

APPH 4200

Physics of Fluids

Fluid Equations of Motion (Ch. 4)

1. Conservation of Mass
2. Navier-Stokes Equation (Force-Momentum)
3. Mechanical and Thermal Energy
4. Entropy
5. Some examples

Equations of Fluid Dynamics

(Conservation Laws)

VARIABLES TO CHARACTERIZE A SIMPLE FLUID

$\rho(\bar{x}, t)$	MASS DENSITY
$\bar{u}(\bar{x}, t)$	FLOW VELOCITY
$P(\bar{x}, t)$	PRESSURE
$T(\bar{x}, t)$	TEMPERATURE

SIX FIELD VARIABLES. DYNAMICS REQUIRES SIX EQUATIONS.
OF MOTION...

CONSERVATION OF MASS
NEWTON'S LAW / FORCE / MOMENTUM
CONSERVATION OF ENERGY
EQUATION OF STATE (RELATING PRESSURE TO TEMP + DENSITY)

ADDITIONALLY, WE CAN SHOW THAT TOTAL
ENTROPY (DISORDER) MUST INCREASE IN TIME AS
REQUIRED BY SECOND LAW OF THERMODYNAMICS.

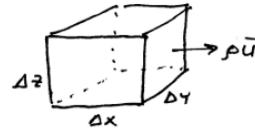
Continuity

CONSERVATION OF Mass

$$\frac{\partial \rho}{\partial t} + \bar{\nabla} \cdot (\rho \bar{u}) = 0$$

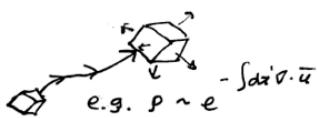
$$\frac{\partial \rho}{\partial t} + (\bar{u} \cdot \bar{\nabla}) \rho = -\rho \nabla \cdot \bar{u}$$

$\underbrace{\quad}_{\frac{\partial \rho}{\partial t}} = -\rho \nabla \cdot \bar{u}$



$$\begin{aligned} \frac{\partial}{\partial t} (\Delta V \rho) &= - \sum_{\text{ALL SURFACES}} \rho \bar{u} \cdot \Delta \bar{A} \\ &= - \frac{\partial}{\partial x_i} (\rho u_i) \Delta x_i \Delta \\ &= - \Delta V \nabla \cdot (\rho \bar{u}) \end{aligned}$$

$$\frac{\partial \rho}{\partial t} = -D_t(\nabla \cdot \bar{u})$$



Ch. 4, Exercise #1

$$u = u(x, t)$$

$$\rho(t) = \rho_0 (2 - \cos \omega t)$$

WHAT IS $u(x, t)$ IF $u(0, t) = U$?

SOLUTION: CONSERVATION OF MASS

$$\frac{\partial \rho}{\partial t} + \bar{u} \cdot \cancel{\bar{\nabla} \rho} + \rho \nabla \cdot \bar{u} = 0$$

$$\text{so } \frac{\partial u}{\partial x} = - \frac{\rho_0 \omega \sin \omega t}{\rho_0 (2 - \cos \omega t)}$$

$$\therefore u(x, t) = \text{constant}(t) - x \left(\frac{\omega \sin \omega t}{2 - \cos \omega t} \right)$$

$$\text{but } u(0, t) = U \therefore u(x, t) = U - x \left(\frac{\omega \sin \omega t}{2 - \cos \omega t} \right)$$

Integral Relations

(Section 4.2)

$$\frac{d}{dt} \int_{V(t)} f dV = \begin{cases} \int_V \frac{\partial f}{\partial t} dV & (\text{fixed } V) \\ \int_V \frac{\partial f}{\partial t} dV + \int_A d\mathbf{A} \cdot \mathbf{U} f \end{cases}$$

Moving "Material" volume, $V(t)$, at velocity \mathbf{U}

Osborn Reynolds

(Reynolds Transport Theory)



1842-1912

Integral Relations

(Section 4.2)

$$\begin{aligned} \frac{d}{dt} \int_{V(t)} (\rho f) dV &= \int_V \frac{\partial(\rho f)}{\partial t} dV + \int_A d\mathbf{A} \cdot \mathbf{U} (\rho f) \\ &= \int_V \left(\rho \frac{\partial f}{\partial t} + f \frac{\partial \rho}{\partial t} + \nabla \cdot \rho f \mathbf{U} \right) dV \\ &= \int_V \left(\rho \frac{\partial f}{\partial t} + f \frac{\partial \rho}{\partial t} + f \nabla \cdot \rho \mathbf{U} + \rho \mathbf{U} \cdot \nabla f \right) dV \\ &= \int_V \rho \left(\frac{\partial f}{\partial t} + \mathbf{U} \cdot \nabla f \right) dV \\ &= \int_V \rho \frac{df}{dt} dV \end{aligned}$$

Moving "Material" volume, $V(t)$, at velocity \mathbf{U}

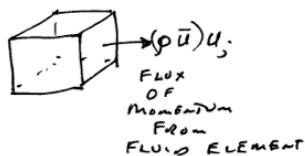
Newton's Law

NEWTON'S LAW
FOR A
PARTICLE

$$\bar{F} = m\bar{a}$$

$$\bar{F} = \frac{d}{dt}(m\bar{v})$$

NEWTON's LAW FOR A FLUID:



$$\begin{aligned}\bar{F} &= \frac{D}{Dt}(\rho\bar{u}) \\ &= \frac{\partial}{\partial t}(\rho\bar{u}) + \nabla \cdot (\rho\bar{u}\bar{u}) \\ &= \frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_j}(\rho u_i u_j) \\ &\rightarrow = \rho \left[\frac{\partial \bar{u}}{\partial t} + (\bar{u} \cdot \nabla) \bar{u} \right] + \bar{u} \underbrace{\left[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho\bar{u}) \right]}_{=0 \text{ CONSERVATION OF MASS}} \\ &= \rho \frac{D\bar{u}}{Dt}\end{aligned}$$

Example: Momentum

$$f = \rho\mathbf{U}$$

LOCAL CONSERVATION OF MOMENTUM

$$\rho \frac{D\bar{u}}{Dt} = \rho \bar{g} + \nabla \cdot \bar{\tau}$$

Moving Volume (with Fluid)

$$\iiint dV \rho \frac{D\bar{u}}{Dt} = \iiint dV [\rho \bar{g} + \nabla \cdot \bar{\tau}]$$

$$\frac{D}{Dt} \iiint dV \rho \bar{u} = \iiint dV \rho \bar{g} + \iint_A dA \cdot \bar{\tau}$$

FIXED VOLUME

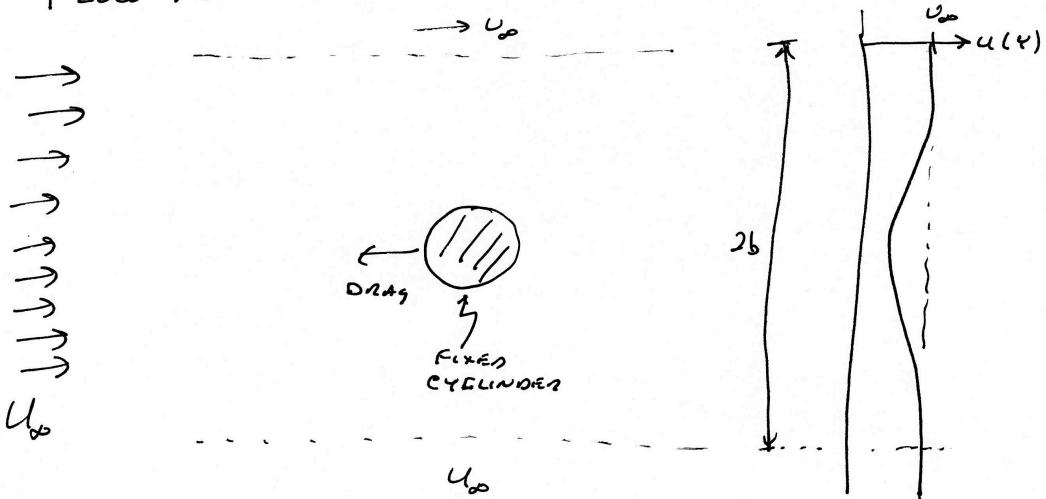
$$\underbrace{\rho \frac{D\bar{u}}{Dt} + \bar{u} \left[\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \bar{u} \right]}_{\frac{\partial}{\partial t} \iiint dV \rho \bar{u}} = \rho \bar{g} + \nabla \cdot \bar{\tau} = (\text{force/m}^3)$$

$$\frac{\partial}{\partial t} \iiint dV \rho \bar{u} = \iiint dV \rho \bar{g} + \iint_A dA \cdot \bar{\tau} - \underbrace{\iint_A dA \cdot \bar{u} \rho \bar{u}}_{\text{MOMENTUM FLOW}}$$

Example 4.1

(fixed volume)

WHAT IS THE DRAG ON A CYLINDER FROM MEASURED FLOW PROFILE DOWNSTREAM?



Integral Conservation Laws

(Fixed Volume)

CONSERVATION OF MASS:

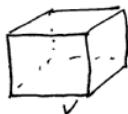
$$\iiint dV \left[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \bar{u}) \right] = 0$$

$$\frac{d}{dt} \iiint dV \rho = - \iint dA \cdot \rho \bar{u}$$

MOMENTUM: $\iiint dV \left[\rho \frac{D \bar{u}}{Dt} \right] = \iiint dV \left[\bar{F} + \nabla \cdot \bar{\bar{e}} \right]$

BUT $\rho \frac{D \bar{u}}{Dt} = \rho \frac{\partial \bar{u}}{\partial t} + (\rho \bar{u} \cdot \bar{\nabla}) \bar{u}$
 $= \frac{2}{\rho} (\rho \bar{u}) - \bar{u} \frac{\partial \rho}{\partial t} + (\rho \bar{u} \cdot \bar{\nabla}) \bar{u}$
 $= \frac{2}{\rho} (\rho \bar{u}) + \bar{u} \nabla \cdot \rho \bar{u} + (\rho \bar{u} \cdot \bar{\nabla}) \bar{u}$
 $= \frac{2}{\rho} (\rho \bar{u}) + \bar{\nabla} \cdot (\rho \bar{u} \bar{u}) \quad \text{so...}$

$$\frac{2}{\rho} \iiint dV \rho \bar{u} = \iiint dV \bar{F} + \iint dA \cdot (\bar{\bar{e}} + \rho \bar{u} \bar{u})$$



CHANGE OF MOMENTUM WITHIN BOX
EQUALS SUM OF VOLUME FORCES
AND SURFACE FORCES AND
FLOW OF MOMENTUM IN/OUT OF BOX.

Example 4.1 (Solution)

- CONSERVATION OF MASS

$$\text{MASS IN} = \text{MASS OUT}$$

$$2b\rho u_0 = m_{\text{top}} + m_{\text{out}} + \rho \int_{-b}^b dy u(y)$$

$$\therefore m_{\text{top}} + m_{\text{out}} = \rho \int_{-b}^b dy (u_0 - u(y))$$

- FORCE / MOMENTUM BALANCE

$$\text{MOMENTUM IN} = \text{MOMENTUM OUT} + \text{DRAG}$$

$$\begin{aligned} \text{DRAG} &= \text{MOMENTUM IN} - \text{MOMENTUM OUT} \\ &= 2b\rho u_0^2 - \rho \int dy u^2 - \rho u_0 \underbrace{\int dy (u_0 - u)}_{m_{\text{top}} + m_{\text{out}}} \\ &= -\rho \int dy u^2 + \rho \int dy u_0 u \\ &= \rho \int dy u (u_0 - u) \end{aligned}$$

Momentum: Body and Surface Forces

$$= (\text{force}/\text{m}^3)$$

$$\rho \left(\frac{\partial \bar{u}}{\partial t} + (\bar{u} \cdot \nabla) \bar{u} \right) = \rho \bar{g} + \nabla \cdot \bar{\tau}$$

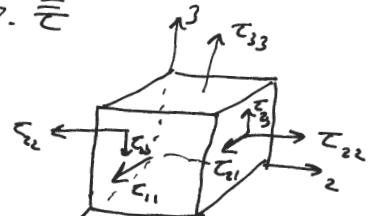
$\bar{\tau}$ = STRESS TENSOR

= USUALLY SYMMETRIC

= HAS NORMAL STRESS \sim PRESSURE

= HAS SHEAR STRESS \sim (OFF DIAGONAL)

GRADIENTS OF STRESS PRODUCE FORCE



τ_{ij} WHICH FACE
DIRECTION

τ_{ij} IS POSITIVE WHEN
DIRECTED IN DIRECTION OF
AXIS

$\tau_{ii} > 0$ IMPLIES TENSILE STRESS

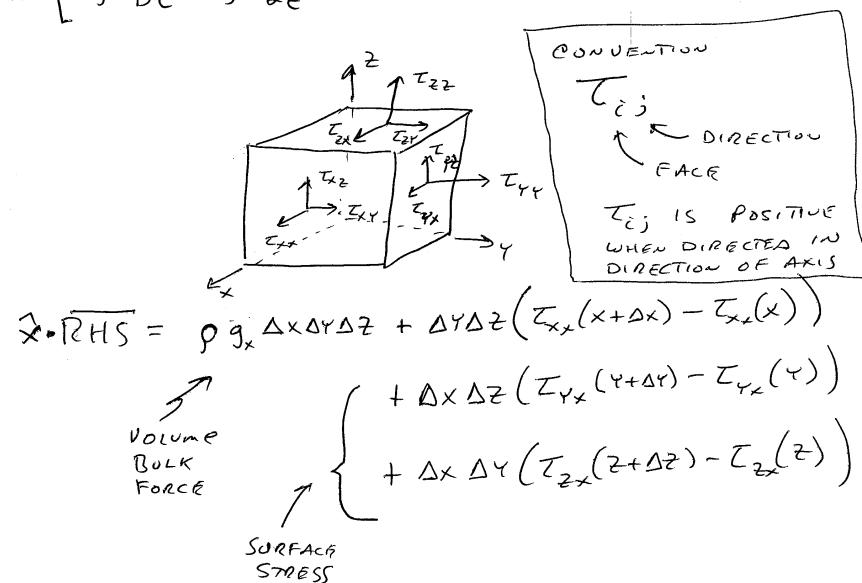
$\tau_{ii} < 0$ IMPLIES COMPRESSIVE STRESS

τ_{ij} ($i \neq j$) ARE SHEAR STRESSES

Momentum x-direction

X-DIRECTION:

$$\hat{x} \cdot \left[\rho \frac{D\vec{\tau}}{Dt} = \rho \left(\frac{2\vec{\tau}}{\partial t} + (\vec{u} \cdot \nabla) \vec{u} \right) = \rho \vec{g} + \nabla \cdot \vec{\tau} \right]$$



Models for Stress

- ISOTROPIC PRESSURE

$$\bar{\tau} = -\rho \bar{\delta} \quad \nabla \cdot \bar{\tau} = -\nabla p$$

- MOVING FLUID WITH VISCOSITY

$$\bar{\tau} = -\rho \bar{\delta} + \bar{\sigma} \quad \text{VISCOSITY}$$

WHAT IS $\bar{\sigma}$?

Energy (Part 1)

$$\rho \frac{D\bar{u}}{Dt} = \rho \bar{g} + \nabla \cdot \bar{\varepsilon}$$

$$\rho \frac{D}{Dt} \left(\frac{1}{2} u^2 \right) = \underbrace{\rho \bar{u} \cdot \bar{g}}_{\substack{\text{WORK DONE} \\ \text{BY} \\ \text{BODY FORCES}}} + \underbrace{\bar{u} \cdot (\nabla \cdot \bar{\varepsilon})}_{\substack{\text{WORK DONE} \\ \text{BY} \\ \text{SURFACE FORCES}}}$$

The Importance of Viscosity

INCOMPRESSIBLE EULER EQUATION

$$\frac{\partial \bar{u}}{\partial t} + (\bar{u} \cdot \nabla) \bar{u} = - \nabla p / \rho + \bar{g} + (\text{viscous force/m}^3)$$

$$\nabla \cdot \bar{u} = 0$$

LET $\bar{\omega} = \nabla \times \bar{u}$. THEN

$$(\bar{u} \cdot \bar{\nabla}) \bar{u} = \bar{\omega} \times \bar{u} + \frac{1}{2} \bar{\nabla} u^2$$

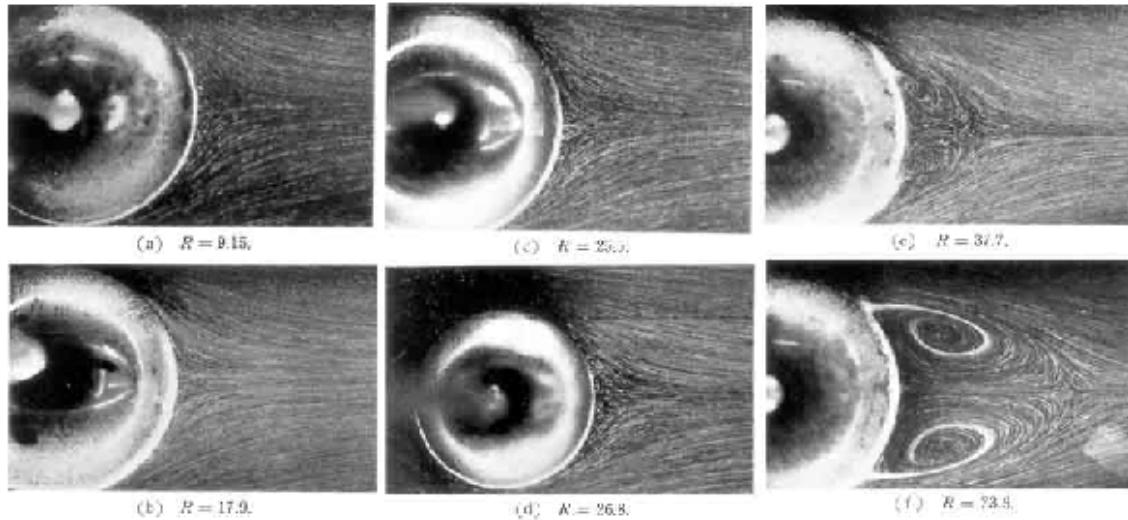
$$\frac{\partial \bar{u}}{\partial t} + \bar{\omega} \times \bar{u} = - \frac{\nabla p}{\rho} + \bar{g} - \frac{1}{2} \bar{\nabla} u^2$$

TAKE CURL OF THIS EQUATION

$$\frac{\partial}{\partial t} \bar{\omega} + \nabla \times (\bar{\omega} \times \bar{u}) = 0 \quad (\text{if } \bar{g} = -\nabla \varphi)$$

IF $\bar{\omega} = 0$ at $t=0$, THEN $\bar{\omega} = 0$ FOREVER!

Creation of Vorticity

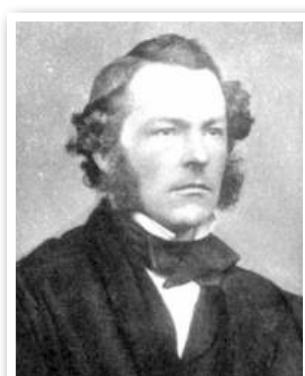


(Note: Flow at thin layer at surface of cylinder vanishes.)

Navier & Stokes



Claude-Louis Henri Navier
(1785–1836)
Elasticity

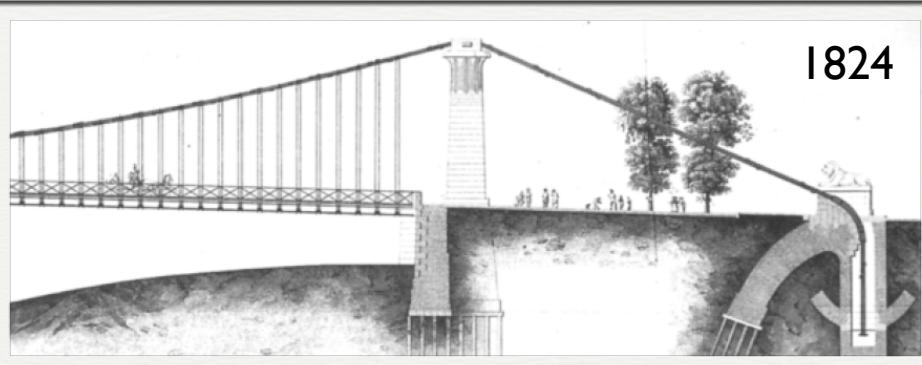


George Stokes
(1819–1903)
Viscous flow, Stokes' Law

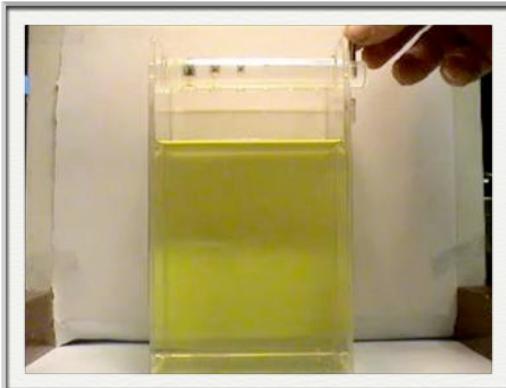
Pont des Invalides



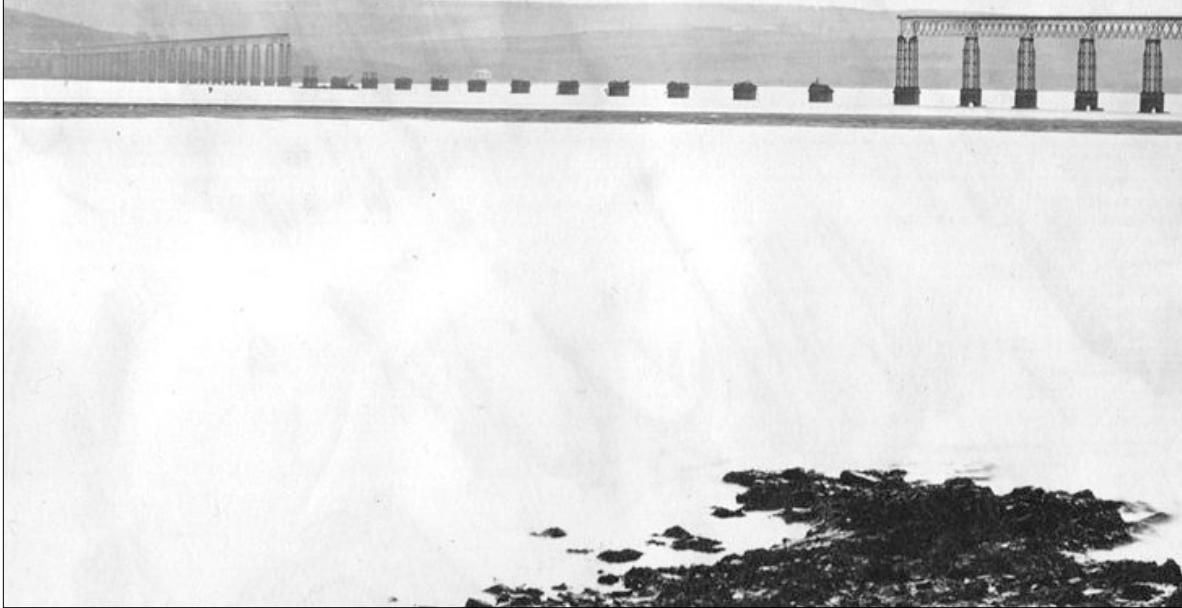
VUE DU PONT DES INVALIDES.



Stokes Flow



Tay Bridge Disaster (1879)



Stokesian Fluid

MATERIAL ISOTROPY AND STRESS SYMMETRY

(e.g. AIR, WATER BUT NOT MAGNETIZED PLASMA)

$$\bar{\sigma} = 2\mu \bar{\epsilon} + \lambda (\nabla \cdot \bar{u}) \bar{\delta}$$

↑ ↑
VISCOOSITY ~ BULK VISCOSITY

STOKES MODELED VISCOSITY VIA KINETIC THEORY OF MONATOMIC ATOMS AND SHOWED $\lambda = -\frac{2}{3}\mu$.

THEN, STRESS TENSOR

$$\bar{\tau} = -p \bar{\delta} + 2\mu \bar{\epsilon} + \frac{2}{3}\mu (\nabla \cdot \bar{u}) \bar{\delta}$$

Navier-Stokes Equation

$$\rho \left(\frac{\partial \bar{u}}{\partial t} + (\bar{u} \cdot \nabla) \bar{u} \right) = -\nabla p + \rho \bar{g} + \nabla \cdot \left[2\mu \bar{\epsilon} - \frac{2}{3} \mu (\bar{u} \cdot \bar{u}) \bar{\delta} \right]$$

ASSUME $\mu \sim$ INDEPENDENT OF \bar{x} . THEN,

$$\nabla \cdot 2\mu \bar{\epsilon} = 2\mu (\nabla \cdot \bar{\epsilon}) \quad \epsilon_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$$

$$(\nabla \cdot \bar{\epsilon})_i = \frac{1}{2} \left(\frac{\partial^2 u_i}{\partial x_j^2} + \frac{\partial^2 u_j}{\partial x_i \partial x_j} \right) = \frac{1}{2} \nabla^2 \bar{u}_i + \frac{1}{2} \bar{\nabla} (\bar{u} \cdot \bar{u})$$

Navier-Stokes & Euler

$$\rho \frac{D \bar{u}}{Dt} = -\nabla p + \rho \bar{g} + \begin{cases} \mu [\nabla^2 \bar{u} + \frac{1}{3} \bar{\nabla} (\bar{u} \cdot \bar{u})] & \text{NAVIER-STOKES} \\ \mu \nabla^2 \bar{u} & \text{INCOMPRESSIBLE NAVIER-STOKES} \\ 0 & \text{EULER EQUATION (IDEAL FLUID)} \end{cases}$$

NAVIER-STOKES EQUATION IS ONE OF THE MOST USEFUL EQUATIONS IN APPLIED PHYSICS: ENGINEERING, ASTRONOMICS, WEATHER, OCEANOGRAPHY, ETC.

MAGNETOHYDRODYNAMICS INCLUDES N.S. PLUS MAXWELL'S EQUATIONS AND BODY FORCE IS $\bar{J} \times \bar{B}$ INSTEAD OF $\rho \bar{g}$.

NS Properties

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = \mathbf{g} - (1/\rho) \nabla p + \nu \nabla^2 \mathbf{u}$$

- Equation for **rate of change** of $\mathbf{u}(x,t)$ (not position) in lab frame
- Nonlinear. Complicated term: $\mathbf{u} \cdot \nabla \mathbf{u}$
- Viscous length scale: ν/U
- NS is a starting point: Once $\mathbf{u}(x,t)$ is known, then other fluid properties can be determined.

Clay Prize: \$1M



Clay Mathematics Institute

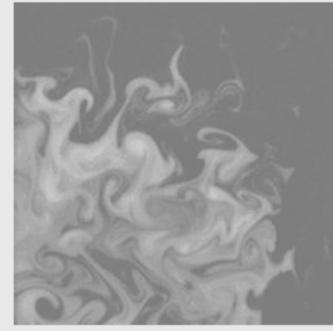
Dedicated to increasing and disseminating mathematical knowledge

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Navier-Stokes Equation

Waves follow our boat as we meander across the lake, and turbulent air currents follow our flight in a modern jet. Mathematicians and physicists believe that an explanation for and the prediction of both the breeze and the turbulence can be found through an understanding of solutions to the Navier-Stokes equations. Although these equations were written down in the 19th Century, our understanding of them remains minimal. The challenge is to make substantial progress toward a mathematical theory which will unlock the secrets hidden in the Navier-Stokes equations.

- [The Millennium Problems](#)
- [Official Problem Description — Charles Fefferman](#)
- [Lecture by Luis Caffarelli \(video\)](#)

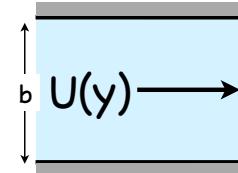


Simple Example:

What is the steady flow between two parallel plates?

Cartesian Coordinates

No slip Boundary Conditions



$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -(1/\rho) \nabla p + \nu \nabla^2 \mathbf{u}$$

$$u_x \frac{\partial u_x}{\partial x} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \nabla^2 u_x$$

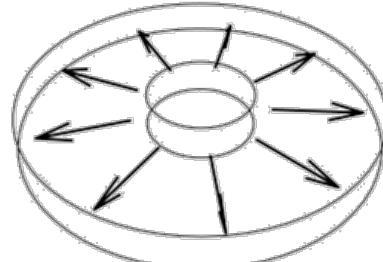
Solution:

$$u_x(y) = 4U_0 \frac{y(b-y)}{b^2}$$

$$\nu \nabla^2 \mathbf{u} = \hat{x} \nu \frac{\partial^2 u_x}{\partial y^2} = -\hat{x} 8 \frac{\nu U_0}{b^2}$$

Not Simple Example:

Steady, 1D Radial Flow, Cylindrical Coordinates



$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -(1/\rho) \nabla p + \nu \nabla^2 \mathbf{u}$$

$$u_r \frac{\partial u_r}{\partial r} = -\frac{1}{\rho} \frac{\partial p}{\partial r} + \nu \left(\frac{\partial^2 u_r}{\partial z^2} + \frac{\partial^2 u_r}{\partial r^2} + \frac{1}{r} \frac{\partial u_r}{\partial r} - \frac{u_r}{r^2} \right)$$

$$\frac{1}{r} \frac{\partial(rU_r)}{\partial r} = 0 \rightarrow U_r(r, z) = c_1(z)/r$$

The Importance of Viscosity

INCOMPRESSIBLE EULER EQUATION

$$\frac{\partial \bar{u}}{\partial t} + (\bar{u} \cdot \nabla) \bar{u} = -\nabla p/\rho + \bar{g}$$

$$\nabla \cdot \bar{u} = 0$$

LET $\bar{\omega} = \nabla \times \bar{u}$. THEN

$$(\bar{u} \cdot \bar{\nabla}) \bar{u} = \bar{\omega} \times \bar{u} + \frac{1}{2} \bar{\nabla} u^2$$

$$\frac{\partial \bar{u}}{\partial t} + \bar{\omega} \times \bar{u} = -\frac{\nabla p}{\rho} + \bar{g} - \frac{1}{2} \bar{\nabla} u^2$$

TAKE CURL OF THIS EQUATION

$$\frac{\partial}{\partial t} \bar{\omega} + \nabla \times (\bar{\omega} \times \bar{u}) = 0 \quad (\text{if } \bar{g} = -\nabla \varphi)$$

IF $\bar{\omega} = 0$ at $t=0$, THEN $\bar{\omega} = 0$ FOREVER!

Energy Conservation

- MECHANICAL ENERGY DENSITY $\frac{1}{2} \rho u^2$
- INTERNAL/THERMAL ENERGY DENSITY $\rho E \sim \rho C_v T$
(NOTE: TEXTBOOK USES "e" NOT "E")
- EQUATION OF STATE
e.g. IDEAL GAS $P = \rho RT$ (WHERE $R = C_p - C_v$)

COMMENTS:

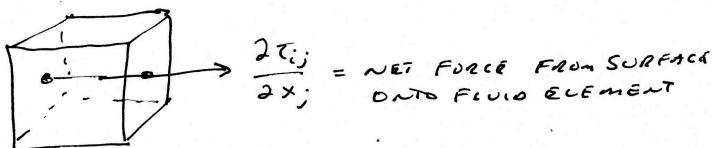
- CHANGE OF TOTAL ENERGY EQUALS SUM OF WORK AND HEAT ADDED.
- DISSIPATION (VISCOSITY) CAN CHANGE MECHANICAL ENERGY INTO THERMAL ENERGY
- THERMAL DIFFUSION CAN MOVE INTERNAL ENERGY FROM PLACE TO PLACE

Mechanical Energy

$$\bar{u} \cdot \left[\rho \frac{D \bar{u}}{Dt} = \rho \bar{g} + \nabla \cdot \bar{\tau} \right]$$

on

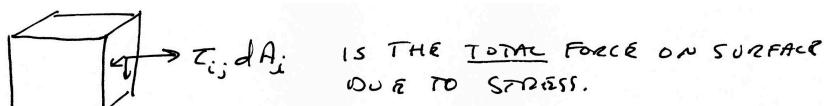
$$\rho \frac{D}{Dt} \left(\frac{1}{2} u^2 \right) = \rho \bar{u} \cdot \bar{g} + \underbrace{\bar{u} \cdot (\nabla \cdot \bar{\tau})}_{\substack{U_i \frac{\partial}{\partial x_j} T_{ij} \\ \text{BODY FORCE}}} + \underbrace{\bar{u} \cdot (\nabla \cdot \bar{\tau})}_{\substack{U_i \frac{\partial}{\partial x_j} T_{ij} \\ \text{SURFACE FORCE}}}$$



$\underline{U}_i \frac{\partial T_{ij}}{\partial x_j}$ = MECHANICAL WORK DONE
ON FLUID BY SURFACE STRESS

~ THE WORK DONE TO MOVE
THE ELEMENT

Total Work Done by Stress



$$50 \quad U_i T_{ij} dA_j = \text{TOTAL WORK DONE BY SURFACE STRESSES}\\ \text{INCLUDING DEFORMATION, STRETCHING,}\\ \text{PINCHING, ROTATION!}$$

$$\text{So } \nabla \cdot (\bar{u} \cdot \bar{\tau}) = \frac{2}{2x_j} (u_i \tau_{ij}) = \text{RATE OF } \underline{\text{TOTAL}} \text{ LOAD ON SURFACE INCLUDING DEFORMATION}$$

$$\frac{\partial}{\partial x_j} (u_i \tau_{ij}) = \underbrace{\tau_{ij} \frac{\partial u_i}{\partial x_j}}_{\text{DEFORMATION WORK}} + \underbrace{u_i \frac{\partial \tau_{ij}}{\partial x_j}}_{\text{MECHANICAL WORK}}$$

NOTE: SINCE τ_{ij} IS SYMMETRIC (USUALLY), THEN $\frac{\partial \tau_{ij}}{\partial x_j} = \tau_{ij} \epsilon_{ij}$.
 SINCE A SYMMETRIC TENSOR CONTRACTED WITH ANOTHER TENSOR
 EXTRACTS ONLY THE SYMMETRIC PART.

Mechanical Energy Density for a Stokes Fluid

$$\rho \frac{D}{Dt} \left(\frac{1}{2} u^2 \right) = \rho u_i g_i + \frac{2}{2x_j} (u_i \tau_{ij}) - \tau_{ij} \epsilon_{ij}$$

with $\tau_{ij} = -P \delta_{ij} + 2\mu \epsilon_{ij} - \frac{2}{3} \mu (\nabla \cdot u) \delta_{ij}$

gives $\tau_{ij} \epsilon_{ij} = -P \epsilon_{ii} + 2\mu \epsilon_{ij} \epsilon_{ij} - \frac{2}{3} \mu (\nabla \cdot u) \epsilon_{ii}$

where $\epsilon_{ii} = \sum_i \epsilon_{ii} = \nabla \cdot \bar{u}$

or

$$\rho \frac{D}{Dt} \left(\frac{1}{2} u^2 \right) = \rho u_i g_i + \underbrace{P(\nabla \cdot u)}_{\text{WORK DUE TO EXPANSION}} + \frac{2}{2x_j} (u_i \tau_{ij}) - \underbrace{\mu \left[2 \epsilon_{ij} \epsilon_{ij} - \frac{2}{3} (\nabla \cdot u)^2 \right]}_{\substack{\text{VISCOSITY DISSIPATION} \\ \text{OF MECHANICAL WORK}}}$$

Conservation of Total Energy Density

$$\rho \frac{D}{Dt} \left(E + \frac{1}{2} \mu^2 \right) = \rho \bar{u} \cdot \bar{g} + \nabla \cdot (\bar{u} \cdot \bar{\epsilon}) - \underbrace{\nabla \cdot \bar{q}}_{\text{HEAT FLUX}}$$

Simplif "Ficks Law" for heat flux is $\bar{q} = -k \nabla T$

Equation for Internal Energy

(Temperature)

SUBTRACT MECHANICAL ENERGY E_{kin} FROM TOTAL ENERGY E_{tot} TO GET

$$\oint \frac{D}{Dt} E = -\rho \nabla \cdot \bar{u} - \nabla \cdot \bar{q} + \mu (2\epsilon_{ij}\epsilon_{ij} - \frac{2}{3}(\nabla \cdot \bar{u})^2) \quad | E = C_v T$$

BUT CONSERVATION OF MASS ALLOWS US TO WRITE THIS IN ANOTHER FORM ...

$$\rho \nabla \cdot \bar{u} = -\frac{\rho}{\rho} \frac{D\rho}{Dt} \quad (\text{since } \frac{D\rho}{Dt} = -\rho \nabla \cdot \bar{u})$$

PLUS, EQUATION OF STATE ...

$$\rho = \rho R T \quad (R = c_p - c_v)$$

$$\frac{D\rho}{Dt} = RT \frac{D\rho}{Dt} + \rho R \frac{DT}{Dt}$$

$$\therefore \rho (\nabla \cdot \bar{u}) = -\frac{\rho}{\rho} \frac{D\rho}{Dt} = -RT \frac{D\rho}{Dt} = -\frac{D\rho}{Dt} + \rho (c_p - c_v) \frac{DT}{Dt}$$

Internal Energy Equation

$$\underbrace{\rho \frac{DE}{Dt} + \rho (c_p - c_v) \frac{DT}{Dt}}_{\rho c_p \frac{DT}{Dt}} = -\nabla \cdot \bar{q} + \frac{D\rho}{Dt} + \mu (2\epsilon_{ij}\epsilon_{ij} - \frac{2}{3}(\nabla \cdot \bar{u})^2)$$

$$\rho c_p \frac{DT}{Dt} = "$$

IF WE NEGLECT $\frac{D\rho}{Dt}$ AND (SMALL) DISSIPATION, WE OBTAIN
THIS USUAL THERMAL CONDUCTION EQUATION ...

$$\rho c_p \frac{DT}{Dt} \approx -\nabla \cdot \bar{q} = \nabla \cdot k \nabla T$$

IF $k \sim \text{constant}$, THEN

$$\frac{DT}{Dt} = \underbrace{\left(\frac{k}{\rho c_p}\right)}_{\chi} \nabla^2 T$$

$\chi = \text{THERMAL DIFFUSIVITY}$

$S \equiv \text{ENTROPY}$ must increase (2nd Law)

$$TdS = dE + PdV$$

BUT $V\phi \sim 1$ so $dV \sim -V \frac{d\phi}{P} \sim -\frac{dP}{P^2}$

This results in

$$T \frac{dS}{dt} = \left(\frac{dE}{dt} - \frac{P}{P^2} \frac{dP}{dt} \right) \quad (\text{Disorder, Direction of Time})$$

$$= -\nabla \cdot \bar{q} + \mu (2\varepsilon_i \epsilon_{ij} - \frac{2}{3} (\nabla \cdot \bar{u})^2)$$

How does S change?

$$\frac{DS}{dt} = -\frac{1}{T} \nabla \cdot \bar{q} + \underbrace{\frac{\mu}{T}}_{\text{ALWAYS POSITIVE IF } \mu > 0} (\text{DISSIPATION})$$

$$= -\nabla \cdot \left(\frac{\bar{q}}{T} \right) - \underbrace{\frac{\bar{q} \cdot \nabla T}{T^2}}_{\text{RE-ARRANGE } \bar{q} \cdot \nabla T < 0} + (..)$$

OR
HEAT MUST FLOW FROM HOT TO COLD

$$\bar{q} = -k \nabla T \quad (k > 0!)$$

Entropy

Summary of Fluid Dynamical Equations

$$\frac{D\phi}{Dt} = -\rho \nabla \cdot \bar{u}$$

$$\rho \frac{D\bar{u}}{Dt} = \rho \bar{g} + \bar{\nabla} \cdot \bar{\tau}$$

$$= \rho \bar{g} - \nabla p + \mu \left[\nabla^2 \bar{u} + \frac{1}{2} \bar{\nabla} (\nabla \cdot \bar{u}) \right]$$

$$\rho = \rho R T$$

$$\rho \frac{DE}{Dt} = \rho C_v \frac{DT}{Dt} = \nabla \cdot k \nabla T - \rho (\nabla \cdot \bar{u}) + \mu (\text{DISSIPATION})$$

$$\rho C_p \frac{DT}{Dt} = \nabla \cdot k \nabla T + \frac{DP}{Dt} + \mu (\text{DISSIPATION})$$

And $\mu > 0$, $k > 0$ for $\delta S \geq 0$

Summary

- Basic fluid dynamics involves “6” field variables: ρ , U_i , P , T
- Conservation of Mass, Energy, Newton’s Law, and an equation of state provide a “closed” set of dynamical equations for a fluid
- Viscous forces are represented by a stress tensor
- Stokes’ model of viscosity gives us one of the most useful and well-known equations of applied science: Navier-Stokes equation.