APPH 4200
Physics of Fluids

Fluid Equations of Motion (Ch. 4)

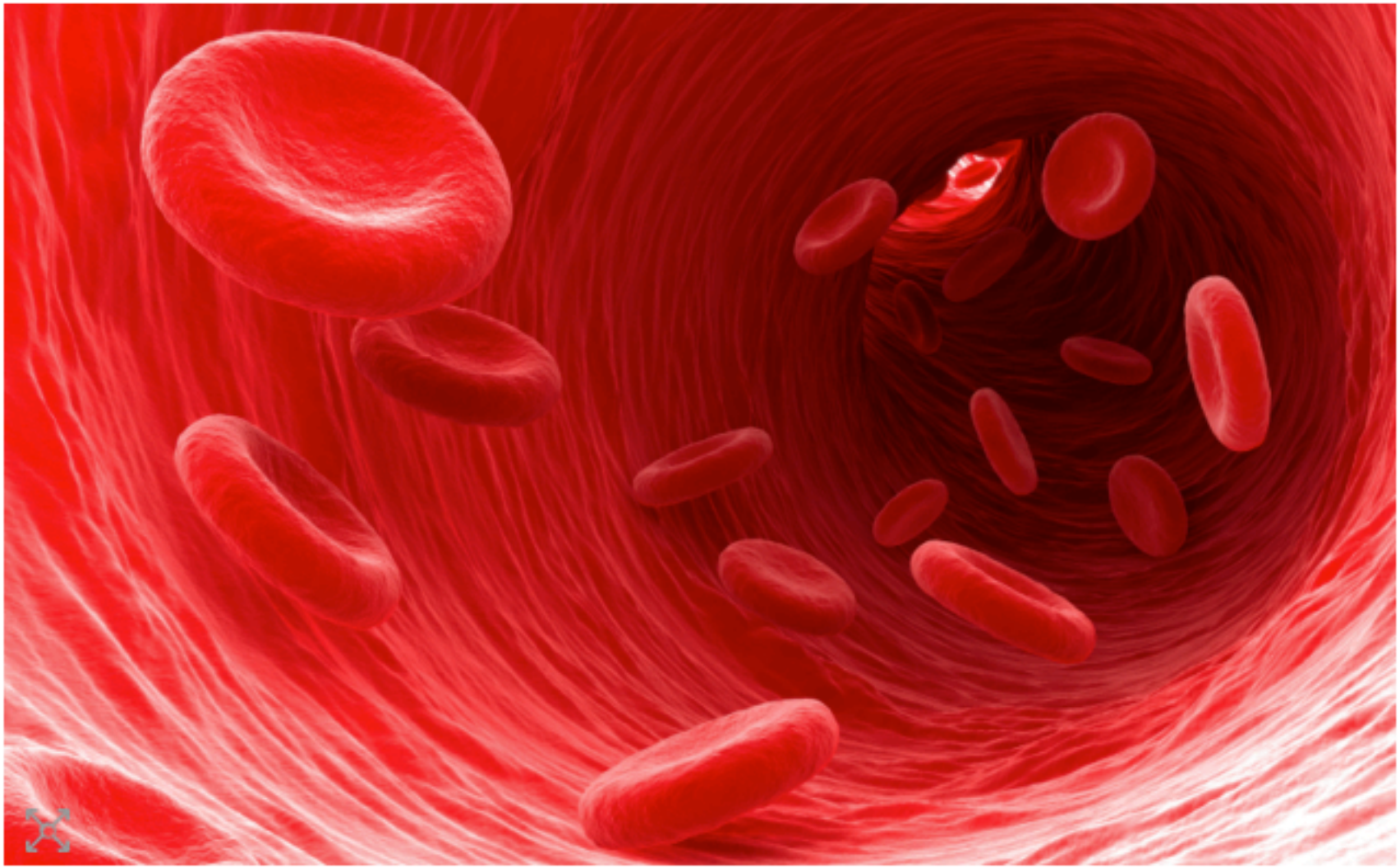
1. Integral E.O.M.s Moving/Fixed Volumes
2. Internal Energy
3. Bernoulli's Principle
4. Equations in co-rotating frames
5. Examples

Fluid Physics in the News...

BIOPHYSICS | RESEARCH UPDATE

Shape-shifting red blood cells respond to shear forces

26 Sep 2018



Flow-Induced Transitions of Red Blood Cell Shapes under Shear

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A recent study of red blood cells (RBCs) in shear flow [Lanotte *et al.*, Proc. Natl. Acad. Sci. U.S.A. **113**, 13289 (2016)] has demonstrated that RBCs first tumble, then roll, transit to a rolling and tumbling stomatocyte, and finally attain polylobed shapes with increasing shear rate, when the viscosity contrast between cytosol and blood plasma is large enough. Using two different simulation techniques, we construct a state diagram of RBC shapes and dynamics in shear flow as a function of shear rate and viscosity contrast, which is also supported by microfluidic experiments. Furthermore, we illustrate the importance of RBC shear elasticity for its dynamics in flow and show that two different kinds of membrane buckling trigger the transition between subsequent RBC states.

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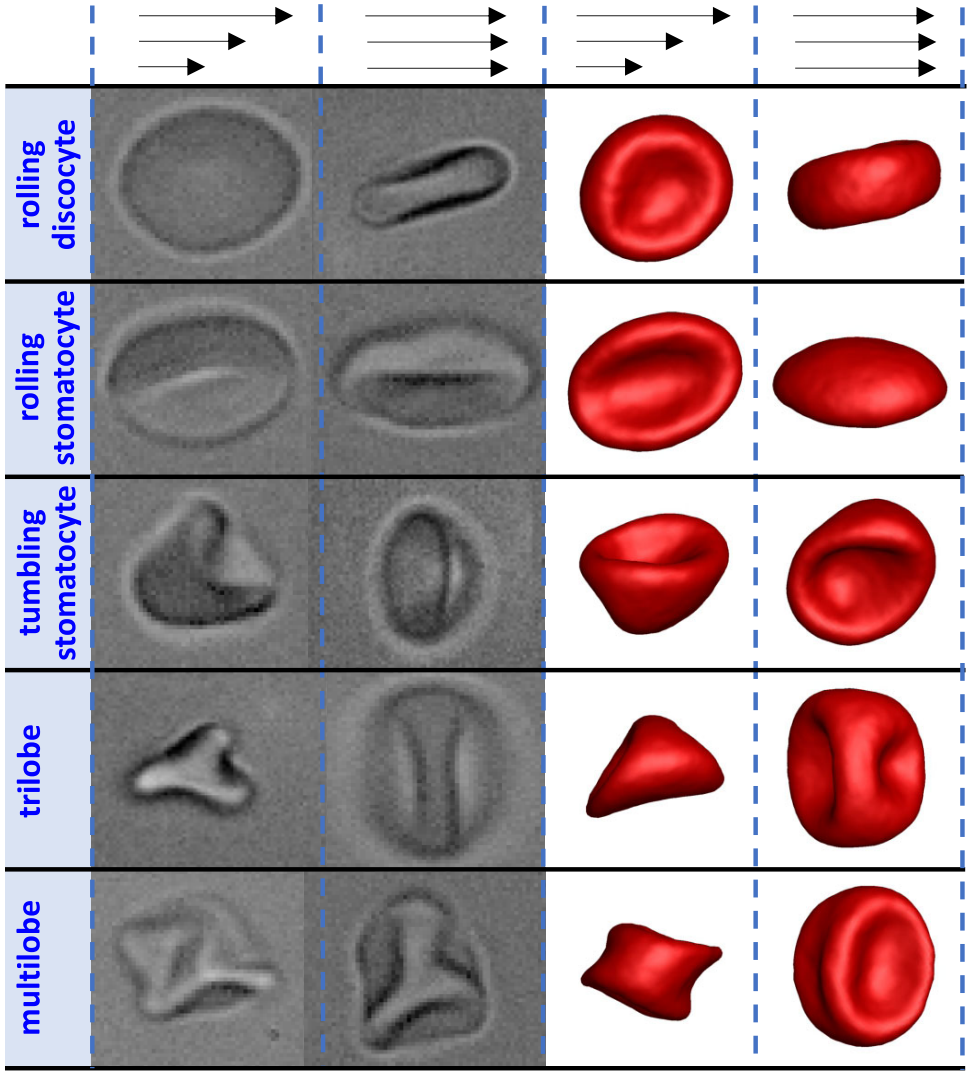


FIG. 1. RBC shapes observed in microfluidic experiments ($\lambda \approx 8$) and SDPD simulations ($\lambda \approx 5$) at various dimensionless shear rates $\dot{\gamma}^* = \dot{\gamma}\tau$ ($\tau \approx 1.2 \times 10^{-3}$ s). The shapes are rolling discocyte, rolling stomatocyte, TB stomatocyte, trilobe, and multilobe, observed at $\dot{\gamma}^* = 0.012$, 0.18, 0.3, 0.9, and 2.15 in experiments and at $\dot{\gamma}^* = 0.014$, 0.18, 0.34, 0.93, and 3.3 in simulations, respectively. Two views, vorticity and flow-gradient directions, are shown by the arrows with unequal and equal lengths, respectively. See also movies S1–S4.

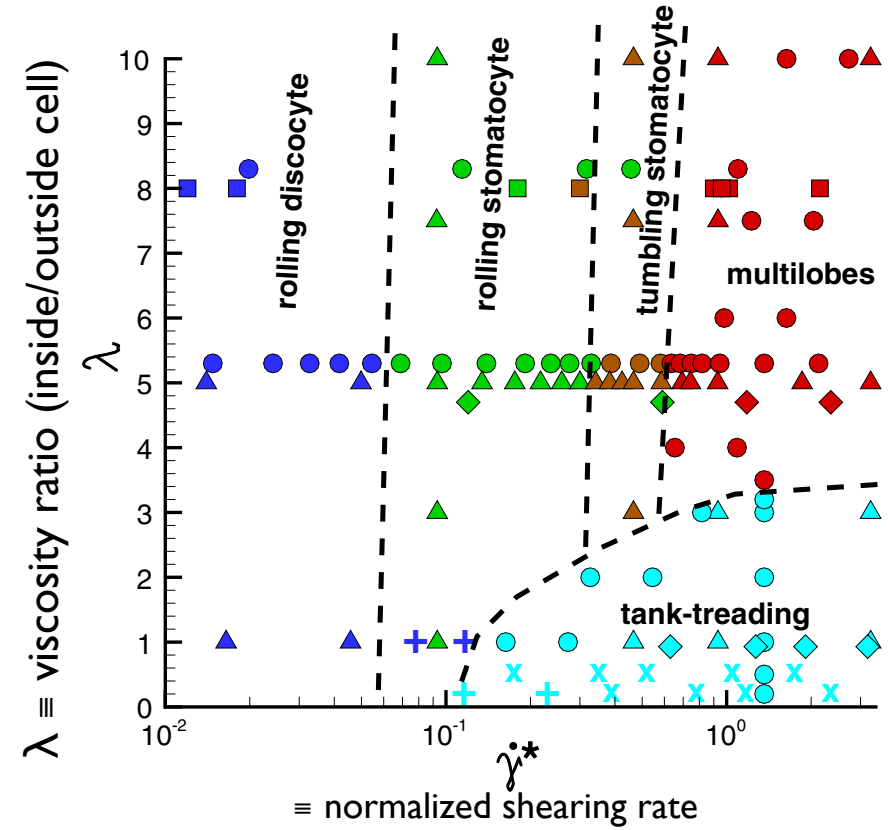


FIG. 2. Shapes and dynamics of RBCs in shear flow as a function of $\dot{\gamma}^*$ and λ. Different areas, representing rolling discocyte, rolling stomatocyte, TB stomatocyte, TT, and multilobes, are based on simulation results, where dashed lines serve as a guide to the eyes. Two sets of simulations are denoted by triangles (SDPD) and circles (YALES2BIO). The colors indicate RBC shape or dynamics. All simulation data are for Föppl-von Kármán number $\alpha = \mu D^2/\kappa = 680$. The two sets of circles at $\lambda \approx 5.3$ and $\lambda \approx 8.3$ correspond to simulations at $\lambda = 5$ and $\lambda = 8$, respectively, and are just shifted up in the diagram for visual clarity. The square symbols ($\lambda \approx 8$) correspond to experiments from Ref. [22], the plus symbols ($\lambda \leq 1$) to data from Ref. [10], and the crosses ($\lambda < 1$) to data from Ref. [38]. Diamond symbols ($\lambda \approx 1$ and $\lambda \approx 5$—shifted down to 4.7) represent most probable states from our microfluidic experiments, since no unique state, but a distribution of different states is obtained for fixed flow conditions, see distributions in Fig. S1. All experimental shear rates are normalized by τ based on average RBC properties (i.e., $D = 6.5 \times 10^{-6}$ m and $\mu = 4.8 \times 10^{-6}$ N/m).

Stokesian Fluid

MATERIAL ISOTROPY AND STRESS SYMMETRY

(e.g. AIR, WATER BUT <u>NOT</u> MAGNETIZED PLASMA)

$$\bar{\bar{\sigma}} = 2\mu \bar{\bar{\epsilon}} + \lambda (\nabla \cdot \bar{u}) \bar{\bar{\delta}}$$

VISCOSITY

~ BULK VISCOSITY

STOKES MODELED VISCOSITY VIA KINETIC THEORY OF MONATOMIC ATOMS AND SHOWED $\lambda = -\frac{2}{3}\mu$.

THEN, STRESS TENSOR

$$\bar{\bar{\tau}} = -p\bar{\bar{\delta}} + 2\mu\bar{\bar{\epsilon}} - \frac{2}{3}\mu(\nabla \cdot \bar{u})\bar{\bar{\delta}}$$

NS Properties

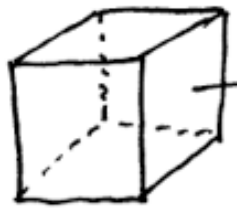
(incompressible)

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = \mathbf{g} - (1/\rho)\nabla p + \nu \nabla^2 \mathbf{u}$$

- Equation for **rate of change** of **u**(**x**,t) (not position) in lab frame
- Nonlinear. Complicated term: $\mathbf{u} \cdot \nabla \mathbf{u}$
- Viscous length scale: ν/U
- NS is a starting point: Once **u**(**x**,t) is known, then other fluid properties can be determined.

Fixed and Material Volumes

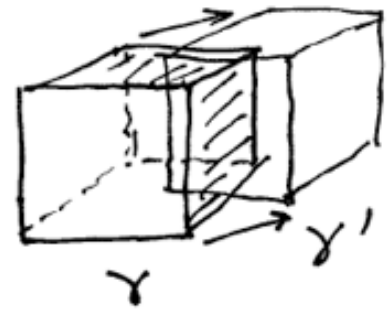
FIXED VOLUME



$\bar{u} f \cdot d\bar{A}$ IS FLUX THRU SURFACE

V

MATERIAL VOLUME MOVING WITH FLUID



$f\,\bar{u} \cdot d\bar{A}$ IS RATE OF CHANGE DUE TO MOVING BOUNDARY

FIXED VOLUME

$$\frac{d}{dt}\iiint_V f\,dV = \iiint_V \frac{\partial f}{\partial t}\,dV$$

IF f IS CONSERVED, THEN CHANGE OF f MUST BE DUE TO FLUX THRU SURFACE

$$\iiint_V \frac{\partial f}{\partial t}\,dV = -\iint_A f\,\bar{u} \cdot d\bar{A}$$

MOVING VOLUME (MATERIAL)

$$\frac{D}{Dt}\iiint_V f\,dV = \iiint_V \frac{\partial f}{\partial t}\,dV + \iint_A f\,\bar{u} \cdot d\bar{A}$$

GAUSS'S LAW

$$= \iiint_V dV \left[\frac{\partial f}{\partial t} + \bar{\nabla} \cdot (f\,\bar{u})\right]$$

Identities For Conserved Quantities

EXAMPLES OF "f":

$f = \rho$ MASS DENSITY

$f = \rho \bar{u}$ MOMENTUM DENSITY

$f = \rho \bar{r} \times \bar{u}$ ANGULAR MOMENTUM DENSITY

$f = \rho E$ INTERNAL ENERGY DENSITY

FIXED VOLUME

$$\frac{d}{dt}\iiint f\,dV = \iiint \frac{\partial f}{\partial t}\,dV$$

BUT EQUATIONS FOR f MUST INCLUDE SURFACE FLUX IF f IS A CONSERVED QUANTITY

E.G.

$$\frac{d}{dt}\iiint f\,dV = -\iint_A f\bar{u}\cdot d\bar{A}$$

$$\frac{d}{dt}\iiint \rho\bar{u}\,dV = -\iint_A \rho\bar{u}\bar{u}\cdot dA + \text{FORCE}$$

MATERIAL VOLUME

$$\frac{D}{Dt}\iiint f\,dV = \iiint_V dV\left[\frac{\partial f}{\partial t} + \nabla\cdot(f\bar{u})\right]$$

$$= \iiint_V dV\left[\frac{\partial f}{\partial t} + (\bar{u}\cdot\bar{\nabla})f + f\nabla\cdot\bar{u}\right]$$

BUT $\nabla\cdot\bar{u} = -\frac{1}{\rho}\frac{D\rho}{Dt}$ SO...

$$= \iiint_V dV\left[\frac{Df}{Dt} - \frac{f}{\rho}\frac{D\rho}{Dt}\right]$$

$$= \iiint_V dV\,\rho\frac{D}{Dt}\left(\frac{f}{\rho}\right)$$

E.G. LET $f = \rho\bar{u}$, THEN

$$\frac{D}{Dt}\iiint \rho\bar{u}\,dV = \iiint_V dV\,\rho\frac{D\bar{u}}{Dt}$$

IF ρ IS CONSERVED!!!

Example: Mechanical Energy

$$f = \frac{1}{2}\rho U^2$$

LOCAL CONSERVATION OF MECHANICAL ENERGY

$$\bar{u}\cdot\left[\rho\frac{D\bar{u}}{Dt} = \rho\bar{g} + \bar{\nabla}\cdot\bar{\bar{\tau}}\right]$$

$$\rho\frac{D}{Dt}\left(\tfrac{1}{2}u^2\right) = \rho\,\bar{u}\cdot\bar{g} + \bar{u}\cdot(\bar{\nabla}\cdot\bar{\bar{\tau}})$$

$$\left\{\begin{array}{l} \text{ADD:}\quad \tfrac{1}{2}u^2\left[\frac{\partial\rho}{\partial t} + \bar{\nabla}\cdot(\rho\bar{u}) = 0\right] \\ \frac{\partial}{\partial t}\left(\tfrac{1}{2}\rho u^2\right) + \bar{\nabla}\cdot\tfrac{1}{2}\rho u^2\,\bar{u} = \end{array}\right\}$$

NOTE:

$$\bar{u}\cdot\bar{\nabla}\cdot\bar{\bar{\tau}} = \bar{\nabla}\cdot\bar{u}\,\bar{\bar{\tau}} - \bar{\bar{\tau}}\cdot(\bar{\nabla}\bar{u})$$

$$u_i\frac{\partial\tau_{ij}}{\partial x_j} = \frac{\partial}{\partial x_j}(u_i\tau_{ij}) - \boxed{\tau_{ij}\frac{\partial u_i}{\partial x_j}}$$

$$= \frac{\partial}{\partial x_j}(u_i\tau_{ij}) - \tau_{ij}\epsilon_{ij}$$

$$\tau_{ij}\epsilon_{ij} = -p(\nabla\cdot\bar{u}) + \boxed{2\mu\left[\epsilon_{ij} - \tfrac{1}{3}\delta_{ij}\nabla\cdot\bar{u}\right]^2}$$

MOVING VOLUME

$$\frac{D}{Dt}\iiint dV\,\tfrac{1}{2}\rho u^2 = \iiint dV\left(\rho\bar{g}\cdot\bar{u} + p\nabla\cdot\bar{u}\right) + \iint_A d\bar{A}\cdot(\bar{u}\cdot\bar{\bar{\tau}}) - \iiint dV\,2\mu\epsilon^2$$

FIXED VOLUME

$$\frac{d}{dt}\iiint dV\,\tfrac{1}{2}\rho u^2 = \iiint dV\left(\rho\bar{g}\cdot\bar{u} + p\nabla\cdot\bar{u}\right) + \iint_A d\bar{A}\cdot(\bar{u}\cdot\bar{\bar{\tau}}) - \iint_A d\bar{A}\cdot\bar{u}\,\tfrac{1}{2}\rho u^2 - \iiint dV\,\boxed{2\mu\,\epsilon^{*2}}$$

deformation work

Energy Conservation

- MECHANICAL ENERGY DENSITY $\frac{1}{2}\rho u^2$
- INTERNAL/THERMAL ENERGY DENSITY $\rho E \sim \rho C_v T$
  (NOTE: TEXTBOOK USES "e" NOT "E")
- EQUATION OF STATE
  e.g. IDEAL GAS $P = \rho R T$ (WHERE $R \equiv C_p - C_v$)

COMMENTS:

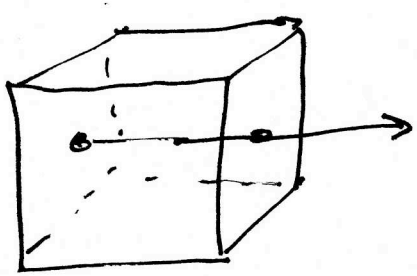
- CHANGE OF TOTAL ENERGY EQUALS SUM OF WORK AND HEAT ADDED.
- DISSIPATION (VISCOSITY) CAN CHANGE MECHANICAL ENERGY INTO THERMAL ENERGY
- THERMAL DIFFUSION CAN MOVE INTERNAL ENERGY FROM PLACE TO PLACE

Mechanical Energy

$$\bar{u}\cdot\left[\rho\frac{D\bar{u}}{Dt} = \rho\bar{g} + \nabla\cdot\bar{\bar{\tau}}\right]$$

or

$$\rho\frac{D}{Dt}\left(\tfrac{1}{2}u^2\right) = \underbrace{\rho\,\bar{u}\cdot\bar{g}}_{\text{BODY FORCE}} + \underbrace{\bar{u}\cdot(\nabla\cdot\bar{\bar{\tau}})}_{\underbrace{u_i\frac{\partial}{\partial x_j}\tau_{ji}}_{\text{SURFACE FORCE}}}$$

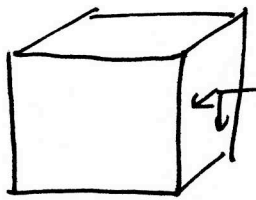


$\frac{\partial \tau_{ij}}{\partial x_j}$ = NET FORCE FROM SURFACE ONTO FLUID ELEMENT

$u_i\frac{\partial \tau_{ij}}{\partial x_j}$ = MECHANICAL WORK DONE ON FLUID BY SURFACE STRESS

~ THE WORK DONE TO <u>MOVE</u> THE ELEMENT

Total Work Done by Stress



$\tau_{ij}\, dA_j$ IS THE <u>TOTAL</u> FORCE ON SURFACE DUE TO STRESS.

SO $u_i \tau_{ij}\, dA_j$ = <u>TOTAL</u> <u>WORK</u> DONE BY SURFACE STRESS INCLUDING DEFORMATION, STRETCHING, PINCHING, ROTATION!

SO $\nabla \cdot (\bar{u} \cdot \bar{\bar{\tau}}) = \frac{\partial}{\partial x_j}(u_i \tau_{ij})$ = RATE OF <u>TOTAL</u> WORK ON SURFACE INCLUDING DEFORMATION

$$\frac{\partial}{\partial x_j}(u_i \tau_{ij}) = \underbrace{\tau_{ij} \frac{\partial u_i}{\partial x_j}}_{\text{DEFORMATION WORK}} + \underbrace{u_i \frac{\partial \tau_{ij}}{\partial x_j}}_{\text{MECHANICAL WORK}}$$

NOTE: SINCE τ_{ij} IS SYMMETRIC (USUALLY), THEN $\tau_{ij} \frac{\partial u_i}{\partial x_j} = \tau_{ij} \epsilon_{ij}$.

SINCE A SYMMETRIC TENSOR CONTRACTED WITH ANOTHER TENSOR EXTRACTS ONLY THE SYMMETRIC PART.

Mechanical Energy Density for a Stokes Fluid

$$\rho \frac{D}{Dt}\left(\tfrac{1}{2}u^2\right) = \rho u_i g_i + \frac{\partial}{\partial x_j}\left(u_i \tau_{ij}\right) - \tau_{ij}\epsilon_{ij}$$

WITH $\tau_{ij} = -p\delta_{ij} + 2\mu\epsilon_{ij} - \frac{2}{3}\mu(\nabla\cdot u)\delta_{ij}$

GIVES $\tau_{ij}\epsilon_{ij} = -p\epsilon_{ii} + 2\mu\epsilon_{ij}\epsilon_{ij} - \frac{2}{3}\mu(\nabla\cdot u)\epsilon_{ii}$

WHERE $\epsilon_{ii} = \sum_i \epsilon_{ii} = \nabla\cdot\bar{u}$

OR

$$\rho \frac{D}{Dt}\left(\tfrac{1}{2}u^2\right) = \rho u_i g_i + \overbrace{p(\nabla\cdot u)}^{\text{WORK DUE TO EXPANSION}} + \frac{\partial}{\partial x_j}\left(u_i \tau_{ij}\right) \underbrace{- \mu\left[2\epsilon_{ij}\epsilon_{ij} - \frac{2}{3}(\nabla\cdot u)^2\right]}_{\text{VISCOUS DISSIPATION OF MECHANICAL WORK}}$$

Conservation of Total Energy Density

$$\rho \frac{D}{Dt}\left(E + \tfrac{1}{2}u^2\right) = \rho\, \bar{u}\cdot\bar{g} + \nabla\cdot(\bar{u}\cdot\bar{\bar{\tau}}) - \underbrace{\nabla\cdot\bar{q}}_{\text{HEAT FLUX}}$$

SIMPLIS "FICKS LAW" FOR HEAT FLUX IS $\bar{q} = -k\nabla T$

Equation for Internal Energy

(Temperature)

SUBTRACT MECHANICAL ENERGY EQN FROM TOTAL ENERGY EQN TO GET

$$\rho \frac{D}{Dt} E = -p \nabla \cdot \bar{u} - \nabla \cdot \bar{q} + \mu \left(2 \epsilon_{ij} \epsilon_{ij} - \frac{2}{3} (\nabla \cdot \bar{u})^2 \right) \quad \Big/ \; E = c_v T$$

BUT CONSERVATION OF MASS ALLOWS US TO WRITE THIS IN ANOTHER FORM...

$$p \nabla \cdot \bar{u} = -\frac{p}{\rho} \frac{D\rho}{Dt} \qquad \left(\text{SINCE } \frac{D\rho}{Dt} = -\rho \nabla \cdot \bar{u} \right)$$

PLUS, EQUATION OF STATE...

$$p = \rho R T \qquad (R \equiv c_p - c_v)$$ (Meyer's Equation for an ideal gas)

$$\frac{Dp}{Dt} = RT \frac{D\rho}{Dt} + \rho R \frac{DT}{Dt}$$

$$\therefore \quad p (\nabla \cdot \bar{u}) = -\frac{p}{\rho} \frac{D\rho}{Dt} = -RT \frac{D\rho}{Dt} = -\frac{Dp}{Dt} + \rho (c_p - c_v) \frac{DT}{Dt}$$

Internal Energy Equation

$$\underbrace{\rho \frac{DE}{Dt} + \rho (C_p - C_v) \frac{DT}{Dt}}_{\rho C_p \frac{DT}{Dt} \;=\; \text{''}} = -\nabla \cdot q + \frac{DP}{Dt} + \mu \left(2 \epsilon_{ij} \epsilon_{ij} - \frac{2}{3} (\nabla \cdot \bar{u})^2 \right)$$

IF WE NEGLECT $\frac{DP}{Dt}$ AND (SMALL) DISSIPATION, WE OBTAIN THE USUAL THERMAL CONDUCTION EQUATION...

$$\rho C_p \frac{DT}{Dt} \approx -\nabla \cdot \bar{q} = \nabla \cdot k \nabla T$$

IF $k \sim$ CONSTANT, THEN

$$\frac{DT}{Dt} = \underbrace{\left(\frac{k}{\rho C_p} \right)}_{\kappa \,=\, \text{THERMAL DIFFUSIVITY}} \nabla^2 T$$

Entropy

(Disorder, Direction of Time)

$S \equiv$ ENTROPY MUST INCREASE (2ND LAW)

$$T\,dS = dE + p\,dV$$

BUT $V\rho \sim 1$ SO $dV \sim -V\dfrac{d\rho}{\rho} \sim -\dfrac{d\rho}{\rho^2}$

THIS RESULTS IN

$$T\frac{dS}{Dt} = \left(\frac{DE}{Dt} - \frac{P}{\rho^2}\frac{D\rho}{Dt}\right)$$

$$= -\nabla\cdot\bar{q} + \mu\left(2\epsilon_{ij}\epsilon_{ij} - \tfrac{2}{3}(\nabla\cdot\bar{v})^2\right)$$

HOW DOES S CHANGE?

$$\frac{DS}{Dt} = -\frac{1}{T}\nabla\cdot\bar{q} + \underbrace{\frac{\mu}{T}(\text{DISSIPATION})}_{\boxed{\text{ALWAYS POSITIVE IF } \mu > 0}}$$

$$= \underbrace{-\nabla\cdot\left(\frac{q}{T}\right)}_{\text{RE-ARRANGE}} - \underbrace{\frac{q\cdot\nabla T}{T^2}}_{q\cdot\nabla T < 0} + (\,\,)$$

OR

HEAT MUST FLOW FROM HOT TO COLD

$$\bar{q} = -k\nabla T \quad \boxed{(k > 0\,!)}$$

Summary of Fluid Dynamical Equations

$$\frac{D\rho}{Dt} = -\rho \nabla \cdot \bar{u}$$

$$\rho \frac{D\bar{u}}{Dt} = \rho \bar{g} + \bar{\nabla} \cdot \bar{\bar{\tau}}$$

$$= \rho \bar{g} - \nabla p + \mu \left[\nabla^2 \bar{u} + \frac{1}{3} \bar{\nabla} (\nabla \cdot \bar{u}) \right]$$

$$p = \rho R T$$

$$\rho \frac{DE}{Dt} = \rho c_v \frac{DT}{dt} = \nabla \cdot k \nabla T - p (\nabla \cdot \bar{u}) + \mu (\text{DISSIPATION})$$

$$\rho c_p \frac{DT}{Dt} = \nabla \cdot k \nabla T + \frac{Dp}{Dt} + \mu (\text{DISSIPATION})$$

AND $\mu > 0$, $k > 0$ FOR $\partial S \geq 0$

Bernoulli's Equation

(Euler Equation)

EULER EQUATION

$$\rho \frac{D\bar{u}}{Dt} = \rho \bar{g} - \nabla p \qquad \bar{g} = -\nabla \varphi_g$$

$$\rho \left(\frac{\partial \bar{u}}{\partial t} + (\bar{u} \cdot \bar{\nabla}) \bar{u} \right) = -\rho \nabla \varphi_g - \nabla p$$

DOT $(\bar{u} \cdot \bar{\nabla}) \bar{u} = \bar{\nabla} \left(\frac{1}{2} u^2 \right) + (\nabla \times \bar{u}) \times \bar{u}$ $\qquad \bar{\Omega} = \nabla \times \bar{u}$ = VORTICITY

SO

$$\frac{\partial \bar{u}}{\partial t} + \bar{\nabla} \left(\frac{1}{2} u^2 + \varphi_g \right) + \frac{1}{\rho} \nabla p = -\bar{\Omega} \times \bar{u}$$

BUT $\boxed{h = E + p/\rho}$ $\quad dh \sim dp/\rho$ IF S = ENTROPY IS CONSTANT

$$\boxed{\int \frac{dp}{\rho} \sim \frac{p}{\rho} \text{ IF } \rho \approx \text{CONSTANT}}$$

SO $\frac{\partial}{\partial x} \int dh = \frac{\partial}{\partial x} \int \frac{dp}{\rho} = \frac{1}{\rho} \frac{\partial p}{\partial x} = \frac{1}{\rho} \nabla p$

SO

$$\boxed{\frac{\partial \bar{u}}{\partial t} + \nabla \left(\varphi_g + \frac{1}{2} u^2 + \int \frac{dp}{\rho} \right) = -\bar{\Omega} \times \bar{u}}$$

$\boxed{\text{IF } \frac{\partial \bar{u}}{\partial t} \to 0}$, THEN $\bar{\Omega}, \bar{u}$ DEFINE A <u>SURFACE</u> WHERE

$$\varphi_g + \frac{1}{2} u^2 + \int \frac{dp}{\rho} = \text{CONSTANT}$$

Bernoulli's Equation

(Conservation of Energy)

CONSERVATION OF ENERGY

$$\rho \frac{D}{Dt}\left(E + \tfrac{1}{2}u^2\right) = \rho\, \bar{g}\cdot\bar{u} + \bar{\nabla}\cdot(\bar{\bar{\tau}}\cdot\bar{u}) - \nabla\cdot\bar{q}$$

BERNOULLI'S PRINCIPLE IS VERY USEFUL BUT IT IS AN <u>APPROXIMATION</u> VALID WHEN $\mu = k \to 0$: VISCOSITY AND THERMAL DIFFUSION CAN BE IGNORED.

SO $\nabla\cdot\bar{q} \to 0$ $\qquad \tau_{ij} \to -p\,\delta_{ij}$

$$\nabla\cdot\bar{\bar{\tau}}\cdot\bar{u} \to -\nabla\cdot(p\bar{u})$$

$\boxed{\text{LET } \bar{g} = -\nabla\varphi_g}$, $\quad \varphi_g$ = GRAVITATIONAL POTENTIAL.

<u>THEN</u>

$$\rho \frac{D}{Dt}\left(E + \tfrac{1}{2}u^2\right) = -\rho\,\bar{u}\cdot\nabla\varphi_g - \nabla\cdot(p\bar{u})$$

BUT $\quad \nabla\cdot(p\bar{u}) = \nabla(\rho p\bar{u}/\rho) = \rho\,\bar{u}\cdot\bar{\nabla}(p/\rho) + \frac{p}{\rho}\nabla\cdot(\rho\bar{u})$

SO IN $\boxed{\text{STEADY STATE}}$

$$0 \cong -\rho\,\bar{u}\cdot\bar{\nabla}\left[E + \frac{p}{\rho} + \tfrac{1}{2}u^2 + \varphi_g\right] \qquad !!$$

$$E + \frac{p}{\rho} + \tfrac{1}{2}u^2 + \varphi_g = \text{CONSTANT ALONG STREAMLINES}$$

$$\boxed{h \equiv E + p/\rho = \text{ENTHALPY}}$$

Bernoulli's Equation

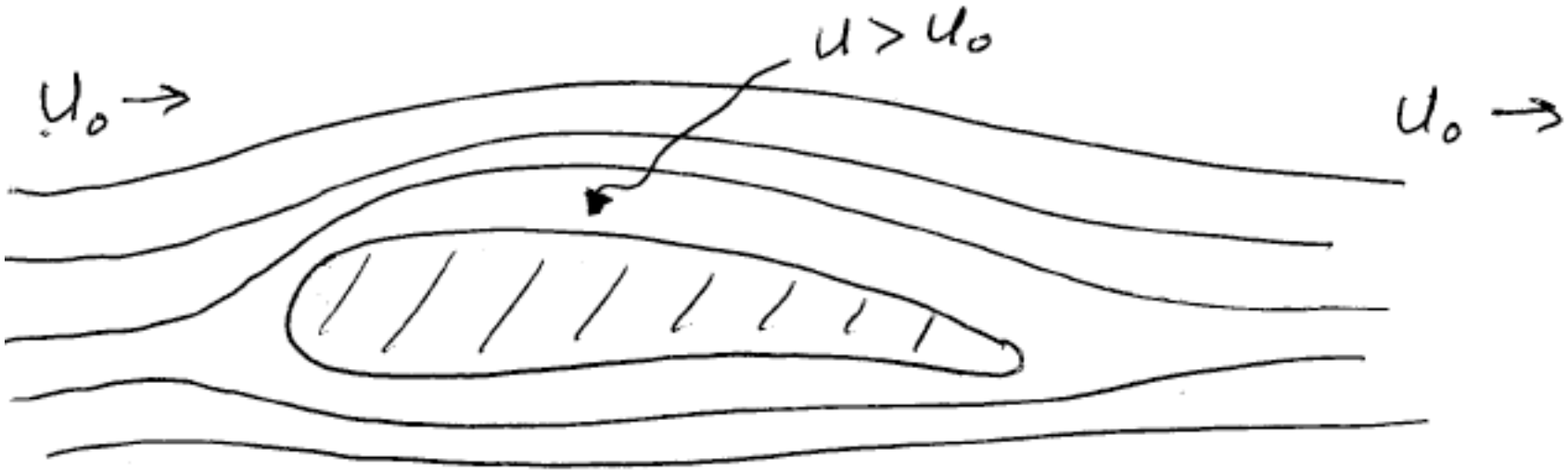
$$\nabla B = \mathbf{u} \times \mathbf{\Omega}$$ (Steady Flow)

$$B \equiv \frac{1}{2}u^2 + \int \frac{dp}{\rho} + gz = \text{constant},$$

$$\text{or} \quad \frac{1}{2}u^2 + \frac{p}{\rho} + gz = \text{constant}$$

(also time-varying irrotational flow)

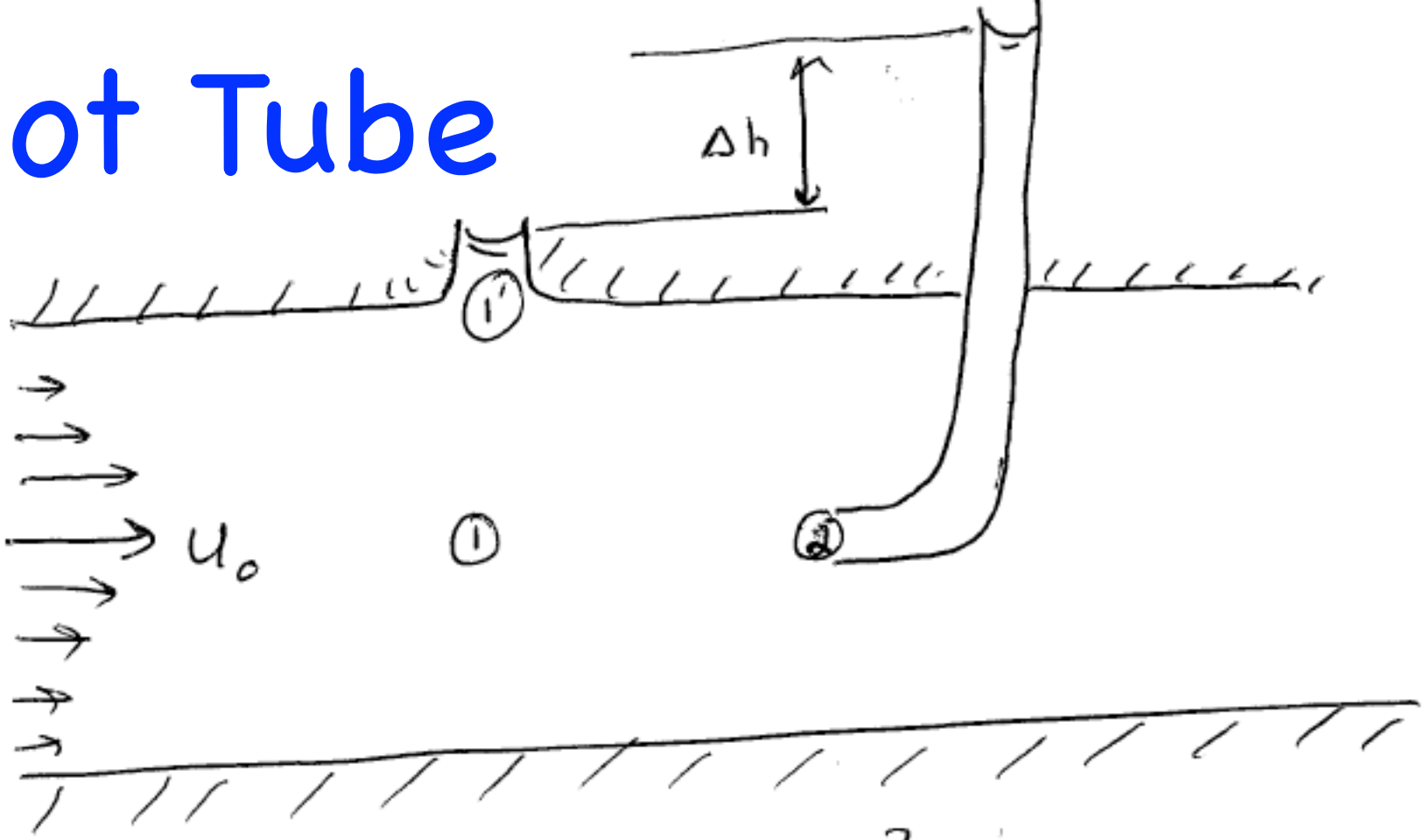
Lift



$$\frac{1}{2}\rho u^2 + P = \text{CONSTANT}$$

$$\text{SO} \quad \Delta P = \frac{1}{2}\rho\left(u^2 - u_0^2\right)$$

Pilot Tube



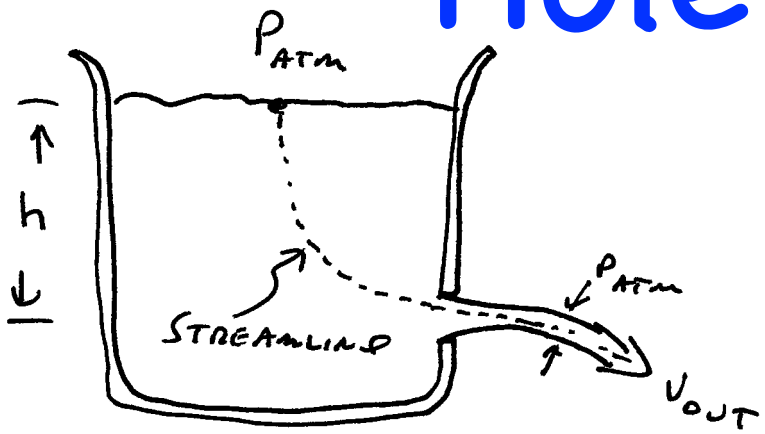
WHAT IS FLOW AT 1?

<u>NO</u> FLOW AT 2 AND 1'

$$\therefore \frac{1}{2}\rho u_1^2 + P_1 = P_2 \Rightarrow u_1 = \sqrt{\frac{2}{\rho}(P_2 - P_1)}$$

BUT $P_2 - P_1 = \rho g \Delta h$!!

Hole in Bucket



ASSUME FLOW IS FAST <u>OUT</u> OF HOLE AND MUCH FASTER THAN FLOW AT LIQUID SURFACE.

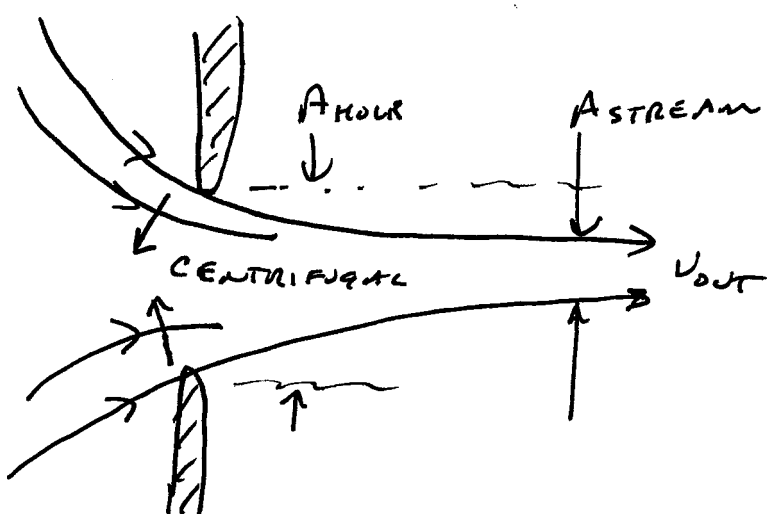
$$P_{ATM} + \frac{1}{2}\rho \cancelto{0}{u_{TOP}^2} = P_{ATM} + \frac{1}{2}\rho u_{OUT}^2 - \rho g h$$

AND $u_{OUT} \sim \sqrt{2gh}$

~ SAME SPEED AS A DROPPED DROPLET.

INTERESTING THAT MASS FLOW IS <u>NOT</u> $\rho A_{HOLE}\sqrt{2gh}$.

BECAUSE OF CENTRIFUGAL FORCE <u>CONTRACTS</u> FLOW STREAM



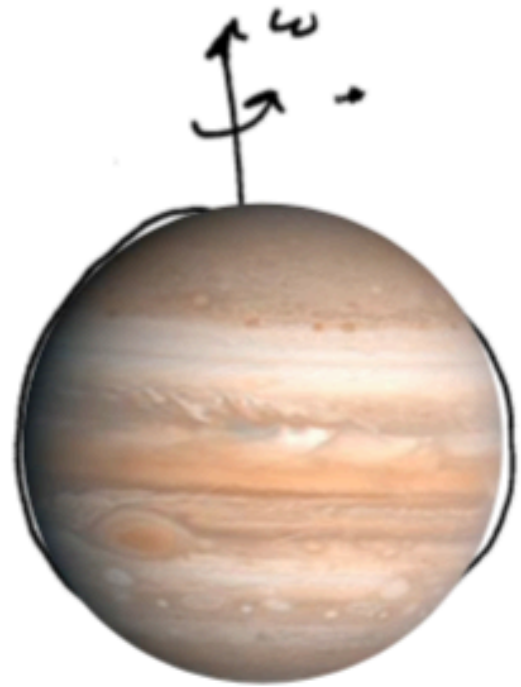
What are E.O.M. in Co-Rotating Frame of Reference?

EARTH

$\omega_E \sim 10^{-5}$ RAD/SEC

$R_E \sim 6.4 \times 10^6$ m

$g_E \sim 9.8$ m/SEC^2



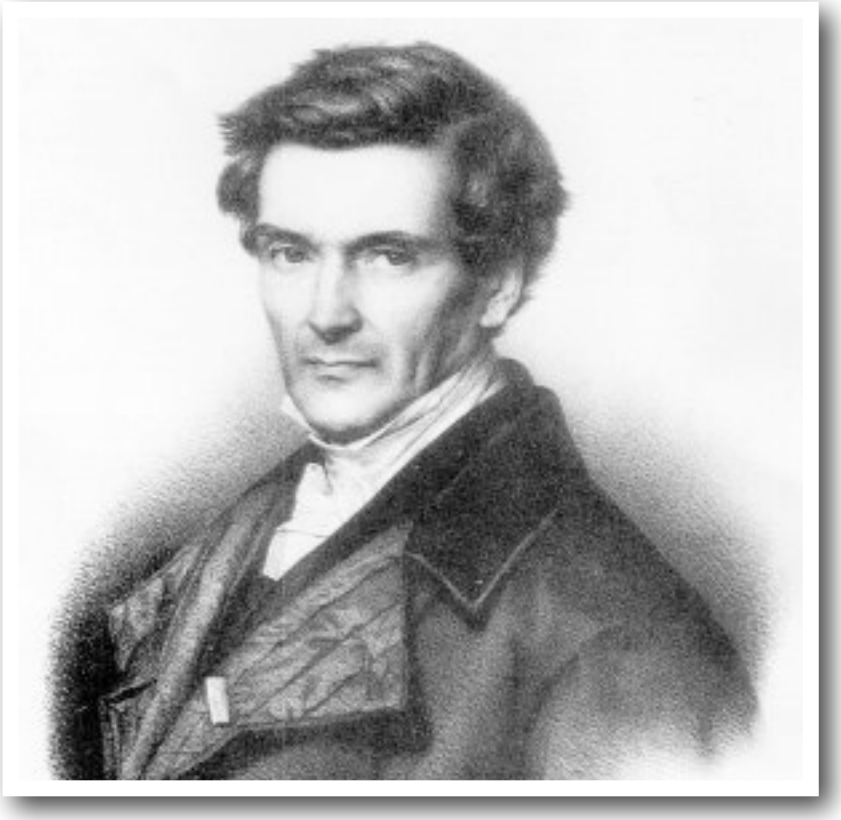
JUPITER

$\omega_J = 18 \times 10^{-5}$ RAD/SEC

$R_J = 71 \times 10^6$ m

$g_J = 23$ m/s^2

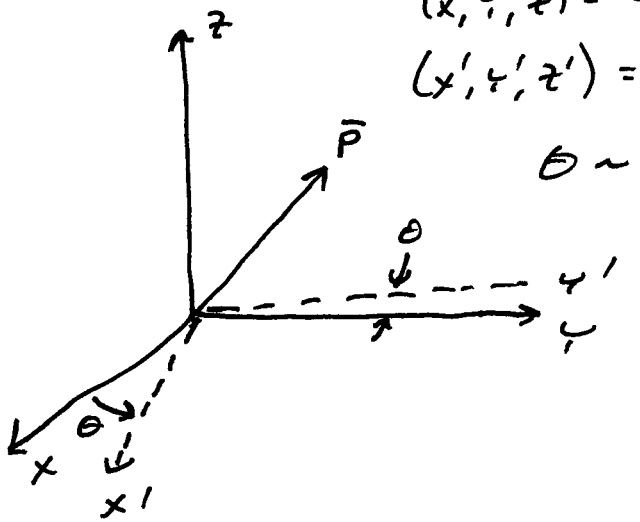
Gaspard-Gustave Coriolis



1792-1843

"On the equations of relative motion of a system of bodies", published 1835

Small Rotation of Frame in δt



(x, y, z) = LAB FRAME

(x', y', z') = CO-ROTATING FRAME

$\theta \sim \omega \delta t$ AS $\delta t \to 0$

$\bar{P}^F$ = VECTOR $\bar{P}$ IN FIXED FRAME OF REFERENCE

$\bar{P}^R$ = VECTOR $\bar{P}$ IN ROTATING FRAME

$$\frac{dP_i^F}{dt} = \lim_{\delta t \to 0} \left(\frac{P_i^F(t+\delta t) - P_i^F(t)}{\delta t} \right) = \lim_{\delta t \to 0} \left[\frac{(C_{ij} P_j^R)(t+\delta t) - (C_{ij} P_j^R)}{\delta t} \right]$$

WHERE $P_i^F(t) = C_{ij}(t)\, P_j^R(t)$

$$C_{ij} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$ WHERE $\theta \sim \omega \delta t$

FOR SMALL δt

$$\cong \begin{pmatrix} 1 & -\omega\delta t & 0 \\ \omega\delta t & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \delta_{ij} + R_{ij}$$

ANTISYMMETRIC ROTATION TENSOR

Co-Rotating Frame

(continued)

THUS

$$\frac{dP_i^F}{dt} = \lim_{\delta t \to 0} \left[\frac{(\delta_{ij} + R_{ij})\left(P_j^R(t) + \delta t \frac{\partial P_j^R}{\partial t}\right) - \delta_{ij} P_j^R}{\delta t} \right]$$

$$= \frac{dP_i^R}{dt} + \left(\frac{R_{ij}}{\delta t}\right) P_j^R = \frac{dP_i^R}{dt} + \bar{\omega} \times \bar{P}^R$$

LET $\bar{P} = \bar{r}$, THEN WE CAN TRANSFORM VELOCITY AND ACCELERATION BETWEEN LAB AND CO-ROTATING FRAMES

$$\bar{u}^F = \bar{u}^R + \bar{\Omega} \times r$$

$$\frac{du^F}{dt} = \left.\frac{d\bar{u}^F}{dt}\right|_{\text{ROTATING FRAME}} + \bar{\Omega} \times \bar{u}^F$$

$$= \frac{d}{dt}\left(\bar{u}^R + \bar{\Omega} \times \bar{r}\right) + \bar{\Omega} \times \left(\bar{u}^R + \bar{\Omega} \times r\right)$$

$$= \frac{du^R}{dt} + \bar{\Omega} \times \bar{u}^R + \bar{\Omega} \times \bar{u}^R + \bar{\Omega} \times (\bar{\Omega} \times \bar{r})$$

Centrifugal and Coriolis Forces

LAB CO-ROTATING

$$\frac{d\bar{u}^F}{dt} = \frac{d\bar{u}^R}{dt} + \underbrace{2\bar{\Omega}\times\bar{u}^R}_{\text{CORIOLIS}} + \underbrace{\bar{\Omega}\times(\bar{\Omega}\times\bar{r})}_{\text{CENTRIFUGAL}}$$

$$\bar{\Omega}\times(\bar{\Omega}\times\bar{r}) = \bar{\Omega}(\bar{\Omega}\cdot\bar{r}) - \bar{r}\,\Omega^2$$

DEFINE $\bar{R}_\perp = \bar{r} - \bar{\Omega}(\bar{\Omega}\cdot\bar{r})/\Omega^2$

THEN $\bar{\Omega}\times\bar{R}_\perp = \bar{\Omega}\times\bar{r}$

$\bar{R}_\perp$ = PERPENDICULAR COMPONENT OF RADIUS

SO

$$\frac{d\bar{u}^F}{dt} = \frac{d\bar{u}^R}{dt} + 2\bar{\Omega}\times\bar{u}^R - \Omega^2\bar{R}_\perp$$

Incompressible Navier-Stokes in Co-Rotating Frame

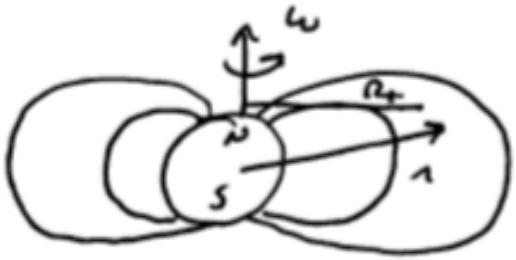
LAB:

$$\frac{d\bar{u}}{dt} = \bar{g} - \frac{\nabla p}{\rho} + \left(\frac{\mu}{\rho}\right)\nabla^2 \bar{u}$$

CO-ROTATING:

$$\frac{d\bar{u}^R}{dt} = \bar{g} + \underbrace{\bar{R}_+ \Omega^2}_{\text{CENTRIFUGAL FORCE}} - \frac{\nabla p}{\rho} - \underbrace{2\bar{\Omega} \times \bar{u}^R}_{\text{CORIOLIS FORCE}} + \left(\frac{\mu}{\rho}\right)\nabla^2 \bar{u}^R$$

Co-Rotating MagnetoSphere



$$\bar{g} = -\hat{r}\, g_0 \left(\frac{R_0}{r}\right)^2$$

$$\Omega^2 \bar{R}_\perp = r \sin\theta\, \Omega^2 \left(\hat{r} \sin\theta + \hat{\theta} \cos\theta\right)$$

$\theta \sim \pi/2$

AT EQUATOR, EFFECTIVE GRAVITY

$$g_{EFF} = g_0 \left(\frac{R_0}{r}\right)^2 - r\Omega^2$$

"GRAVITY" REVERSES DIRECTION IN CO-ROTATING FRAME WHEN

$$r = \sqrt[3]{\frac{g_0 R_0^2}{\Omega^2}} = \begin{cases} \text{EARTH} = 25\, R_E \\ \text{JUPITER} = 2.2\, R_J \end{cases}$$

CENTRIFUGAL FORCE AT SURFACE OF EARTH IS VERY SMALL...

$$\frac{R_E \Omega^2}{g_E} \lesssim 10^{-4}$$

Coriolis Force

MOTION FROM NORTH POLE APPEARS TO MOVE WESTWARD.

CORIOLIS FORCE RESPONSIBLE FOR WINDS, CIRCULATION OF ATMOSPH. ABOUT H, L PRESSURE

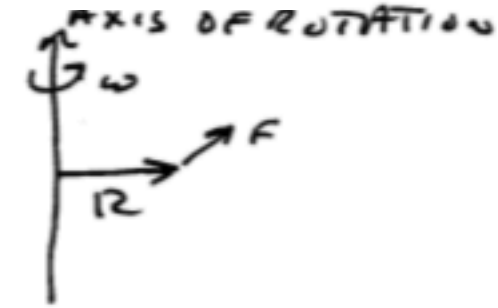
NORTHERN HEMISPHERE

HOW BIG IS CORIOLIS FORCE?

$$\frac{2\bar{\Omega}\times\bar{u}}{g_E} \sim \frac{2\omega_E \cos\theta}{g_E} u_{WIND} \sim 2\times10^{-5} \text{ AT POLES}$$

WITH $u_{NS} \sim 20$ M.P.H ~ 9 m/sec

Angular Momentum



$\bar{T}$ = TORQUE = $\bar{R} \times$ (FORCE)

$\bar{L}$ = ANGULAR MOMENTUM = $\bar{R} \times$ (MOMENTUM)

$$\frac{d\bar{L}}{dt} = \bar{T}$$

FOR A FLUID ABOUT SOME AXIS ...

$$\bar{L} = \iiint_U \bar{R} \times \bar{u}\rho \, dU$$

$$\bar{T} = \iiint_U \bar{R} \times (\rho\bar{g}) \, dU + \iint_A \bar{R} \times (\bar{\bar{\tau}} \cdot d\bar{A})$$

FOR A VOLUME, THERE CAN BE AN ANGULAR MOMENTUM FLUX

SO $$\bar{T} = \frac{d}{dt}\iiint_V R \times (\rho\bar{u}) \, dU + \iint_A (R \times \rho\bar{u})(\bar{u} \cdot d\bar{A})$$

Example 4.2

WHAT TORQUE MUST BE SUPPLIED TO STOP ROTATION OF LAWN SPRINKLER?

$$T_z = \hat{z} \cdot \iint_A \underbrace{(\bar{R} \times \bar{u}\rho)}_{\rho a u \cos\alpha} \underbrace{(\bar{u} \cdot d\bar{A})}_{2Au}$$

= RATE OF ANGULAR MOMENTUM OUT FLOW DUE TO WATER STREAMS

$$= 2 a u^2 \rho \cos\alpha A$$

Ch. 4 Exercises

- 4.3/Angular momentum & stress tensor
- 4.4/Differential form of E.O.M.
- 4.5/Stokes' expression for stress tensor
- 4.8/Rocket equation
- 4.9/Trust generated from a "propeller"

Exercise 4.3

3. Consider conservation of angular momentum derived from the angular momentum principle by the word statement: Rate of increase of angular momentum in volume V = net influx of angular momentum across the bounding surface A of V + torques due to surface forces + torques due to body forces. Here, the only torques are due to the same forces that appear in (linear) momentum conservation. The possibilities for body torques and couple stresses have been neglected. The torques due to the surface forces are manipulated as follows. The torque about a point O due to the element of surface force $\tau_{mk} dA_m$ is $\int \epsilon_{ijk} x_j \tau_{mk} dA_m$, where x is the position vector from O to the element dA. Using Gauss' theorem, we write this as a volume integral,

$$\int_V \varepsilon_{ijk} \frac{\partial}{\partial x_m}(x_j \tau_{mk}) dV = \varepsilon_{ijk} \int_V \left(\frac{\partial x_j}{\partial x_m} \tau_{mk} + x_j \frac{\partial \tau_{mk}}{\partial x_m} \right) dV$$
$$= \varepsilon_{ijk} \int_V \left(\tau_{jk} + x_j \frac{\partial \tau_{mk}}{\partial x_m} \right) dV,$$

where we have used $\partial x_j / \partial x_m = \delta_{jm}$. The second term is $\int_V \mathbf{x} \times \nabla \cdot \boldsymbol{\tau} \, dV$ and combines with the remaining terms in the conservation of angular momentum to give $\int_V \mathbf{x} \times$ (Linear Momentum: equation (4.17)) $dV = \int_V \epsilon_{ijk} \tau_{jk} \, dV$. Since the left-hand side $= 0$ for any volume V, we conclude that $\varepsilon_{ijk} \tau_{kj} = 0$, which leads to $\tau_{ij} = \tau_{ji}$.

Exercise 4.3

PROVE THAT THE STRESS TENSOR IS SYMMETRIC

SMALL VOLUME

$$\text{TOTAL TORQUE FROM SURFACE} = \iint_A R \times (\bar{\bar{\tau}} \cdot d\bar{A})$$

$$T_i \sim \iint_A \epsilon_{ijk} x_j (\bar{\bar{\tau}} \cdot d\bar{A})_k$$

$$\sim \iint_A \epsilon_{ijk} x_j \tau_{km} dA_m$$

USE GAUSS' THEOREM...

$$T_i \sim \iiint_V \epsilon_{ijk} \frac{\partial}{\partial x_m} (x_j \tau_{km}) dV$$

$$\sim \iiint_V \epsilon_{ijk} \left[\underbrace{\frac{\partial x_j}{\partial x_m}}_{\delta_{jm}} \tau_{km} + x_j \frac{\partial \tau_{km}}{\partial x_m} \right] dV$$

<u>NOW</u> AS $V \to 0$

ANY TERM/MOMENT WITH x_i WILL VANISH.

$$T_i = 0 = \iiint_{V \to 0} \epsilon_{ijk} \tau_{kj} dV$$

OR $\epsilon_{ijk} \tau_{kj} = 0 = \epsilon_{ijk} (\tau_{kj} - \tau_{jk}) = 0$

Exercise 4.4

4. Near the end of Section 7 we derived the equation of motion (4.15) by starting from an integral form for a material volume. Derive equation (4.15) by starting from the integral statement for a *fixed region*, given by equation (4.22).

Exercise 4.4

DERIVE EQUATION FOR FORCE/MOMENTUM FROM INTEGRAL OVER FIXED VOLUME...

$$\text{FORCE} = \frac{d}{dt}(\text{MOMENTUM}) + (\text{FLUX OF MOMENTUM FROM SURFACE})$$

$$\iiint_V \rho \bar{g}\, dV + \iint_A \bar{\bar{\tau}} \cdot d\bar{A} = \frac{d}{dt}\iiint_V \rho \bar{u}\, dV + \iint_A \rho \bar{u}(\bar{u} \cdot d\bar{A})$$

USE GAUSS' THEOREM...

$$\iiint_V (\rho \bar{g} + \nabla \cdot \bar{\bar{\tau}})\, dV = \iiint_V dV \left[\frac{\partial}{\partial t}(\rho u) + \nabla \cdot (\rho \bar{u} \bar{u})\right]$$

MUST BE TRUE FOR ALL VOLUMES....

$$\rho \bar{g} + \bar{\nabla} \cdot \bar{\bar{\tau}} = \rho \frac{\partial \bar{u}}{\partial t} + \underbrace{\bar{u} \frac{\partial \rho}{\partial t} + \bar{u}\, \nabla \cdot (\rho \bar{u})}_{=0} + \rho(\bar{u} \cdot \bar{\nabla})\bar{u}$$

$$\therefore \quad \rho \frac{\partial \bar{u}}{\partial t} + \rho(\bar{u} \cdot \bar{\nabla})\bar{u} = \rho \bar{g} + \bar{\nabla} \cdot \bar{\bar{\tau}}$$

QED

Exercise 4.5

5. Verify the validity of the second form of the viscous dissipation given in equation (4.60). [*Hint*: Complete the square and use $\delta_{ij}\delta_{ij} = \delta_{ii} = 3$.]

Exercise 4.5

VERIFY STOKES' EXPRESSION FOR STRESS TENSOR

PROVE:

$$\tau_{ij} = 2\mu \epsilon_{ij}\epsilon_{ij} - \frac{2}{3}\mu(\nabla\cdot u)^2 \stackrel{?}{=} 2\mu\left[\epsilon_{ij} - \frac{1}{3}(\nabla\cdot u)\delta_{ij}\right]^2$$

COMPLETE RHS SQUARE

$$\text{RHS} = 2\mu\left[\epsilon_{ij}\epsilon_{ij} - \frac{2}{3}\epsilon_{ij}\delta_{ij}(\nabla\cdot u) + \frac{1}{9}(\nabla\cdot u)^2\delta_{ij}\delta_{ij}\right]$$

BUT $\delta_{ij}\delta_{ij} = \sum_i \delta_{ii} = 3$ AND $\epsilon_{ij}\delta_{ij} = \sum_i \epsilon_{ii} = \nabla\cdot\bar{u}$

SO

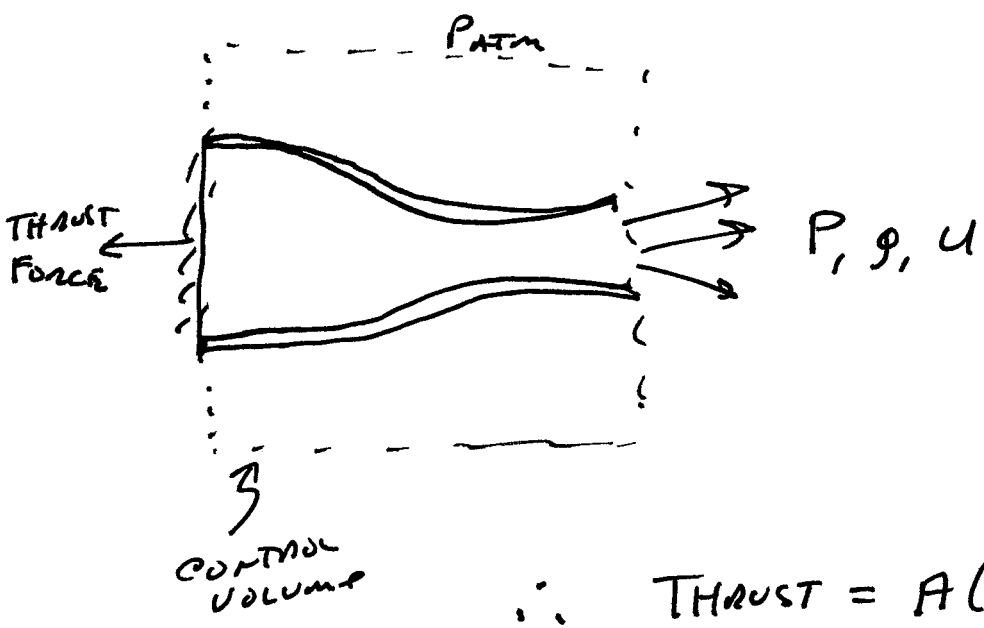
$$\text{RHS} = 2\mu\left[\epsilon_{ij}\epsilon_{ij} - \frac{2}{3}(\nabla\cdot u)^2 + \frac{1}{3}(\nabla\cdot u)^2\right]$$

$$= 2\mu\left[\epsilon_{ij}\epsilon_{ij} - \frac{1}{3}(\nabla\cdot u)^2\right] \quad \text{QED}$$

Exercise 4.8

8. Show that the thrust developed by a stationary rocket motor is $F = \rho A U^2 + A(p - p_{\text{atm}})$, where p_{atm} is the atmospheric pressure, and p, ρ, A, and U are, respectively, the pressure, density, area, and velocity of the fluid at the nozzle exit.

Exercise 4.8



$$\text{THRUST} - \iint_A \tau_{ij}\, dA_j = \text{MOMENTUM FLUX} = (\rho u) A u$$

$$\downarrow$$

$$-A(P - P_{ATM})$$

$$\therefore \quad \text{THRUST} = A(P - P_{ATM}) + (\rho u) A u$$

Exercise 4.9

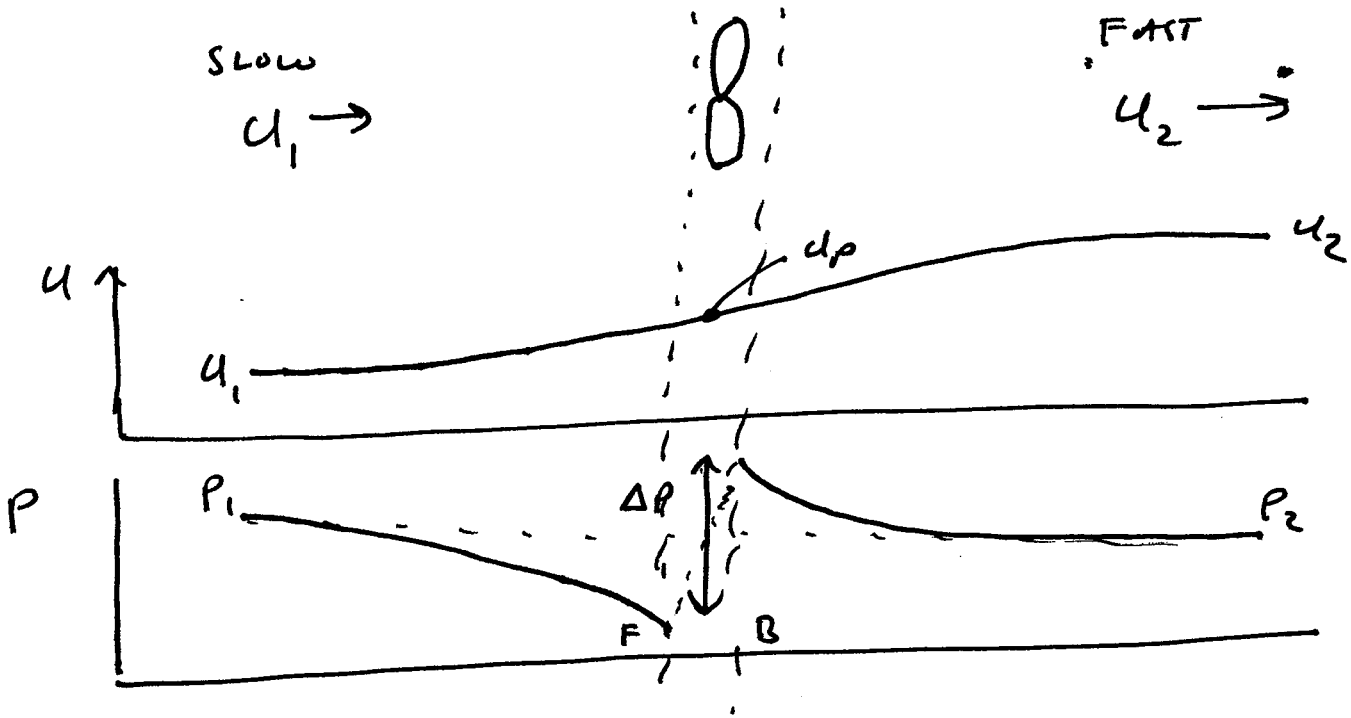
9. Consider the propeller of an airplane moving with a velocity U_1. Take a reference frame in which the air is moving and the propeller [disk] is stationary. Then the effect of the propeller is to accelerate the fluid from the upstream value U_1 to the downstream value $U_2 > U_1$. Assuming incompressibility, show that the thrust developed by the propeller is given by

$$F = \frac{\rho A}{2}(U_2^2 - U_1^2),$$

where A is the projected area of the propeller and ρ is the density (assumed constant). Show also that the velocity of the fluid at the plane of the propeller is the average value $U = (U_1 + U_2)/2$. [*Hint*: The flow can be idealized by a pressure jump, of magnitude $\Delta p = F/A$ right at the location of the propeller. Also apply Bernoulli's equation between a section far upstream and a section immediately upstream of the propeller. Also apply the Bernoulli equation between a section immediately downstream of the propeller and a section far downstream. This will show that $\Delta p = \rho(U_2^2 - U_1^2)/2$.]

Exercise 4.9

MOVING WITH PROPELLER



$\Delta p = p_B - p_F$ = PRESSURE INCREASE DUE TO PROPELLER

NOTE: $p_1 \sim p_2$

BERNOULLI'S PRINCIPLE

$$\frac{p}{\rho} + \frac{u_1^2}{2} = \frac{p_F}{\rho} + \frac{u_F^2}{2}$$

$$\frac{p}{\rho} + \frac{u_2^2}{2} = \frac{p_B}{\rho} + \frac{u_B^2}{2}$$

$$\Delta p = p_B - p_F = \frac{\rho}{2}\left(u_2^2 - u_1^2\right)$$

BUT FORCE $= A\Delta p$

$$= \frac{A}{2}\rho\left(u_2^2 - u_1^2\right)$$

ALSO FORCE $= \underbrace{\rho u_p A}_{\text{MASS FLOW}}\left(u_2 - u_1\right)$ = NET MOMENTUM FLUX

$$\therefore \quad u_p = \frac{\text{FORCE}}{\rho A(u_2 - u_1)} = \frac{1}{2}\left(u_2 + u_1\right)$$

Summary

- Basic fluid dynamics involves "6" field variables: ρ, U_i, P, T
- Conservation of Mass, Energy, Newton's Law, and an equation of state provide a "closed" set of dynamical equations for a fluid Integral or differential formulation of equations of motion (E.O.M.) are equivalent
- Integral form of momentum-force equation can be combined with Bernoulli's Principle for a powerful way to compute flow/force parameters.
- Apparent forces, centrifugal and Coriolis, appear in a co-rotating frame of reference