APPH 4200
Physics of Fluids

Review (Ch. 3) & Fluid Equations of Motion (Ch. 4)

1. Chapter 3 (more notes/examples)
2. Vorticity and Circulation
3. Navier-Stokes Equation

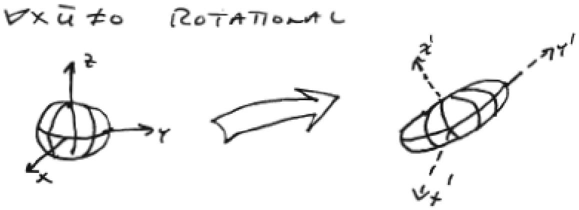
Summary:
Cauchy-Stokes Decomposition

ANY FLUID MOTION CAN BE RESOLVED INTO

- TRANSLATION
- DILATATION / EXPANSION ALONG PRINCIPAL AXES
- ROTATION

$\nabla \times \bar{u} = 0$ IRROTATIONAL

$\nabla \times \bar{u} \neq 0$ ROTATIONAL



Velocity Gradient Tensor

$$\frac{\partial u_i}{\partial x_j} = \epsilon_{ij} + \frac{1}{2} R_{ij}$$

DEFORMATION TENSOR (SYMMETRIC)

ROTATION TENSOR (ANTI-SYMMETRIC)

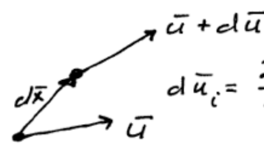
$$\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

$$R_{ij} = \left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)$$

$$R_{ij} = -\epsilon_{ijk}\Omega_k$$

$$\Omega_k = -\frac{1}{2}\epsilon_{ijk}R_{ij}$$

$= \text{VORTICITY}$



$$d\bar{u}_i = \frac{\partial u_i}{\partial x_j} d\bar{x}_j$$

Problem 3.1

1. A two-dimensional steady flow has velocity components

$$u = y \qquad v = x.$$

Show that the streamlines are rectangular hyperbolas

$$x^2 - y^2 = \text{const}.$$

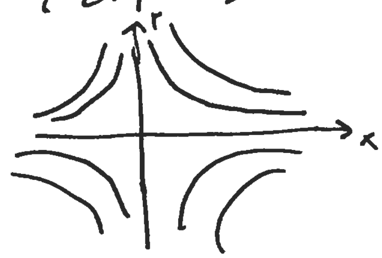
Sketch the flow pattern, and convince yourself that it represents an irrotational flow in a 90° corner.

Problem 3.1

FIND STREAMLINES WHEN $\bar{u} = (y, x)$

$$\frac{dx}{y} = \frac{dy}{x} = ds$$

$$x\,dx = y\,dy \Rightarrow \quad x^2 = y^2 + c$$



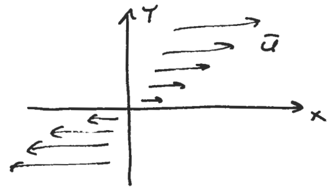
Problem 3.3

3. If a velocity field is given by $u = ay$, compute the circulation around a circle of radius $r = 1$ about the origin. Check the result by using Stokes' theorem.

Problem 3.3

LET $\bar{u} = (aY, 0, 0)$

$$\Gamma = \oint \bar{u} \cdot d\bar{s} = \iint \nabla \times \bar{u} \cdot d\bar{A}$$



FIND CIRCULATION AROUND UNIT CIRCLE

$$\oint \bar{u} \cdot d\bar{s} = \int_0^{2\pi} d\theta \; \bar{u} \cdot \hat{t}$$

$$\hat{t} = \hat{y}\cos\theta - \hat{x}\sin\theta$$

$$\bar{u} \cdot \hat{t} = -aY\sin\theta = -a\sin^2\theta$$

SO $\oint \bar{u} \cdot d\bar{s} = -a \int_0^{2\pi} \sin^2\theta \, d\theta = -a\pi$

ALSO

$$\iint \nabla \times \bar{u} \cdot dA = -a \iint \hat{z} \cdot d\bar{A} = -a\pi$$

SINCE $\nabla \times \bar{u} = -\hat{z} a$

Problem 3.4

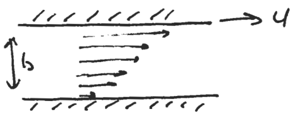
4. Consider a plane Couette flow of a viscous fluid confined between two flat plates at a distance b apart (see Figure 9.4c). At steady state the velocity distribution is

$$u = Uy/b \qquad v = w = 0,$$

where the upper plate at $y = b$ is moving parallel to itself at speed U, and the lower plate is held stationary. Find the rate of linear strain, the rate of shear strain, and vorticity. Show that the streamfunction is given by

$$\psi = \frac{Uy^2}{2b} + \text{const.}$$

Problem 3.4



$\bar{u} = (uy/b, 0, 0)$

LINEAR STRAIN RATE = $\frac{\partial u_i}{\partial x_i}$

$\epsilon_{xx} = \epsilon_{yy} = 0$

SHEAR STRAIN RATE = $\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) = \epsilon_{ij}$

$\epsilon_{xy} = \epsilon_{yx} = \frac{1}{2}\, u/b$

VORTICITY = $\nabla \times \bar{u} = \hat{z}\left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right)$

$\bar{\omega} = -\hat{z}\, u/b$

WHAT IS THE STREAM FUNCTION?

SINCE $\nabla \cdot \bar{u} = 0$, $\bar{u} = \hat{z} \times \nabla\psi = \left(\frac{\partial\psi}{\partial y}, -\frac{\partial\psi}{\partial x}, 0\right)$

$= \nabla z \times \nabla\psi$

$= \nabla \times (z\nabla\psi)$

SO

$\frac{uy}{b} = \frac{\partial\psi}{\partial y} \Rightarrow \psi(y) = \frac{uy^2}{2b}$ + CONSTANT

Problem 3.5

5. Show that the vorticity for a plane flow on the xy-plane is given by

$$\omega_z = -\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}\right).$$

Using this expression, find the vorticity for the flow in Exercise 4.

Problem 3.5

FIND VORTICITY, $\bar{\omega}$, FOR FLOW ON X-Y PLANE

$\bar{\omega} = \nabla \times \bar{u}$ IF $\bar{u} = (u, v, 0)$ THEN

$\bar{u} = -\nabla z \times \nabla \psi = -\hat{z} \times \nabla \psi$

$\therefore \quad \bar{\omega} = -\nabla \times (\hat{z} \times \nabla \psi) = -\hat{z}\, \nabla^2 \psi + \nabla \psi \overbrace{(\nabla \cdot \hat{z})}^{0}$

FOR THE FLOW IN EXERCISE #4

$\omega_z = -4/b$

Problem 3.7

7. Determine an expression for ψ for a Rankine vortex (Figure 3.17b), assuming that $u_\theta = U$ at $r = R$.

Problem 3.7

FIND STREAMFUNCTION FOR A RANKINE VORTEX

$$u_\theta(r) = \begin{cases} U(r/R) & r < R \\ U(R/r) & r > R \end{cases}$$

$$\bar{u} = -\hat{z} \times \nabla \psi \Rightarrow \boxed{u_\theta = -\frac{\partial \psi}{\partial r}}$$

SO

$$\psi(r) = \begin{cases} -\frac{r^2}{2R} U & r \leq R \\ -UR \ln\left(\frac{r}{R}\right) - \frac{RU}{2} & r > R \end{cases}$$

NOTE:

$\nabla^2 \psi \propto \omega_z$ FOR $r < R$ (ROTATIONAL)

$\nabla^2 \psi = 0$ FOR $r > R$ (IRROTATIONAL)

Problem 3.8

8. Take a plane polar element of fluid of dimensions dr and $r\,d\theta$. Evaluate the right-hand side of Stokes' theorem

$$\int \boldsymbol{\omega} \cdot d\mathbf{A} = \int \mathbf{u} \cdot d\mathbf{s},$$

and thereby show that the expression for vorticity in polar coordinates is

$$\omega_z = \frac{1}{r}\left[\frac{\partial}{\partial r}(r u_\theta) - \frac{\partial u_r}{\partial \theta}\right].$$

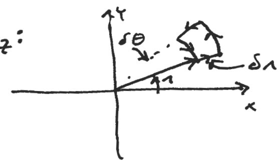
Also, find the expressions for ω_r and ω_θ in polar coordinates in a similar manner.

Problem 3.8

FIND EXPRESSIONS FOR VORTICITY IN CYLINDRICAL COORDINATES.

$\bar{\omega} = \nabla \times \bar{u} \qquad \iint \bar{\omega} \cdot d\bar{A} = \oint \bar{u} \cdot d\bar{s}$

ω_z:

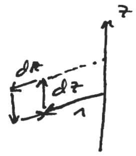


$$\oint \bar{u}\cdot d\bar{s} = u_r\, dr + (r+\delta r)\, d\theta\, u_\theta(r+\delta r) - u_r(\theta+\partial\theta)\, dr - r\, d\theta\, u_\theta$$

$$= -dr\, d\theta \frac{\partial u_r}{\partial \theta} + dr\, d\theta \left[u_\theta + r \frac{\partial u_\theta}{\partial r} \right]$$

$$= r\, dr\, d\theta \left[\frac{1}{r} \frac{\partial}{\partial r}(r u_\theta) - \frac{1}{r} \frac{\partial u_r}{\partial \theta} \right]$$

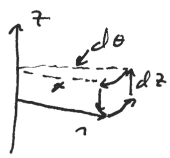
ω_θ:



$$\oint u \cdot ds = dz\, u_z + dr\, u_r(z+\delta z) - dz\, u_z(r+\delta R) - dr\, u_r$$

$$= -dr\, dz \frac{\partial u_z}{\partial r} + dr\, dz \frac{\partial u_r}{\partial z} = dr\, dz \left[\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r} \right]$$

ω_r:



$$\oint u \cdot ds = r\, d\theta\, u_\theta + dz\, u_z(\theta+\delta\theta) - r\, d\theta\, u_\theta(z+\delta z) - dz\, u_z$$

$$= dz\, d\theta \frac{\partial u_z}{\partial \theta} - r\, d\theta\, dz \frac{\partial u_\theta}{\partial z}$$

$$= -r\, d\theta\, dz \left[\frac{1}{r} \frac{\partial u_z}{\partial \theta} - \frac{\partial u_\theta}{\partial z} \right]$$

Q.E.D.

Problem 3.10

10. Show that the vorticity field for *any* flow satisfies

$$\nabla \cdot \omega = 0.$$

Problem 3.10

$\nabla \cdot \bar{\omega} = 0$ WHERE $\bar{\omega} = \nabla \times \bar{u}$

BUT $\nabla \cdot \nabla \times \bar{u} = 0$ FOR ALL $\bar{u}$

Equations of Fluid Dynamics

(Conservation Laws)

- Continuity (Mass)
- Navier-Stokes (Force, Momentum)
- Energy

Integral Relations

(Section 4.2)

$$\frac{d}{dt}\int_{V(t)} f dV = \begin{cases} \int_V \frac{\partial f}{\partial t} dV \text{ (fixed V)} \\ \int_V \frac{\partial f}{\partial t} dV + \int_A d\mathbf{A}\cdot\mathbf{U} f \end{cases}$$

Moving volume, V(t), at velocity U

(Reynolds Transport Theorem)

Integral Relations

(Section 4.2)

$$\begin{aligned} \tfrac{d}{dt}\int_{V(t)} (\rho f) dV &= \int_V \tfrac{\partial(\rho f)}{\partial t} dV + \int_A d\mathbf{A}\cdot\mathbf{U}(\rho f) \\ &= \int_V \left(\rho\tfrac{\partial f}{\partial t} + f\tfrac{\partial \rho}{\partial t} + \nabla\cdot\rho f\mathbf{U}\right) dV \\ &= \int_V \left(\rho\tfrac{\partial f}{\partial t} + \boxed{f\tfrac{\partial \rho}{\partial t} + f\nabla\cdot\rho\mathbf{U}} + \rho\mathbf{U}\cdot\nabla f\right) dV \\ &= \int_V \rho\left(\tfrac{\partial f}{\partial t} + \mathbf{U}\cdot\nabla f\right) dV \\ &= \int_V \rho\tfrac{df}{dt} dV \end{aligned}$$

Conserved material, ρ

Continuity

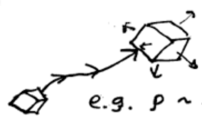
CONSERVATION OF Mass

$$\frac{\partial \rho}{\partial t} + \bar{\nabla}\cdot(\rho \bar{u}) = 0$$

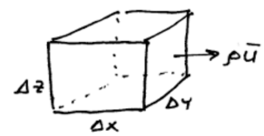
$$\underbrace{\frac{\partial \rho}{\partial t} + (\bar{u}\cdot\bar{\nabla})\rho}_{} = -\rho \nabla\cdot\bar{u}$$

$$\frac{D\rho}{Dt} = -\rho \nabla\cdot\bar{u}$$

$$\frac{D\rho}{\rho} = -Dt(\nabla\cdot\bar{u})$$



e.g. $\rho \sim e^{-\int dt' \nabla\cdot\bar{u}}$



$$\frac{\partial}{\partial t}(\Delta V \rho) = -\sum_{\substack{\text{ALL}\\ \text{SURFACES}}} \rho\bar{u}\cdot\Delta\bar{A}$$

$$= -\frac{\partial}{\partial x_i}(\rho u_i)\Delta x_i\cdot\Delta$$

$$= -\Delta V\, \nabla\cdot(\rho\bar{u})$$

Newton's Law

NEWTON'S LAW FOR A PARTICLE

$$\bar{F} = m\bar{a}$$

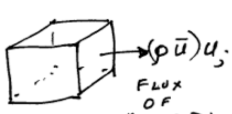
$$\bar{F} = \frac{d}{dt}(m\bar{v})$$

NEWTON'S LAW FOR A FLUID:

$$\bar{F} = \frac{D}{Dt}(\rho\bar{u})$$

$$= \frac{\partial}{\partial t}(\rho\bar{u}) + \bar{\nabla}\cdot(\rho\bar{u}\,\bar{u})$$

$$= \frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_j}(\rho u_i u_j)$$



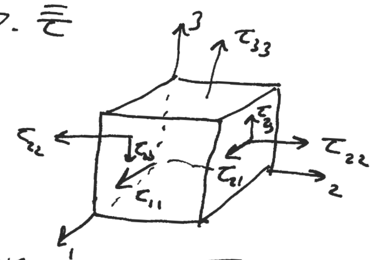
FLUX OF MOMENTUM FROM FLUID ELEMENT

$$= \rho\left[\frac{\partial\bar{u}}{\partial t} + (\bar{u}\cdot\bar{\nabla})\bar{u}\right] + \bar{u}\underbrace{\left[\frac{\partial\rho}{\partial t} + \bar{\nabla}\cdot(\rho\bar{u})\right]}_{=0 \text{ CONSERVATION OF MASS}}$$

$$= \rho\frac{D\bar{u}}{Dt}$$

Momentum

$$\rho\left(\frac{\partial \bar{u}}{\partial t} + (\bar{u}\cdot\nabla)\bar{u}\right) = \rho\bar{g} + \nabla\cdot\bar{\bar{\tau}}$$



$\bar{\bar{\tau}}$ = STRESS TENSOR

= USUALLY SYMMETRIC

- = HAS NORMAL STRESS ~ PRESSURE

= HAS SHEAR STRESS ~ (OFF DIAGONAL)

GRADIENTS OF STRESS PRODUCE FORCE

τ_{ij}: i = WHICH FACE, j = DIRECTION

τ_{ij} IS POSITIVE WHEN DIRECTED IN DIRECTION OF AXIS

$\tau_{ii} > 0$ IMPLIES TENSILE STRESS

$\tau_{ii} < 0$ IMPLIES COMPRESSIVE STRESS

τ_{ij} $(i \neq j)$ ARE SHEAR STRESSES

Models for Stress

- ISOTROPIC PRESSURE

$$\bar{\bar{\tau}} = -p\bar{\bar{\delta}} \qquad \nabla\cdot\bar{\bar{\tau}} = -\nabla p$$

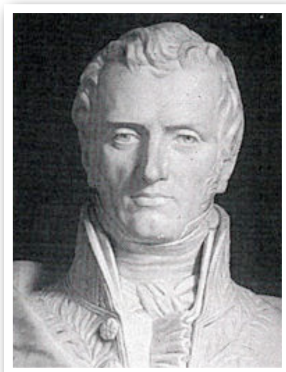
- MOVING FLUID WITH VISCOSITY

$$\bar{\bar{\tau}} = -p\bar{\bar{\delta}} + \bar{\bar{\sigma}}$$

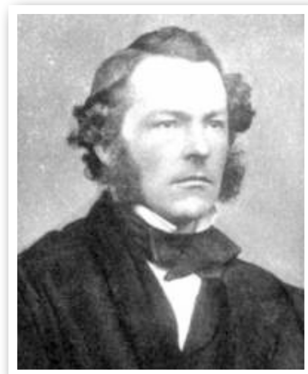
$\bar{\bar{\sigma}}$ = VISCOUS STRESS

WHAT IS $\bar{\bar{\sigma}}$?

Navier & Stokes



Claude-Lewis Henri Navier
(1785-1836)



George Stokes
(1819-1903)

Stokesian Fluid

MATERIAL ISOTROPY AND STRESS SYMMETRY
(e.g. AIR, WATER BUT <u>NOT</u> MAGNETIZED PLASMA)

$$\bar{\bar{\sigma}} = 2\mu\bar{\bar{\epsilon}} + \lambda(\nabla\cdot\bar{u})\bar{\bar{\delta}}$$

VISCOSITY

~ BULK VISCOSITY

STOKES MODELED VISCOSITY VIA KINETIC THEORY OF MONATOMIC ATOMS AND SHOWED $\lambda = -\frac{2}{3}\mu$.

THEN, STRESS TENSOR

$$\bar{\bar{\tau}} = -p\bar{\bar{\delta}} + 2\mu\bar{\bar{\epsilon}} - \frac{2}{3}\mu(\nabla\cdot\bar{u})\bar{\bar{\delta}}$$

Navier-Stokes Equation

$$\rho\left(\frac{\partial \bar{u}}{\partial t} + (\bar{u}\cdot\nabla)\bar{u}\right) = -\nabla p + \rho\bar{g} + \nabla\cdot\left[2\mu\bar{\bar{\epsilon}} - \frac{2}{3}\mu(\nabla\cdot\bar{u})\bar{\bar{\delta}}\right]$$

ASSUME μ ~ INDEPENDENT OF $\bar{x}$. THEN,

$$\nabla\cdot 2\mu\bar{\bar{\epsilon}} = 2\mu(\nabla\cdot\bar{\bar{\epsilon}}) \qquad \epsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

$$(\nabla\cdot\bar{\bar{\epsilon}})_i = \frac{1}{2}\left(\frac{\partial^2 u_i}{\partial x_j^2} + \frac{\partial u_j}{\partial x_i\,\partial x_j}\right) = \frac{1}{2}\nabla^2\bar{u} + \frac{1}{2}\bar{\nabla}(\nabla\cdot\bar{u})$$

Navier-Stokes & Euler

$$\rho\frac{D\bar{u}}{Dt} = -\nabla p + \rho\bar{g} + \begin{cases} \mu\left[\nabla^2\bar{u} + \frac{1}{3}\bar{\nabla}(\nabla\cdot\bar{u})\right] & \text{NAVIER-STOKES EQUATION} \\ \mu\nabla^2\bar{u} & \text{INCOMPRESSIBLE N.S.} \\ 0 & \text{EULER EQUATION} \end{cases}$$

Energy

$$\rho \frac{D\bar{u}}{Dt} = \rho \bar{g} + \bar{\nabla} \cdot \bar{\bar{\tau}}$$

$$\rho \frac{D}{Dt}\left(\tfrac{1}{2}u^2\right) = \underbrace{\rho\, \bar{u} \cdot \bar{g}}_{\text{WORK DONE BY BODY FORCES}} + \underbrace{\bar{u} \cdot (\nabla \cdot \bar{\bar{\tau}})}_{\text{WORK DONE BY SURFACE FORCES}}$$

The Importance of Viscosity

INCOMPRESSABLE EULER EQUATION

$$\frac{\partial \bar{u}}{\partial t} + (\bar{u} \cdot \nabla)\bar{u} = -\nabla p / \rho + \bar{g}$$

$$\nabla \cdot \bar{u} = 0$$

LET $\bar{\Omega} = \nabla \times \bar{u}$. THEN

$$(\bar{u} \cdot \bar{\nabla})\bar{u} = \bar{\Omega} \times \bar{u} + \tfrac{1}{2}\bar{\nabla} u^2$$

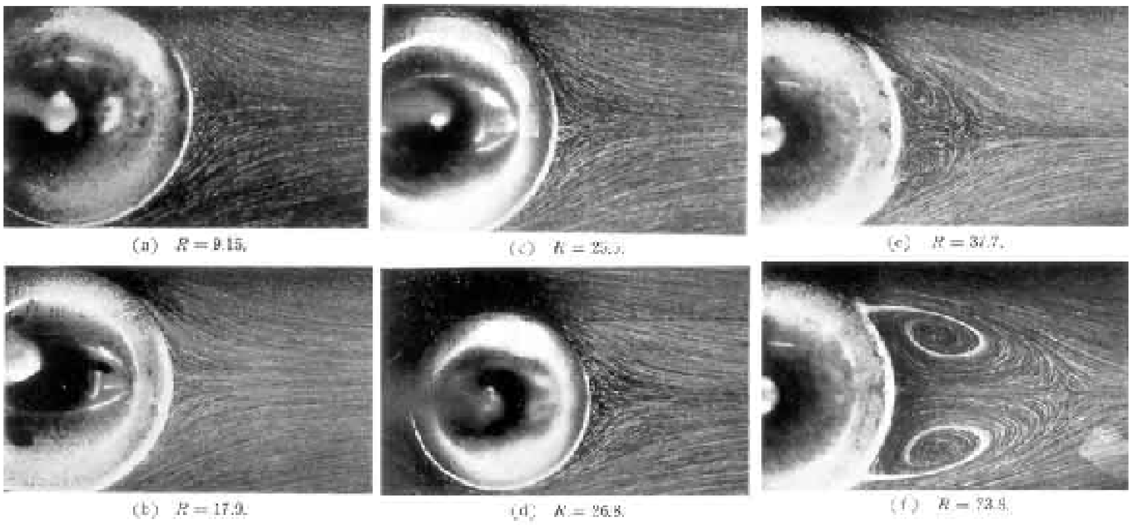
$$\frac{\partial \bar{u}}{\partial t} + \bar{\Omega} \times \bar{u} = -\frac{\nabla p}{\rho} + \bar{g} - \tfrac{1}{2}\bar{\nabla} u^2$$

TAKE CURL OF THIS EQUATION

$$\frac{\partial}{\partial t}\bar{\Omega} + \nabla \times (\bar{\Omega} \times \bar{u}) = 0 \qquad (\text{IF } \bar{g} = -\nabla\varphi)$$

IF $\bar{\Omega} = 0$ at $t = 0$, THEN $\bar{\Omega} = 0$ FOREVER!

Creation of Vorticity



(Note: Flow at thin layer at surface of cylinder vanishes.)

Summary

- The equations of fluid dynamics are dynamical conservation equations:
- Mass conservation
- Momentum changes via total forces (body and surface forces)
- Energy conservation