- Vorticity is solenoidal, \( \nabla \cdot \Omega = 0 \). Vorticity must either close upon itself (like a torus) or close at material boundaries (like a twirling spoon.)