

APPH 4200

Physics of Fluids

Kinematics: Describing Fluid Flow (Ch. 3)

1. Quick Review
2. Lagrangian (material) & Eulerian (field)
3. Streamlines & Pathlines
4. Point Deformations: Stretching, Pinching, and Rotating
5. Principal Axes (**Cauchy-Stokes Decomposition**)

Ch. 2 Question 10

PROVE $\nabla \cdot \nabla \times \bar{u} = 0$ FOR ANY VECTOR FIELD.

METHOD #1

$$\nabla \cdot \nabla \times \bar{u} = \sum_{i,j,k} \epsilon_{ijk} \frac{\partial^2 u_j}{\partial x_i \partial x_k}$$

$$\text{BUT } \epsilon_{isk} = -\epsilon_{iks} \text{ WHILE } \frac{\partial^2 u_j}{\partial x_i \partial x_k} = \frac{\partial^2 u_j}{\partial x_k \partial x_i}$$

SO THE SUM MUST VANISH

METHOD #2



$$\begin{aligned} \iiint dV \nabla \cdot \nabla \times \bar{u} &= \iint \nabla \times \bar{u} \cdot d\bar{s} \\ &= \iint_{\text{TOP}} \nabla \times \bar{u} \cdot d\bar{s} + \iint_{\text{BOTTOM}} \nabla \times \bar{u} \cdot d\bar{s} \\ &= \oint_{\text{TOP}} d\bar{l} \cdot \bar{u} + \oint_{\text{BOT}} d\bar{l} \cdot \bar{u} \end{aligned}$$

BUT THE SENSE OF THESE TWO LINE INTEGRALS ARE OPPOSITE, SO THE SUM MUST VANISH.

Ch. 2 Question 11

PROVE $\nabla \times \nabla \varphi = 0$ FOR ANY WELL-BEHAVED FUNCTION φ .

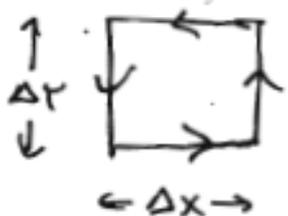
METHOD #1

$$(\nabla \times \nabla \varphi)_i = \sum_{j,k} \epsilon_{ijk} \frac{\partial^2 \varphi}{\partial x_j \partial x_k}$$

BUT $\epsilon_{ijk} = -\epsilon_{ikj}$ AND $\frac{\partial^2 \varphi}{\partial x_j \partial x_k}$ IS SYMMETRIC;

SO SUM MUST VANISH.

METHOD #2



$$\iint ds \cdot \nabla \times \nabla \varphi = \oint_{\partial D} d\vec{l} \cdot \nabla \varphi$$

$$= \Delta x \left(\frac{\partial \varphi}{\partial x} \Big|_y - \frac{\partial \varphi}{\partial x} \Big|_{y+\Delta y} \right)$$

$$+ \Delta y \left(\frac{\partial \varphi}{\partial y} \Big|_{x+\Delta x} - \frac{\partial \varphi}{\partial y} \Big|_x \right)$$

$$= -\Delta x \Delta y \frac{\partial^2 \varphi}{\partial x \partial y} + \Delta x \Delta y \frac{\partial^2 \varphi}{\partial y \partial x}$$

$$= 0$$

SINCE LINE SEGMENTS ADDED IN OPPOSITION.

Kinematics of Fluid Flow (Ch. 3)

- Streamlines, pathlines, and convective (material) derivative
- Translations, Deformation, and Rotation of a fluid element (Cauchy-Stokes Decomposition)

Kinematics of Fluid Flow (Ch. 3)

BASIC IDEA OF "FLOW" IS THAT AN INITIAL POINT \bar{x}_0 AT $t=0$ IS AT A LATER TIME AT $\bar{x}(t)$

$$\bar{x}(t) = \bar{x}(\bar{x}_0, t)$$

A FUNCTION OF INITIAL POSITION AND TIME.

\bar{x}_0 IS A MATERIAL COORDINATE.

$\bar{x}(\bar{x}_0, t)$ IS A LAGRANGIAN DESCRIPTION OF MOTION.

MOTION IS SIMPLE, AND CAN BE "INVERTED"

$$\bar{x}_0 = \bar{x}_0(\bar{x}, t)$$

FOLLOWING A PATH BACKWARDS IN TIME.

Eulerian Description

(Field Variables)

FLUID DYNAMICS IS MOST COMMONLY DESCRIBED USING FIELD QUANTITIES

$$\rho(\vec{x}, t) \quad \vec{u}(\vec{x}, t) \quad T(\vec{x}, t)$$

A PROPERTY AT \vec{x} EVOLVES IN TIME.

THIS IS THE EULERIAN DESCRIPTION.

LAGRANGIAN \Leftrightarrow EULERIAN

ARE DIFFERENT
WAYS TO DESCRIBE THE
SAME THING. THEY ARE
EQUIVALENT!

EULERIAN

$$\rho(\vec{x}, t)$$

$$\rho(\vec{x}, t) = \rho[\vec{x}(\vec{x}_0, t), t]$$

LAGRANGIAN

$$\rho(\vec{x}_0, t) = \rho[\vec{x}_0(\vec{x}, t), t]$$

$$\rho(\vec{x}_0, t)$$

Time Derivatives

$\left. \frac{d}{dt} \right|_{x_0} \sim$ How DOES THE MATERIAL PROPERTIES EVOLVE ALONG A PATHLINE?

$\left. \frac{\partial}{\partial t} \right|_x \sim$ How DOES THE MATERIAL PROPERTIES EVOLVE AT FIXED POINT \bar{x} ?

THUS

$$\bar{u} = \frac{d\bar{x}}{dt} = \frac{\partial \bar{x}}{\partial t}(x_0, t)$$

AND

$$\begin{aligned} \frac{df}{dt} &= \frac{\partial}{\partial t} f(\bar{x}_0, t) = \frac{\partial}{\partial t} f[x_0(\bar{x}_0, t), t] \\ &= \frac{\partial f}{\partial x_i} \frac{\partial x_i}{\partial t} \Big|_{x_0} + \frac{\partial f}{\partial t} \Big|_x \\ &= u_i \frac{\partial f}{\partial x_i} + \frac{\partial f}{\partial t} \end{aligned}$$

“advective” + “unsteady”

i.e. CONVECTIVE OR MATERIAL DERIVATIVE.

Streamlines and Pathlines

WHAT ARE THE STREAMLINES AND PATHLINES
FOR A FLOW GIVEN BY

$$\vec{U} = \left(\frac{x}{1+a_x t}, \frac{y}{1+a_y t}, \frac{z}{1+a_z t} \right)$$

NOTE: STREAMLINES ARE COMPUTED AT A
FIXED TIME, t .
PATHLINES ARE COMPUTED ALONG
A TRAJECTORY OF A FLUID ELEMENT

STREAMLINES

$$\frac{dx}{ds} = \frac{x}{1+a_x t}$$

$$x(s) \sim e^{\frac{s}{1+a_x t}}$$

PATH LINES

$$\frac{dx}{dt} = \frac{x(t)}{1+a_x t}$$

$$x(t) \sim (1+a_x t)^{1/a_x} \\ (\text{IF } a_x \neq 0)$$

Streamlines and Pathlines (cont.)

EXAMPLE: $a_x = 1$ $a_y = \frac{1}{2}$

STREAMLINES

$$x(s) = x_0 e^{s/(1+t)}$$

$$y(s) = y_0 e^{s/(1+t/2)}$$

$t \sim$ PARAMETER

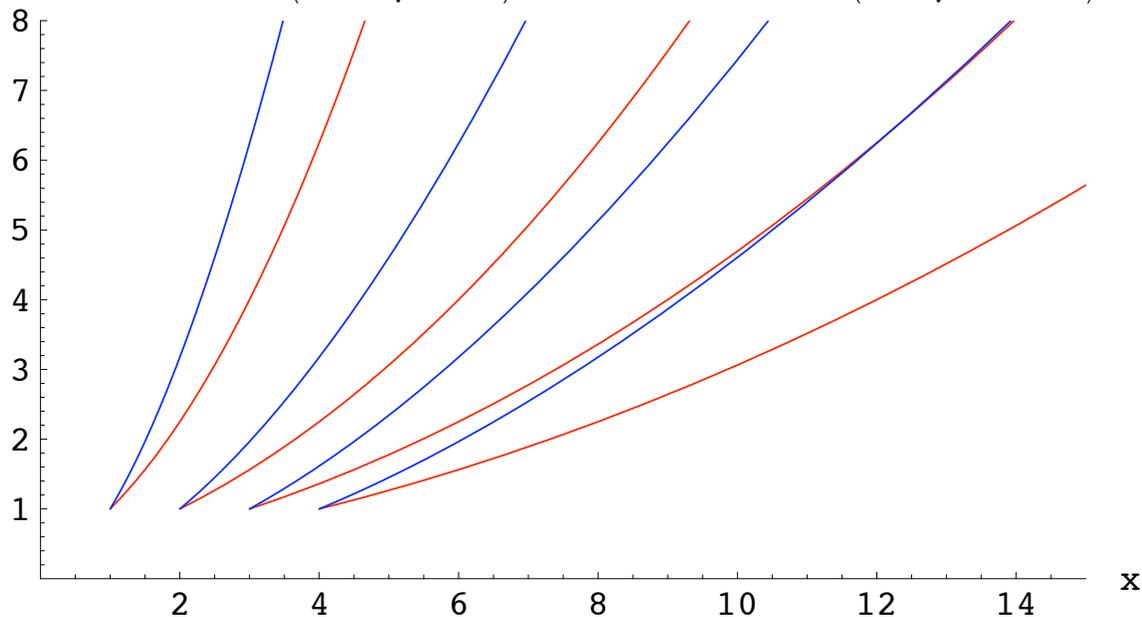
PATHLINES

$$x(t) = x_0(1+t)$$

$$y(t) = y_0(1+t/2)^2$$

$t \sim$ TIME VARIABLE

Streamlines (Blue, $t=4$) and Pathlines (Red, $0 < t < 4$)



Streaklines or "Smoke Lines"

"STREAKLINES" SHOW THE PATH OF "SMOKE" RELEASED FROM A SINGLE POINT INTO A FLOW.

RELEASE SMOKE AT (x_0, y_0)

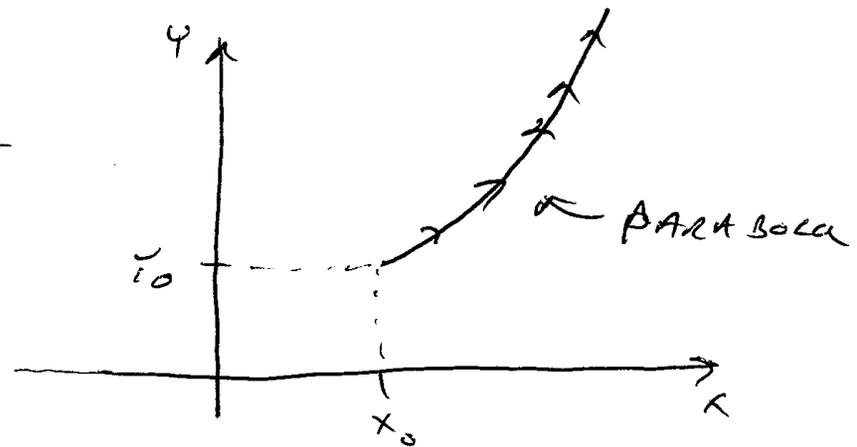
SMOKE RELEASED AT $t = t'$ FOLLOWS THE PATH

$$\begin{aligned} x(t) &= x_0 (1 + (t - t')) \\ y(t) &= y_0 (1 + (t - t')/2)^2 \end{aligned} \quad \text{WITH } \bar{u} = \left(\frac{x}{1+t}, \frac{y}{1+t/2} \right) \text{ FOR } t > t'$$

NOW t' = PARAMETER OF CURVE = THE RELEASE TIME

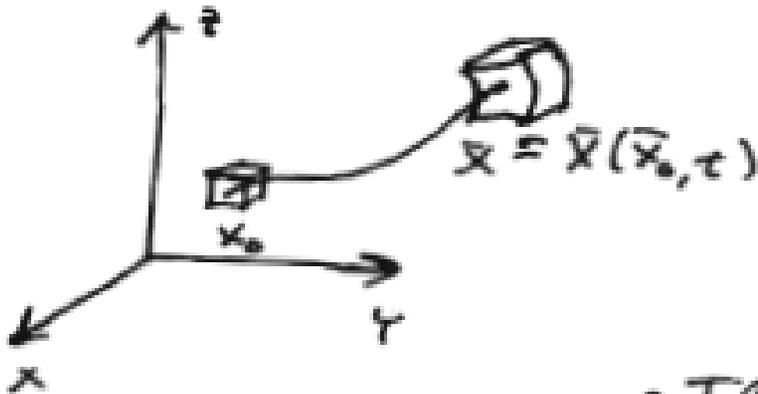
WHEN WE ELIMINATE:

$$\frac{y(t)}{y_0} = \left(1 + \frac{(x(t) - x_0)}{2} \right)^2$$



Deformation and Flow:

Translation, Stretching, Pinching, and Rotating



WHAT HAPPENS TO A FLUID ELEMENT AS IT FLOWS?

- TRANSLATION
- EXPANSION / SHRINKAGE (DILATATION)
- STRETCHING OR PINCHING
- SHEARING OR DISTORTION
- ROTATION

In class problem...

Simple Comments about Velocity Gradient Tensor

RIGID TRANSLATION IS CHARACTERIZED AS

$$\vec{x} = \vec{x}_0 + \vec{u} t$$

$$\frac{d\vec{x}}{dt} = \vec{u} = \text{CONSTANT}$$

$$\text{SO } \frac{\partial u_i}{\partial x_j} = 0$$

SYMMETRIC AND ANTI-SYMMETRIC PARTS

$$\frac{\partial u_i}{\partial x_j} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) + \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} \right)$$

$$\equiv \epsilon_{ij} + \mathcal{R}_{ij}$$

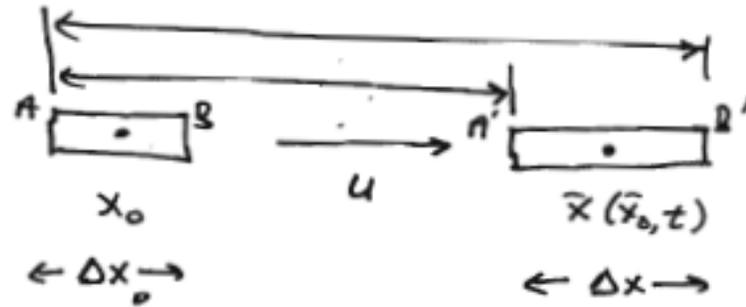
$$\equiv \epsilon_{ij} + \frac{1}{2} R_{ij} \quad (R_{ij} = \text{ROTATION TENSOR})$$

STRAIN RATE TENSOR

$$du_i = \left(\epsilon_{ij} + \frac{1}{2} R_{ij} \right) dx_j$$

Simple Example:

Expansion/ Contraction



$$\begin{aligned}\Delta x_i &= x_i(x_0 + \Delta x, t) - x_i(x_0, t) \\ &= \frac{\partial x_i(t)}{\partial x_{0j}} \Delta x_{0j}\end{aligned}$$

DISPLACEMENT GRADIENT TENSOR

Vector Notation

TIME DERIVATIVE....

$$\begin{aligned}\frac{d}{dt} \Delta x_i(t) &= \frac{d}{dt} \left(\frac{\partial x_i(t)}{\partial x_{0j}} \right) \Delta x_{0j} \\ &= \frac{\partial u_i(t)}{\partial x_{0j}} \Delta x_{0j} \\ &= \frac{\partial u_i}{\partial x_{0j}} \frac{\partial x_{0j}}{\partial x_h} \Delta x_h \\ &= \frac{\partial u_i}{\partial x_j} \Delta x_j\end{aligned}$$

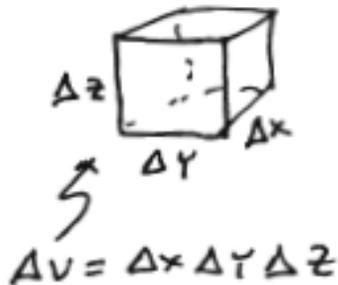
VELOCITY GRADIENT TENSOR

Stretching along one Axis

$$\frac{d \Delta x_1(t)}{dt} = \frac{2u_1}{2x_1} \Delta x_1$$

$$\frac{1}{\Delta x_1} \frac{d \Delta x_1}{dt} \stackrel{\text{OR}}{=} \epsilon_{11} = \begin{cases} \text{IF } \epsilon_{11} > 0 & \Delta x_1 \sim e^{\epsilon_{11} t} \\ \text{IF } \epsilon_{11} < 0 & \Delta x_1 \sim e^{\epsilon_{11} t} \end{cases}$$

A CUBE...



$$\frac{d}{dt} \Delta V = \frac{d}{dt} (\Delta x \Delta y \Delta z)$$

$$= \epsilon_{11} \Delta V + \epsilon_{22} \Delta V + \epsilon_{33} \Delta V$$

$$= (\nabla \cdot \mathbf{u}) \Delta V$$

SO

$$(\nabla \cdot \mathbf{u}) t$$

$$\Delta V(t) \sim e$$

IF $\nabla \cdot \mathbf{u} > 0$, THEN EXPANSION 

IF $\nabla \cdot \mathbf{u} = 0$, VOLUME IS CONSTANT

IF $\nabla \cdot \mathbf{u} < 0$, THEN DILATATION

Rotation

WHAT IF $\frac{\partial u_i}{\partial x_j} = \mathcal{R}_{ij}$ (ANTI-SYMMETRIC)
 $= \frac{1}{2} R_{ij}$

WHERE $\omega_i = -\epsilon_{ijk} R_{jk}$
 $= -\frac{1}{2} \epsilon_{ijk} \mathcal{R}_{jk}$

THEN $u_i = \epsilon_{ijk} \omega_j x_k + u_{0i}$
 $\bar{u} = \bar{\omega} \times \bar{x} + \bar{u}_0$ (RIGID ROTATION)

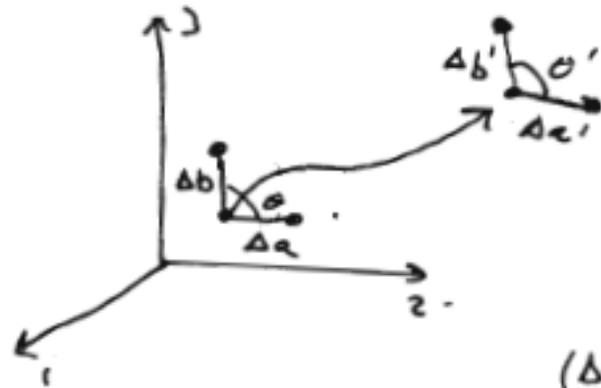
$$\mathcal{R}_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} \right)$$

$$\frac{\partial u_i}{\partial x_m} = \epsilon_{ijk} \omega_j \frac{\partial x_k}{\partial x_m} = \epsilon_{ijm} \omega_j$$

SO THAT

$$\mathcal{R}_{ij} = \frac{1}{2} (\epsilon_{ikj} \omega_k - \epsilon_{ilk} \omega_k)$$
$$= \epsilon_{ikj} \omega_k \quad (\text{CHECK!})$$

Bending, Distorting, Shearing



SHEAR FLOWS CAUSE
DISTORTIONS

$$\Delta \bar{a} \cdot \Delta \bar{b} = dx_{ai} dx_{bi}$$

$$(\Delta a \Delta b \cos \theta) =$$

DIFFERENTIATE WRT TIME...

$$\frac{d}{dt} (\Delta a \Delta b \cos \theta) = dU_{ai} dx_{bi} + dx_{ai} dU_{bi}$$

$$\cos \theta \left\{ \frac{d\Delta a}{dt} \Delta b + \Delta a \frac{d\Delta b}{dt} \right\}$$

$$- \Delta a \Delta b \sin \theta \frac{d\theta}{dt} = \frac{\partial U_i}{\partial x_j} dx_{aj} dx_{bi} +$$

$$dx_{ai} \frac{\partial U_i}{\partial x_j} dx_{bj}$$

$$= \left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right) dx_{aj} dx_{bi}$$

Example Shearing

$$\cos \theta \left\{ \frac{d\Delta a}{dt} \Delta b + \Delta a \frac{d\Delta b}{dt} \right\} - \Delta a \Delta b \sin \theta \frac{d\theta}{dt} = \left(\frac{\partial y_i}{\partial x_j} + \frac{\partial y_j}{\partial x_i} \right) dx_{a_j} dx_{b_i}$$

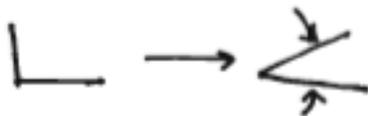
LET $dx_a \sim \Delta a$ BE ALIGNED ALONG Y-AXIS

$dx_b \sim \Delta b$ ALIGNED ALONG X-AXIS

$$\theta \sim \frac{\pi}{2}$$

AND

$$\begin{aligned} \frac{d\theta}{dt} &= - \left(\frac{\partial y_i}{\partial x_j} + \frac{\partial y_j}{\partial x_i} \right) \underbrace{\frac{dx_{a_j}}{da}}_{=1} \underbrace{\frac{dx_{b_i}}{db}}_{=1} \\ &= -2 \epsilon_{12} \quad (\text{SHEAR STRAIN RATE}) \end{aligned}$$

IF $\epsilon_{12} > 0$, 

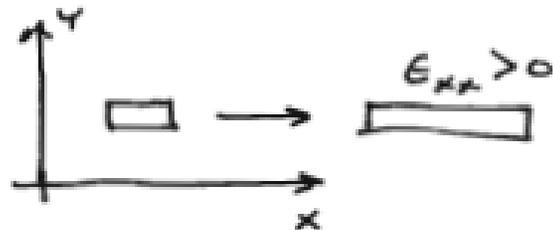
IF $\epsilon_{12} < 0$, 

Simple Example #1

$$\bar{u} = (\beta x, 0, 0)$$

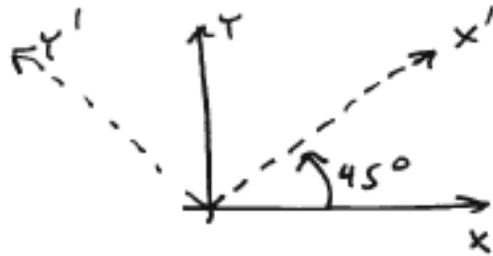
$$\bar{\epsilon} = \begin{pmatrix} \beta & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

A FLUID ELEMENT ALIGNED WITH \hat{x}
WILL STRETCH ($\beta > 0$) OR COMPRESS
(WHEN $\beta < 0$)



Simple Example #1

(from a different point of view)



$$x'_i = c_{ij} x_j$$

$$x_i = c_{ij}^+ x'_j$$

$$\bar{u}' = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \bar{u}$$

$$\bar{u} = (\beta x, 0, 0)$$

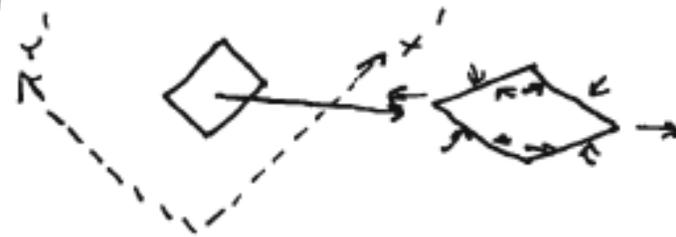
$$\bar{u}' = \left(\frac{1}{\sqrt{2}} \beta x, -\frac{1}{\sqrt{2}} \beta x, 0 \right)$$

$$x = (x' - y') \frac{1}{\sqrt{2}}$$

$$\bar{u}' = \left\{ \frac{\beta}{2} (x' - y'), -\frac{\beta}{2} (x' - y'), 0 \right\}$$

$$\bar{\epsilon}' = \begin{pmatrix} \beta/2 & -\beta/2 & 0 \\ -\beta/2 & \beta/2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\epsilon'_{ij} = c_{il} c_{jm} \epsilon_{lm}$$



Simple Example #2

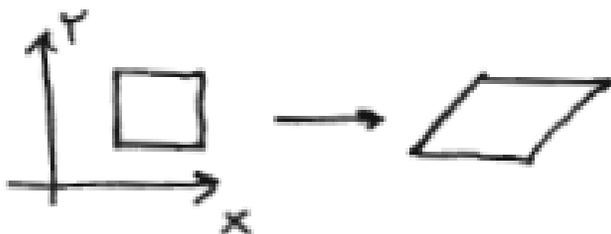
$$\bar{u} = (\beta \gamma, 0, 0)$$

$$\frac{\partial u_i}{\partial x_j} = \epsilon_{ij} + \mathcal{R}_{ij} = \epsilon_{ij} + \frac{1}{2} R_{ij}$$

$$\epsilon_{ij} = \begin{pmatrix} 0 & \beta/2 & 0 \\ \beta/2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \beta/2 & 0 \\ -\beta/2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

SYMMETRIC ANTI-SYMMETRIC

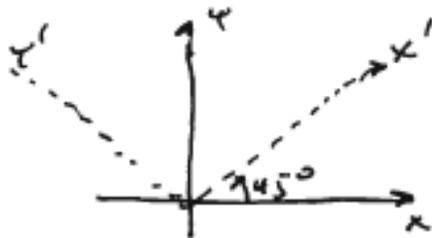
ROTATION CORRESPONDS TO $\bar{\omega} = (0, 0, -\beta)$



SUM OF ROTATION
AND SHEAR BENDING

Simple Example #2

(from a different point of view)

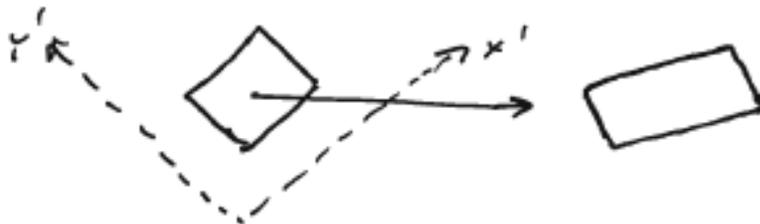


$$u_{x'}' = \frac{B}{2} (x' + y')$$

$$u_{y'}' = -\frac{B}{2} (x' - y')$$

$$\frac{2u_{i'}}{2x_j'} = \epsilon_{ij}' + \mathcal{R}_{ij}'$$

$$= \begin{pmatrix} B/2 & 0 & 0 \\ 0 & -B/2 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & B/2 & 0 \\ -B/2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$



A SUM OF ROTATION
AND EXPANSION!

Principal Axes for Symmetric Tensors

A SYMMETRIC TENSOR CAN BE MADE
DIAGONAL BY ORIENTING AXES ALONG
THE PRINCIPAL AXES.

LET $\bar{\bar{E}}$ BE SYMMETRIC.

EIGEN SYSTEM ANALYSIS IS USED TO FIND
PRINCIPAL AXES.

\bar{b} = EIGEN VECTORS ALONG PRINCIPAL AXES

$\bar{\bar{E}} \cdot \bar{b} = \lambda \bar{\delta} \cdot \bar{b}$ THERE WILL BE 3 EIGENVECTORS
AND THREE EIGENVALUES

\bar{b}_i λ_i

SOLUTION

$$\det | \bar{\bar{E}} - \lambda \bar{\delta} | = 0$$

$$\bar{\bar{E}} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \quad \bar{C} = \begin{pmatrix} b_{1x} & b_{2x} & b_{3x} \\ b_{1y} & b_{2y} & b_{3y} \\ b_{1z} & b_{2z} & b_{3z} \end{pmatrix}$$

Example Principal Axes #2

$$\bar{u} = (A/2, 0, 0) \text{ THIS IS NOT IRROTATIONAL.}$$

$$\nabla \times \bar{u} = (0, 0, -\beta) \neq 0$$

$$\bar{\epsilon} = \begin{pmatrix} 0 & D/2 & 0 \\ D/2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \bar{\mu} = \begin{pmatrix} 0 & D/2 & 0 \\ -A/2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\det |\bar{\epsilon} - \lambda \bar{\delta}| = \det \begin{vmatrix} -\lambda & D/2 & 0 \\ D/2 & -\lambda & 0 \\ 0 & 0 & -\lambda \end{vmatrix} = 0$$

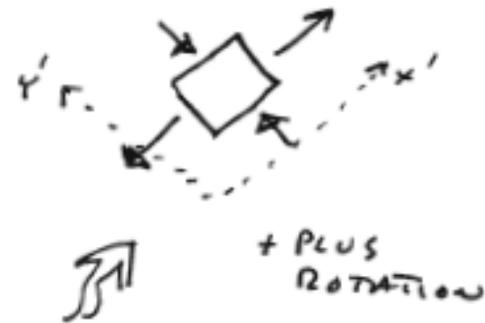
$$\text{or } \lambda = 0, \lambda = \pm D/2$$

$$\bar{b}_1 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \right)$$

$$\bar{b}_2 = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \right)$$

$$\bar{b}_3 = (0, 0, 1)$$

$$\bar{\epsilon}'_{ij} = C_{ie} C_{jm} \epsilon_{em} = \begin{pmatrix} D/2 & 0 & 0 \\ 0 & -D/2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$



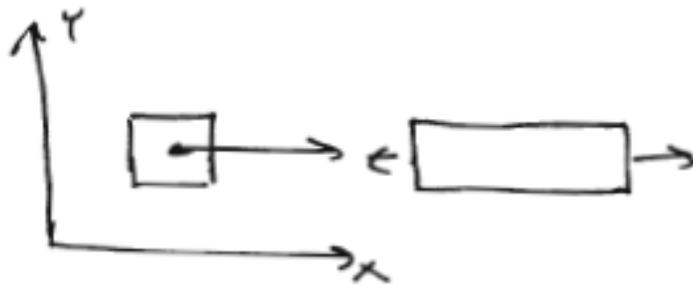
Example Principal Axes #1

$$\bar{u} = (\beta x, 0, 0) \quad \text{THIS IS IRROTATIONAL}$$

$$\nabla \times \bar{u} = 0$$

$$\bar{\epsilon} = \begin{pmatrix} \epsilon & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \bar{j} = 0$$

THESE AXES ALREADY ARE PRINCIPAL AXES
SINCE $\bar{\epsilon}$ IS DIAGONAL.



Summary:

Cauchy-Stokes Decomposition

ANY FLUID MOTION CAN BE RESOLVED
INTO

- TRANSLATION
- DILATATION / EXPANSION ALONG PRINCIPAL AXES
- ROTATION

$\nabla \times \bar{u} = 0$ IRRATIONALAL



$\nabla \times \bar{u} \neq 0$ ROTATIONAL

