1. Review Chapter 3

2. Navier–Stokes Equation

Material or Convective Derivative

\[
\frac{D}{Dt} = \frac{d}{dt} \equiv \frac{\partial}{\partial t} + (\mathbf{u} \cdot \nabla)
\]

e.g. \[ \frac{Df}{Dt} = \frac{df}{dt} + (\mathbf{u} \cdot \nabla) f \]
Velocity Gradient Tensor

\[
\frac{2u_i}{2x_j} = \varepsilon_{ij} + \frac{1}{2} R_{ij}
\]

\[
\varepsilon_{ij} = \frac{1}{2} \left( \frac{2u_i}{2x_j} + \frac{2u_j}{2x_i} \right)
\]

\[
R_{ij} = \left( \frac{2u_i}{2x_j} - \frac{2u_j}{2x_i} \right)
\]

\[
R_{ij} = \varepsilon_{ijk} \frac{dx_k}{dx_i}
\]

\[
\frac{dx_i}{u_i} = \frac{dy}{u_Y} = \frac{dz}{u_Z} = ds
\]

Path Lines:

\[
\frac{dx}{u_x} = \frac{dy}{u_y} = \frac{dz}{u_z} = ds
\]

Visualizing Flow

\[
\vec{u} + d\vec{u}
\]

\[
\frac{d\vec{u}}{d\vec{x}} = \frac{2u_x}{2x_j} d\vec{x}_j
\]
Characterizing Flow

Vorticity $\vec{\omega} = \nabla \times \vec{U} = -\frac{1}{\rho} \epsilon_{ijk} \vec{R}_{ij}$

Circulation $\Gamma = \oint \vec{U} \cdot d\vec{s}$

$= \iint \nabla \times \vec{U} \cdot d\vec{A}$

$= \iint \vec{\omega} \cdot d\vec{A}$

Problem 3.1

1. A two-dimensional steady flow has velocity components

$u = y \quad v = x.$

Show that the streamlines are rectangular hyperbolas

$x^2 - y^2 = \text{const}.$

Sketch the flow pattern, and convince yourself that it represents an irrotational flow in a 90° corner.
Problem 3.1

Find streamlines when \( \overrightarrow{u} = (\gamma, x) \)

\[
\frac{dx}{\gamma} = \frac{dy}{x} = ds
\]

\( x \, dx = \gamma \, dy \Rightarrow x^2 = \gamma^2 + c \)

---

Problem 3.2

2. Consider a steady axisymmetric flow of a compressible fluid. The equation of continuity in cylindrical coordinates \((R, \varphi, x)\) is

\[
\frac{\partial}{\partial R} (\rho R u_R) + \frac{\partial}{\partial x} (\rho R u_x) = 0.
\]

Show how we can define a streamfunction so that the equation of continuity is satisfied automatically.
Problem 3.2

Consider axisymmetric flow

Conservation of mass is

\[
\frac{2}{2r} \left( \rho RU_z \right) + \frac{2}{2r} \left( \rho RU_r \right) = 0
\]

\[
\frac{\partial \psi}{\partial r} = 0
\]

Find a stream function.

Try \( \rho \mathbf{U} = \nabla \phi \times \nabla \phi (r, z) \) \( \nabla \phi = \frac{\phi}{r} \)

which satisfies \( \nabla \cdot (\rho \mathbf{U}) = 0 \)

Then

\[
\frac{\partial \psi}{\partial z} = \rho RU_z \quad \frac{\partial \psi}{\partial r} = -\rho RU_r
\]
Problem 3.3

\[ \mathbf{\hat{u}} = (x, y, u) \]

\[ n = \oint \mathbf{\hat{n}} \cdot d\mathbf{s} = \iint \nabla \times \mathbf{\hat{u}} \cdot d\mathbf{A} \]

Find circulation around unit circle

\[ \oint \mathbf{\hat{u}} \cdot d\mathbf{s} = \int_0^{2\pi} d\theta \, \mathbf{\hat{u}} \cdot \hat{\mathbf{\hat{e}}} \]

\[ \hat{\mathbf{\hat{e}}} = \hat{x} \cos \theta - \hat{y} \sin \theta \]

\[ \mathbf{\hat{u}} \cdot \hat{\mathbf{\hat{e}}} = -x \cos \theta - y \sin \theta \]

So \[ \oint \mathbf{\hat{u}} \cdot d\mathbf{s} = \int_0^{2\pi} \hat{x} \cos \theta \, d\theta = -\pi \]

Also

\[ \iint \nabla \times \mathbf{\hat{u}} \cdot d\mathbf{A} = -\pi \]

\[ \mathbf{\hat{e}} \times \mathbf{\hat{u}} = -\hat{z} \]

Problem 3.4

4. Consider a plane Couette flow of a viscous fluid confined between two flat plates at a distance \( b \) apart (see Figure 9.4c). At steady state the velocity distribution is

\[ u = Uy/b , \quad v = w = 0 , \]

where the upper plate at \( y = b \) is moving parallel to itself at speed \( U \), and the lower plate is held stationary. Find the rate of linear strain, the rate of shear strain, and vorticity. Show that the streamfunction is given by

\[ \psi = \frac{Uy^2}{2b} + \text{const.} \]
Problem 3.4

\[ \mathbf{v} = \left( v, \frac{u v}{b}, 0 \right) \]

**Linear Strain Rate**:
\[ \varepsilon_{xx} = \varepsilon_{yy} = 0 \]
\[ \varepsilon_{xy} = \varepsilon_{yx} = \frac{1}{2} \left( \frac{2 u v}{b^2} \right) \]

**Shear Strain Rate**:
\[ \gamma_{xy} = \frac{1}{b} \left( \frac{2 u v}{b^2} - \frac{2 v u}{b^2} \right) \]

**Vorticity**:
\[ \omega = \frac{1}{b} \left( \frac{2 u v}{b^2} - \frac{2 v u}{b^2} \right) \]

**What is the stream function?**

Since \( \nabla \cdot \mathbf{v} = 0 \),
\[ \mathbf{v} = \nabla \times \mathbf{A} = \hat{\mathbf{z}} \left( \frac{2 y}{b^2}, -\frac{2 y}{b^2}, 0 \right) \]

So
\[ \frac{u v}{b} = \frac{2 y}{b^2} \Rightarrow \psi(r) = \frac{u v}{b} + \text{constant} \]

Problem 3.5

5. Show that the vorticity for a plane flow on the \( xy \)-plane is given by

\[ \omega_z = -\left( \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} \right). \]

Using this expression, find the vorticity for the flow in Exercise 4.
Problem 3.5

Find vorticity, $\mathbf{\omega}$, for flow on $x-y$ plane.

$$\mathbf{\omega} = \nabla \times \mathbf{u}$$

If $\mathbf{u} = (u, v, 0)$ then

$$\mathbf{u} = -\frac{\partial z}{\partial y} \mathbf{i} + \frac{\partial x}{\partial y} \mathbf{j} - \frac{\partial y}{\partial y} \mathbf{k} = -\frac{\partial z}{\partial y} \mathbf{i} + \frac{\partial x}{\partial y} \mathbf{j}$$

$$\therefore \mathbf{\omega} = -\nabla \times \left( \frac{\partial z}{\partial y} \mathbf{i} + \frac{\partial x}{\partial y} \mathbf{j} \right) = -\frac{\partial}{\partial y} \left( \frac{\partial x}{\partial y} \right) \mathbf{i} + \frac{\partial}{\partial y} \left( \frac{\partial z}{\partial y} \right) \mathbf{j}$$

For the flow in Exercise 4.

$$\omega_z = -\frac{u}{b}$$

Problem 3.7

7. Determine an expression for $\psi$ for a Rankine vortex (Figure 3.17b), assuming that $u_\theta = U$ at $r = R$. 
Problem 3.7

**Find stream function for a Rankine vortex**

Where the upper plate at \( y = b \) is moving parallel to itself at speed \( U \), and the lower plate is held stationary. Find the rate of linear strain, the rate of shear strain, and vorticity. Show that the streamfunction is given by

\[
\psi(y) = \begin{cases} 
U(y) & y < R \\
U(R) & y > R
\end{cases}
\]

So

\[
\psi(y) = \begin{cases} 
-\frac{U^2}{2b} & y \leq R \\
u R \left(\frac{y}{R}\right) - \frac{RY}{2} & y > R
\end{cases}
\]

*Note:*

\( \nabla^2 \psi = \omega_z \) for \( y < R \) (rotational)

\( \nabla^2 \psi = 0 \) for \( y > R \) (irrotational)

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Problem 3.8

8. Take a plane polar element of fluid of dimensions \( dr \) and \( r \, d\theta \). Evaluate the right-hand side of Stokes' theorem

\[
\int \omega \cdot dA = \int u \cdot ds,
\]

and thereby show that the expression for vorticity in polar coordinates is

\[
\omega_z = \frac{1}{r} \left[ \frac{\partial}{\partial r}(ru_\theta) - \frac{\partial u_r}{\partial \theta} \right].
\]

Also, find the expressions for \( \omega_r \) and \( \omega_\theta \) in polar coordinates in a similar manner.
Problem 3.8

9. The velocity field of a certain flow is given by

\[ u = 2xy^2 + 2xz^2, \quad v = x^2y, \quad w = x^2z. \]

Consider the fluid region inside a spherical volume \( x^2 + y^2 + z^2 = a^2 \). Verify the validity of Gauss' theorem

\[ \int \nabla \cdot \mathbf{u} \, dV = \int \mathbf{u} \cdot dA, \]

by integrating over the sphere.
Problem 3.9

Let's consider the flow \( \mathbf{U} = (2x^2 + 2xy, y^2, x^2) \). We need to compute the divergence of \( \mathbf{U} \) over the sphere.

\[
\nabla \cdot \mathbf{U} = 2x^2 + 2x^2 + y^2 = 2x^2 - x^2 + y^2 = 2(2 - \cos^2 \phi \sin^2 \theta + \sin^2 \phi).
\]

Using spherical coordinates:

\[
x = \rho \cos \phi \sin \theta, \\
y = \rho \sin \phi, \\
z = \rho \cos \phi.
\]

The divergence is then:

\[
\frac{\partial U_x}{\partial x} + \frac{\partial U_y}{\partial y} + \frac{\partial U_z}{\partial z} = \frac{2 \rho^2 \sin \phi \cos \phi \sin \theta}{\rho^2} = \frac{2 \sin \phi \cos \phi \sin \theta}{\rho}.
\]

But at the surface \( \rho = 4 \):

\[
\oint_{\partial V} \mathbf{U} \cdot d\mathbf{A} = \frac{8 \pi}{3} \quad \text{Q.E.D.}
\]

Problem 3.10

10. Show that the vorticity field for any flow satisfies

\[
\nabla \cdot \omega = 0.
\]
Problem 3.10

\[ \nabla \cdot \vec{\omega} = 0 \quad \text{where} \quad \vec{\omega} = \nabla \times \vec{u} \]

But \( \nabla \cdot (\nabla \times \vec{u}) = 0 \) for all \( \vec{u} \)

Problem 3.11

11. A flow field on the xy-plane has the velocity components

\[ u = 3x + y \quad v = 2x - 3y. \]

Show that the circulation around the circle \((x - 1)^2 + (y - 6)^2 = 4\) is \(4\pi\).
Problem 3.11

Find the circulation for the flow \( \mathbf{U} = (3x + y, 2x - 3z) \) about the point \((x, y) = (1, 6)\) with radius \(r = 2\).

Let's translate axis to \((x', y') = (1, 6)\)

\[
\oint \mathbf{U} \cdot d\mathbf{r} = \oint \mathbf{U}' \cdot d\mathbf{r}' = 2 \int_0^{2\pi} \mathbf{U} \cdot \hat{z}' = 2 \int_0^{2\pi} d\theta (\mathbf{U} \cdot \hat{z}')
\]

\[
\mathbf{U}' = (3(x' + 1) + y' + 6, 2(x' + 1) - 3(y' + 6)) = (3x' + y' + 9, 2x' - 3y' - 16)
\]

\[
x' = 2 \cos \theta \quad y' = 2 \sin \theta
\]

\[
\int = 2 \int_0^{2\pi} \left[ -5 \cos (3x' + y' + 9) + \cos (2x' - 3y' - 16) \right] = 2 \int_0^{2\pi} \left[ 12 \cos 5 \sin 6 - 6 \sin 6 + 9 \cos 6 - 16 \cos 0 \right] = 4\pi
\]

Problem 3.12

12. Consider the solid-body rotation

\[
u_\theta = \omega r \quad u_r = 0.
\]

Take a polar element of dimension \(r \, d\theta\) and \(dr\), and verify that the circulation is vorticity times area. (In Section 11 we performed such a verification for a circular element surrounding the origin.)
Problem 3.12

Verify that circulation is vorticity times area for the flow \( \omega_0 = \omega_0 x \) (solid body rotation).

Note: vorticity \( \omega \times u = \frac{1}{2} \omega_0 (\omega_0 \cdot \omega_0) = 2 \omega_0 \).

\[ \Gamma = \int \overrightarrow{u} \cdot d\overrightarrow{a} \]

\[ \Gamma = \frac{\omega_0 (r+5\gamma)}{l^2 + 2\gamma} \]

\[ = 2 \omega_0 r \delta \theta = 2 \omega_0 \times \frac{r \delta \theta}{\text{vorticity area}} \]

Problem 3.13

13. Using the indicial notation (and without using any vector identity) show that the acceleration of a fluid particle is given by

\[ \mathbf{a} = \frac{\partial \mathbf{u}}{\partial t} + \nabla \left( \frac{1}{2} q^2 \right) + \omega \times \mathbf{u}, \]

where \( q \) is the magnitude of velocity \( \mathbf{u} \) and \( \omega \) is the vorticity.
Using invarics, show
\[
\frac{d\mathbf{u}}{dt} = \frac{2\mathbf{u}}{2\varepsilon} + \nabla \left( \frac{1}{2} \mathbf{u} \cdot \nabla \mathbf{u} \right) + \mathbf{\omega} \times \mathbf{u}
\]

But \[
\frac{2\mathbf{u}}{2\varepsilon} = \mathbf{u} \cdot \nabla \mathbf{u} = \frac{2u_i}{2\varepsilon} + \partial_j \frac{2u_i}{\partial x_j}.
\]

But \[
u_j \frac{2u_i}{\partial x_j} = u_j \left( \frac{2u_i}{\partial x_j} - \frac{2u_j}{\partial x_i} \right) + u_j \frac{2u_i}{\partial x_i} - \mathbf{\omega} \times \mathbf{u} \frac{2u_i}{\partial x_i} \left( \frac{1}{2} u_j^2 \right)
\]

To show this...

\[
\mathbf{\omega} = \nabla \times \mathbf{u} = e_{ijk} \frac{2u_k}{\partial x_j}
\]

\[
\left( \frac{2u_k}{\partial x_j} \right) = e_{ijk} \frac{2u_k}{\partial x_j}
\]

\[
\mathbf{\omega} \times \mathbf{u} = e_{ijk} \mathbf{\omega}_j u_k
\]

\[
\mathbf{\omega} \times \mathbf{u} = e_{ijk} \frac{2u_k}{\partial x_j}
\]

\[
= - \left( \delta_{kj} \frac{2u_k}{\partial x_i} - \delta_{ki} \frac{2u_k}{\partial x_j} \right) u_j \frac{2u_k}{\partial x_i} - u_k \frac{2u_k}{\partial x_i} + u_j \frac{2u_k}{\partial x_j}
\]

Which can be written as above. (6.20)

**Problem 3.14**

14. The definition of the streamfunction in vector notation is

\[
\mathbf{u} = -\mathbf{k} \times \nabla \psi,
\]

where \(\mathbf{k}\) is a unit vector perpendicular to the plane of flow. Verify that the vector definition is equivalent to equations (3.35).
Problem 3.14

Verify that \( \mathbf{u} = -\mathbf{k} \times \nabla \psi \) (with \( \mathbf{k} = \mathbf{\hat{z}} \)) is equivalent to the condition \( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \) with \( \mathbf{u} = (u, v) \)...

But \( (\mathbf{u}) = (-\mathbf{k} \times \nabla \psi) = -\varepsilon_{ijk} \frac{\partial \psi}{\partial x^i} \) or \( \mathbf{u} = \left( \frac{\partial \psi}{\partial y}, -\frac{\partial \psi}{\partial x} \right) \)

Thus \( \frac{\partial}{\partial x} \left( \frac{\partial \psi}{\partial y} \right) - \frac{\partial}{\partial y} \left( \frac{\partial \psi}{\partial x} \right) = 0 \) is satisfied.

Equations of Fluid Dynamics

(Conservation Laws)

- Continuity (Mass)
- Navier-Stokes (Force, Momentum)
- Energy
Continuity

**Conservation of Mass**

\[
\frac{2\rho}{2\varepsilon} + \nabla \cdot (\rho \mathbf{u}) = 0
\]

\[
\frac{2\rho}{2\varepsilon} + (\mathbf{u} \cdot \nabla) \rho = -\rho \nabla \cdot \mathbf{u}
\]

\[
\frac{\partial \rho}{\partial t} = -\rho \nabla \cdot \mathbf{u}
\]

\[
\frac{\partial \rho}{\partial t} = -\mathbf{F} \cdot (\nabla \cdot \mathbf{u})
\]

\[
\text{e.g. } \rho = \epsilon
\]

Newton's Law

**Newton's Law for a Particle**

\[
\mathbf{F} = m \mathbf{a}
\]

\[
\mathbf{F} = \frac{d}{dt} (m \mathbf{v})
\]

**Newton's Law for a Fluid**

\[
\mathbf{F} = \frac{D}{Dt} (\rho \mathbf{u})
\]

\[
= \frac{2}{2\varepsilon} (\rho \mathbf{u}) + \nabla \cdot (\rho \mathbf{u} \mathbf{u})
\]

\[
= \frac{2}{2\varepsilon} (\rho \mathbf{u} \mathbf{u}) + \frac{3}{2\varepsilon} (\rho \mathbf{u}_i \mathbf{u}_j)
\]

\[
\text{Flux of Momentum from Fluid Element}
\]

\[
= \rho \left[ \frac{2\mathbf{u}}{2\varepsilon} + (\mathbf{u} \cdot \nabla) \mathbf{u} \right] + \mathbf{u} \left[ \frac{2\rho}{2\varepsilon} + \nabla / \rho \right]
\]

\[
= \rho \left( \frac{D \mathbf{u}}{Dt} \right)
\]
Momentum

\[ \rho \left( \frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla) \vec{u} \right) = \rho \vec{g} + \nabla \cdot \vec{\tau} \]

\( \vec{\tau} = \text{STRESS TENSOR} \)
\( = \text{USUALLY SYMMETRIC} \)
\( = \text{HAS NORMAL STRESS \& PRESSURE} \)
\( = \text{HAS SHEAR STRESS \& (OFF DIAGONAL)} \)
\( \text{GRADIENTS OF STRESS PRODUCE FORCE} \)

\( \tau_{ij} > 0 \) IMPLIES TENSILE STRESS
\( \tau_{ij} < 0 \) IMPLIES COMPRESSIVE STRESS
\( \tau_{ij} \) \( (i \neq j) \) ARE SHEAR STRESSES

Models for Stress

- ISOTROPIC PRESSURE

\[ \vec{\tau} = -\rho \vec{g} \quad \nabla \cdot \vec{\tau} = -\nabla \rho \]

- MOVING FLUID WITH VISCOSITY

\[ \vec{\tau} = -\rho \vec{g} + \vec{\sigma} \]

\( \vec{\sigma} \) VISCOUS STRESS

WHAT IS \( \vec{\sigma} \)?
Navier & Stokes

Claude-Lewis Henri Navier (1785-1836)

George Stokes (1819-1903)

Stokesian Fluid

**Material Isotropy and Stress Symmetry**

(e.g., air, water but not magnetized plasma)

\[
\vec{\sigma} = 2\mu \vec{\varepsilon} + \lambda (\nabla \cdot \vec{u}) \vec{\delta}
\]

\[\begin{array}{c|c}
\text{Viscosity} & \text{Bulk Viscosity} \\
\hline
\lambda & \frac{2}{3} \mu
\end{array}\]

Stokes modeled viscosity via kinetic theory of monatomic atoms and showed \( \lambda = \frac{2}{3} \mu \).

Then, stress tensor

\[\vec{\tau} = -\rho \vec{\delta} + 2\mu \vec{\varepsilon} + \frac{2}{3} \lambda (\nabla \cdot \vec{u}) \vec{\delta}\]
Navier-Stokes Equation

\[ \rho \left( \frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla) \vec{u} \right) = -\nabla p + \rho \vec{g} + \nabla \left[ 2 \mu \frac{\partial \vec{e}}{\partial \vec{e}} - \frac{2}{3} \mu (\nabla \cdot \vec{u}) \vec{e} \right] \]

Assume \( \mu \) independent of \( x \). Then,

\[ \nabla \cdot 2 \mu \vec{e} = 2 \mu (\nabla \cdot \vec{e}) \]

\[ \varepsilon_{ij} = \frac{1}{\varepsilon} \left( \frac{2 \mu \varepsilon_{ij}}{\partial x_j} + \frac{2 \mu \varepsilon_{ij}}{\partial x_i} \right) \]

\( \nabla \cdot \frac{\partial \vec{e}}{\partial \vec{e}} = \frac{1}{\varepsilon} \left( \frac{2 \partial \vec{u}}{\partial x_j} + \frac{2 \partial \vec{u}}{\partial x_i} \right) \approx \frac{1}{\varepsilon} \nabla^2 \vec{u} + \frac{1}{\varepsilon} \nabla (\nabla \cdot \vec{u}) \]

Navier-Stokes & Euler

\[ \rho \frac{D \vec{u}}{Dt} = -\nabla p + \rho \vec{g} + \begin{cases} \mu \left[ \nabla^2 \vec{u} + \frac{1}{3} \nabla (\nabla \cdot \vec{u}) \right] & \text{Navier-Stokes' Equation} \\ \mu \nabla^2 \vec{H} & \text{Incompressible N.S.} \\ 0 & \text{Euler Equation} \end{cases} \]
Energy

\[ \rho \frac{D \vec{u}}{Dt} = \rho \vec{g} + \nabla \cdot \vec{t} \]

\[ \frac{D}{Dt} \left( \frac{1}{2} u^2 \right) = \rho \vec{u} \cdot \vec{g} + \vec{u} \cdot \left( \nabla \cdot \vec{t} \right) \]

Work Done by Body Forces

Work Done by Surface Forces

The Importance of Viscosity

Incompressible Euler Equation

\[ \frac{2 \ddot{u}}{\partial t} + (\vec{u} \cdot \nabla) \vec{u} = -\nabla \rho \vec{g} + \vec{g} \]

\[ \nabla \cdot \vec{u} = 0 \]

Let \( \vec{\mathcal{S}} = \nabla \times \vec{u} \). Then

\[ (\vec{u} \cdot \nabla) \vec{u} = \vec{\mathcal{S}} \times \vec{u} + \frac{1}{2} \nabla u^2 \]

\[ \frac{2 \ddot{u}}{\partial t} + \vec{\mathcal{S}} \times \vec{u} = -\nabla \rho + \vec{g} - \frac{1}{2} \nabla u^2 \]

Take curl of this equation

\[ \frac{2}{\partial t} \vec{\mathcal{S}} + \nabla \times (\vec{\mathcal{S}} \times \vec{u}) = 0 \quad (\text{if } \vec{g} = -\nabla \phi) \]

If \( \vec{\mathcal{S}} = 0 \) at \( t = 0 \), then \( \vec{\mathcal{S}} = 0 \) forever!
Creation of Vorticity

(Note: Flow at thin layer at surface of cylinder vanishes.)

Summary

- The equations of fluid dynamics are dynamical conservation equations:
- Mass conservation
- Momentum changes via total forces (body and surface forces)
- Energy conservation