Scalors, Vectors, & Tensors

- Scalars: mass density ($\rho$), temperature ($T$), concentration ($S$), charge density ($\rho_q$)
- Vectors: flow ($U$), force ($F$), magnetic field ($B$), current density ($J$), vorticity ($\Omega$)
- Tensors: stress ($\tau$), strain rate ($\varepsilon$), rotation ($R$), identity ($I$)

How to work and operate with tensors...
Vector Identities

Notation: $f, g$, are scalars; $\mathbf{A}, \mathbf{B}$, etc., are vectors; $\mathbf{T}$ is a tensor; $I$ is the unit dyad.

1. $\mathbf{A} \cdot \mathbf{B} \times \mathbf{C} = \mathbf{A} \times \mathbf{B} \cdot \mathbf{C} = \mathbf{B} \cdot \mathbf{C} \times \mathbf{A} = \mathbf{B} \times \mathbf{C} \cdot \mathbf{A} = \mathbf{C} \cdot \mathbf{A} \times \mathbf{B} = \mathbf{C} \times \mathbf{A} \cdot \mathbf{B}$
2. $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{C} \times \mathbf{B}) \times \mathbf{A} = (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C}$
3. $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) + \mathbf{B} \times (\mathbf{C} \times \mathbf{A}) + \mathbf{C} \times (\mathbf{A} \times \mathbf{B}) = 0$
4. $(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{C} \times \mathbf{D}) = (\mathbf{A} \cdot \mathbf{C})(\mathbf{B} \cdot \mathbf{D}) - (\mathbf{A} \cdot \mathbf{D})(\mathbf{B} \cdot \mathbf{C})$
5. $(\mathbf{A} \times \mathbf{B}) \times (\mathbf{C} \times \mathbf{D}) = (\mathbf{A} \times \mathbf{B})\mathbf{C} - (\mathbf{A} \times \mathbf{B})\mathbf{D}$
6. $\nabla (fg) = \nabla (gf) = f \nabla g + g \nabla f$
7. $\nabla \cdot (f \mathbf{A}) = f \nabla \cdot \mathbf{A} + \mathbf{A} \cdot \nabla f$
8. $\nabla \times (f \mathbf{A}) = f \nabla \times \mathbf{A} + \nabla f \times \mathbf{A}$
9. $\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot \nabla \times \mathbf{A} - \mathbf{A} \cdot \nabla \times \mathbf{B}$
10. $\nabla \times (\mathbf{A} \times \mathbf{B}) = \mathbf{A} (\nabla \cdot \mathbf{B}) - \mathbf{B} (\nabla \cdot \mathbf{A}) + (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B}$
11. $\mathbf{A} \times (\nabla \times \mathbf{B}) = (\nabla \mathbf{B}) \cdot \mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B}$
12. $\nabla (\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}) + (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A}$
13. $\nabla^2 f = \nabla \cdot \nabla f$
14. $\nabla^2 \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla \times \nabla \times \mathbf{A}$
15. $\nabla \times \nabla f = 0$
16. $\nabla \cdot \nabla \times \mathbf{A} = 0$

Green’s Theorem Variants

If $V$ is a volume enclosed by a surface $S$ and $d\mathbf{S} = n dS$, where $n$ is the unit normal outward from $V$,

27. $\int_V dV \nabla f = \int_S d\mathbf{S} f$
28. $\int_V dV \nabla \cdot \mathbf{A} = \int_S d\mathbf{S} \cdot \mathbf{A}$
29. $\int_V dV \nabla \cdot \mathbf{T} = \int_S d\mathbf{S} \cdot \mathbf{T}$
30. $\int_V dV \nabla \times \mathbf{A} = \int_S d\mathbf{S} \times \mathbf{A}$
31. $\int_V dV (f \nabla^2 g - g \nabla^2 f) = \int_S d\mathbf{S} \cdot (f \nabla g - g \nabla f)$
32. $\int_V dV (\mathbf{A} \cdot \nabla \times \nabla \times \mathbf{B} - \mathbf{B} \cdot \nabla \times \nabla \times \mathbf{A})$

$= \int_S d\mathbf{S} \cdot (\mathbf{B} \times \nabla \times \mathbf{A} - \mathbf{A} \times \nabla \times \mathbf{B})$
Stokes’ Theorem Variants

If $S$ is an open surface bounded by the contour $C$, of which the line element is $dl$,

(33) \[ \int_S dS \times \nabla f = \oint_C d l \]

(34) \[ \int_S dS \cdot \nabla \times A = \oint_C d l \cdot A \]

(35) \[ \int_S (dS \times \nabla) \times A = \oint_C d l \times A \]

(36) \[ \int_S dS \cdot (\nabla f \times \nabla g) = \oint_C f dg = - \oint_C g df \]

What is a Tensor?

**Definition:** A tensor (vector) is an entity that transforms like a tensor (vector) under coordinate transformation due to rotation. That is like a position vector, $\mathbf{x}$. In other words: a vector has a direction and magnitude ...
Rotation about the z-Axis

\[ \vec{x}' = (\cos \theta + y \sin \theta, -x \sin \theta + y \cos \theta, z) \]

or
\[ \vec{x}' = \begin{pmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \]

\[ x'_i = \sum_{c \in 1, \ldots, n} c_{ij} x_i \]

Definition: if \( u'_i = c_{ij} u_i \), then \( \vec{u} \) is a vector.

Rotation Matrix is Orthogonal

Transpose: \( (c_{ij})^T = c_{ji} \)

Note: \( (c_{ij})^T \cdot c_{ij} = \delta_{ij} \)

\[ \begin{pmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \cos^2 \theta + \sin^2 \theta & \cos \theta \sin \theta - \cos \theta \sin \theta & 0 \\ \cos \theta \sin \theta - \cos \theta \sin \theta & \cos^2 \theta + \sin^2 \theta & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \delta_{ij} \text{ Identity Tensor} \]

So \( x'_i = (c_{ij})^T x_j = c_{ji} x_i \)

And \( x'_j = c_{ij} x_i \)
Ch 2: Problem 4

4. Show that

\[ C \cdot C^T = C^T \cdot C = \delta, \]

where \( C \) is the direction cosine matrix and \( \delta \) is the matrix of the Kronecker delta. Any matrix obeying such a relationship is called an orthogonal matrix because it represents transformation of one set of orthogonal axes into another.

Vector Examples

- If \( \vec{a} \) is a vector, then \( \alpha \vec{a} \) is a vector if \( \alpha \) is a scalar.
- If \( \vec{a}, \vec{b}, \) and \( \vec{c} \) are vectors, then
  \[ \vec{a} + \vec{b} \text{ is a vector} \]
  \[ \vec{a} - \vec{b} \text{ is a vector} \]
  \[ \vec{a} + (\vec{b} + \vec{c}) = (\vec{a} + \vec{b}) + \vec{c} \text{ is a vector} \]
**Scalar Product**

\[ \overrightarrow{a} \cdot \overrightarrow{b} = \overrightarrow{b} \cdot \overrightarrow{a} = |\overrightarrow{a}| |\overrightarrow{b}| \cos \theta \]

**A scalar invariant under rotation.**

*IF \( \overrightarrow{a} \cdot \overrightarrow{b} = 0 \), THEN \( \overrightarrow{a} \) AND \( \overrightarrow{b} \) ARE MUTUALLY ORTHOGONAL \( (\theta = \pm 90^\circ) \)*

**Vector Product**

\[ \overrightarrow{c} = \overrightarrow{a} \times \overrightarrow{b} \]

\[ = |\overrightarrow{a}| |\overrightarrow{b}| \sin \theta \]

**Note:** Right handed definition!

\[ (\overrightarrow{a} \times \overrightarrow{b}) = \hat{e}_i \hat{e}_j \hat{e}_k \]

\[ \hat{e}_k = \begin{vmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \]

\( E_{ijk} \) is permutation tensor

\( = \) isotropic 3\( ^{rd} \) order tensor
Permutation Tensor

\[ \epsilon_{ijk} = \begin{cases} 
1 & \text{if } i,j,k \text{ are cyclic} \\
0 & \text{if any } i,j,k \text{ are repeated} \\
-1 & \text{if } i,j,k \text{ are anti-cyclic} 
\end{cases} \]

\[ \epsilon_{123} = \epsilon_{312} = \epsilon_{231} \]

\[ \epsilon_{123} = -\epsilon_{132} \]

\[ \epsilon_{113} = 0 \]

\( \epsilon_{ijk} \) is an isotropic, 3rd order tensor

\[ \epsilon_{kmm} = \epsilon_{imk} = \epsilon_{ikm} \epsilon_{ijk} \]

(\( \epsilon_{ijk} \) transforms to itself.)

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Co-Planar and Not

If \( \vec{a}, \vec{b}, \) and \( \vec{c} \) are co-planar, then

\[ \vec{c} = \lambda \vec{a} + \mu \vec{b} \]

\[ c_i = \lambda a_i + \mu b_i \]

or

\[ \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} \begin{pmatrix} \lambda \\ \mu \\ -1 \end{pmatrix} = 0 \]

This homogeneous equation has a non-trivial solution only if

\[ \text{DET} \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} = 0 \quad (\text{same as } \vec{c} \cdot (\vec{a} \times \vec{b})) \]

If \( \vec{a}, \vec{b}, \vec{c} \) are not co-planar, then

\[ \lambda \vec{a} + \mu \vec{b} + \nu \vec{c} = \vec{d} \]

And \( \vec{a}, \vec{b}, \vec{c} \) can serve as a “basis” to represent a vector \( \vec{d} \).
Triple Scalar Product

\[ \vec{a} \cdot (\vec{b} \times \vec{c}) = \epsilon_{ijk} a_i b_j c_k \]
\[ = \vec{b} \cdot (\vec{c} \times \vec{a}) \]
\[ = \vec{c} \cdot (\vec{a} \times \vec{b}) \]

\[ \approx \text{Volume of Parallel Epiped with sides } \vec{a}, \vec{b}, \vec{c} \]

If \( \vec{a} \cdot (\vec{b} \times \vec{c}) = 0 \), the vectors are co-planar.

\[ \epsilon_{ijk} \text{ Identity} \]

\[ \epsilon_{ijk} \epsilon_{klm} = \delta_{il} \delta_{jm} - \delta_{im} \delta_{jl} \]

Check:

- If any \( i = j, j = k, \) or \( j = m \), then \( \phi \)
- If both cyclic:
  \[ \epsilon_{ijk} \epsilon_{klm} = \begin{cases} 1 & \text{if } 123 \rightarrow 1 \rightarrow 2 \rightarrow 3 \rightarrow 1 \rightarrow 2 \rightarrow 3 \rightarrow 1 \\ 0 & \text{otherwise} \end{cases} \]
- If both anti-cyclic:
  \[ \epsilon_{ijk} \epsilon_{klm} = \begin{cases} 1 & \text{if } 123 \rightarrow 2 \rightarrow 3 \rightarrow 1 \rightarrow 2 \rightarrow 3 \rightarrow 1 \\ 0 & \text{otherwise} \end{cases} \]
- If one is anti-cyclic:
  \[ \epsilon_{ijk} \epsilon_{klm} = \begin{cases} 1 & \text{if } 123 \rightarrow 2 \rightarrow 3 \rightarrow 1 \rightarrow 2 \rightarrow 3 \rightarrow 1 \\ 0 & \text{otherwise} \end{cases} \]

Formula:

\[ (\vec{a} \times \vec{b}) \cdot (\vec{a} \times \vec{b}) = |\vec{a}|^2 |\vec{b}|^2 - (\vec{a} \cdot \vec{b})^2 \]

\[ \epsilon_{ijk} a_i b_j c_k \epsilon_{mnl} a_m b_n c_l = (\delta_{im} \delta_{jn} - \delta_{im} \delta_{j} \delta_{kn}) a_i b_j c_k \]
\[ = a_i^2 b_j^2 c_k^2 - (a_i b_j c_k)^2 \]
**Triple Vector Product**

\[
\overrightarrow{a} \times (\overrightarrow{b} \times \overrightarrow{c}) = \epsilon_{ijk} \epsilon_{kmn} \overrightarrow{a}_i \overrightarrow{b}_j \overrightarrow{c}_m \\
= b_i (a_j c_k) - c_i (a_j b_k) \\
= (\overrightarrow{a} \cdot \overrightarrow{c}) \overrightarrow{b} - (\overrightarrow{a} \cdot \overrightarrow{b}) \overrightarrow{c}
\]

If \( \overrightarrow{a} \) is a unit vector, and \( \overrightarrow{\alpha} \) a vector, then

\[
\overrightarrow{a} = (\overrightarrow{a} \cdot \overrightarrow{\alpha}) \overrightarrow{a} + \overrightarrow{\alpha} \times (\overrightarrow{a} \times \overrightarrow{\alpha})
\]

**Proof:**

\[
\overrightarrow{\alpha} \times (\overrightarrow{a} \times \overrightarrow{\alpha}) = \overrightarrow{\alpha} (\overrightarrow{a} \cdot \overrightarrow{\alpha}) - \overrightarrow{\alpha} (\overrightarrow{a} \cdot \overrightarrow{\alpha}) = \overrightarrow{0}
\]

Thus, \( \overrightarrow{\alpha} \) can be resolved into a component along \( \overrightarrow{\alpha} \) and one perpendicular to \( \overrightarrow{\alpha} \).

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**Rigid Rotation**

Rotation of \( \overrightarrow{x} \) about \( \overrightarrow{v} \)

\[
|d\overrightarrow{x}| = |x \sin \omega dt|
\]

\[
\frac{d\overrightarrow{\omega}}{dt} = \overrightarrow{\omega} = \overrightarrow{\omega} \times \overrightarrow{x} = \text{velocity of rotation}
\]

\( \overrightarrow{\omega} = \overrightarrow{\omega} \times \overrightarrow{x} \)
Cartesian Tensors

The indicial notation avoids all the problems mentioned in the preceding. The algebraic manipulations are especially simple. The ordering of terms is unnecessary because $A_{ij}B_{kl}$ means the same thing as $B_{ki}A_{ij}$. In this notation we deal with components only, which are scalars. Another major advantage is that one does not have to remember formulas except for the product $\varepsilon_{ijk}e_{klm}$, which is given by equation (2.19). The disadvantage of the indicial notation is that the physical meaning of a term becomes clear only after an examination of the indices. A second disadvantage is that the cross product involves the introduction of $\varepsilon_{ijk}$. This, however, can frequently be avoided by writing the $i$-component of the vector product of $u$ and $v$ as $(u \times v)_i$ using a mixture of boldface and indicial notations. In this book we shall use boldface, indicial and mixed notations in order to take advantage of each. As the reader might have guessed, the algebraic manipulations will be performed mostly in the indicial notation, sometimes using the comma notation.

Exercises

1. Using indicial notation, show that

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}. $$

(Hint: Call $\mathbf{d} = \mathbf{b} \times \mathbf{c}$. Then $(\mathbf{a} \times \mathbf{d})_m = \varepsilon_{pqm}a_p b_q = \varepsilon_{pqm}a_p \mathbf{c}_q$. Using equation (2.19), show that $(\mathbf{a} \times \mathbf{d})_m = (\mathbf{a} \cdot \mathbf{c})b_m - (\mathbf{a} \cdot \mathbf{b})c_m$.)

2. Show that the condition for the vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ to be coplanar is

$$\varepsilon_{ijk}a_{ib}c_{jk} = 0.$$ 

3. Prove the following relationships:

$$\delta_{ij} = 3$$

$$\varepsilon_{pqm}\varepsilon_{pqj} = 6$$

$$\varepsilon_{pqm} = 2\delta_{ij}.$$ 

4. Show that

$$C \cdot \mathbf{CT} = \mathbf{CT} \cdot \mathbf{C} = \delta,$$

where $\mathbf{C}$ is the direction cosine matrix and $\delta$ is the matrix of the Kronecker delta. Any matrix obeying such a relationship is called an orthogonal matrix because it represents transformation of one set of orthogonal axes into another.

5. Show that for a second-order tensor $\mathbf{A}$, the following three quantities are invariant under the rotation of axes:

$$I_1 = A_{ii}$$

$$I_2 = \begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix} + \begin{vmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{vmatrix} + \begin{vmatrix} A_{11} & A_{13} \\ A_{31} & A_{33} \end{vmatrix}$$

$$I_3 = \det(A_{ij}).$$

[Hint: Use the result of Exercise 4 and the transformation rule (2.12) to show that $I_1' = A_{ii}' = A_{ii} = I_1$. Then show that $A_{ij}A_{ji}$ and $A_{ij}A_{jk}A_{ki}$ are also invariants. In fact, all contracted scalars of the form $A_{ij}A_{jk} \cdots A_{mi}$ are invariants. Finally, verify that

$$I_2 = \frac{1}{2}[I_1^2 - A_{ij}A_{ji}]$$

$$I_3 = A_{ij}A_{jk}A_{ki} - I_1A_{ij}A_{ji} + I_2A_{ii}. $$

Because the right-hand sides are invariant, so are $I_2$ and $I_3$.]
**Ch 2: Problem 5**

\[ I_1 = A_{ii} \]
\[ I_2 = \begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix} + \begin{vmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{vmatrix} + \begin{vmatrix} A_{11} & A_{13} \\ A_{31} & A_{33} \end{vmatrix} \]
\[ I_3 = \det(A_{ij}). \]

**Chapter 2: Problem 5**

Kundu & Cohen, Fluid Dynamics

**Part a**
- \[ \text{Out[1]} = \text{Sum}[a[i, 1], (1, 2)] \]
- \[ \text{Out[2]} = a[1, 2] + a[2, 1] + a[1, 2] + a[2, 1] + a[2, 1] + a[1, 2] + a[2, 1] + a[1, 2] \]

**Part b**
- \[ \text{Out[1]} = \text{Sum}[a[i, j] a[j, k], (i, j, k)] \]

**Part c**
- \[ \text{MatrixForm} \]
- \[ \text{Det} - \text{rhs} = \text{Simplify} \]
- \[ \text{Out[3]} = 0 \]

\[ \text{Out[1]} = \text{Expand}[\text{Sum}[a[i, j] a[j, k] a[k, l], (i, j, k, l)] - \text{Out[2]} - \text{Out[3]}] \]


\[ \text{Out[3]} = 0 \]
Symmetric and Antisymmetric

1. If \( A_{ij} = A_{ji} \), then \( \overline{A} \) is symmetric.
2. If \( A_{ij} = -A_{ji} \), then \( \overline{A} \) is antisymmetric.

All tensors can be decomposed as...

\[
A_{ij} = \frac{1}{2} (A_{ij} + A_{ji}) + \frac{1}{2} (A_{ij} - A_{ji})
\]

Symmetric part

\[
\frac{1}{2} (A_{ij} + A_{ji})
\]

Antisymmetric part

\[
\frac{1}{2} (A_{ij} - A_{ji})
\]

Symmetric tensor has 6 independent values

\[
S_{ij} = \begin{pmatrix}
1 & 4 & 5 \\
4 & 2 & 6 \\
5 & 6 & 3
\end{pmatrix}
\]

Antisymmetric tensor has 3 independent values

\[
A_{ij} = \begin{pmatrix}
0 & -1 & 2 \\
1 & 0 & -3 \\
-2 & 3 & 0
\end{pmatrix}
\]

Antisymmetric Tensor

Every antisymmetric tensor is associated with a vector...

\[
\overline{R} = \begin{pmatrix}
\omega_1 & -\omega_2 & \omega_3 \\
\omega_2 & \omega_3 & -\omega_1 \\
-\omega_3 & \omega_1 & \omega_2
\end{pmatrix}
\]

\[
\overline{\omega} = \begin{pmatrix}
\omega_1 \\
\omega_2 \\
\omega_3
\end{pmatrix}
\]

\[
R_{ij} = -\epsilon_{ijk} \omega_k
\]

\[
\omega_k = -\frac{1}{2} \epsilon_{ijk} R_{ij}
\]

Example: Rotation around z-axis

\[
\overline{R} = \begin{pmatrix}
\cos \omega & 0 & -\omega \sin \omega \\
0 & 1 & 0 \\
\omega \sin \omega & 0 & \cos \omega
\end{pmatrix}
\]

\[
\overline{\omega} = \begin{pmatrix}
0 \\
0 \\
\omega
\end{pmatrix}
\]

\[
\vec{x}(t+\delta t) = \vec{x}(t) + \delta t \overline{R} \vec{x}(t)
\]
Time Derivative

IF \( A_{ij}(t) \) IS A TENSOR, THEN SO ARE ALL TIME DERIVATIVES...

\[
\frac{d}{dt} A_{ij}(t) = \sum_{k} \frac{\partial A_{ij}}{\partial x^k} \frac{dx^k}{dt}
\]

Also

\[
\ddot{u}(t) = \frac{d^2 x(t)}{dt^2}
\]

is a vector

\[
\frac{d}{dt} (\ddot{a} \cdot \ddot{b}) = \frac{d\ddot{a}}{dt} \cdot \ddot{b} + \ddot{a} \cdot \frac{d\ddot{b}}{dt}
\]

Proof: IF ACCELERATION IS PERPENDICULAR TO VELOCITY, THEN \( |\ddot{u}| \) IS A CONSTANT

\[
\frac{d}{dt} (u^2) = \frac{d}{dt} (\ddot{u} \cdot \ddot{u}) = 2 \ddot{u} \cdot \frac{d\ddot{u}}{dt} = 0
\]

Vector Fields and Trajectory Lines

\( \dddot{u}(x_1, x_2, x_3, t) \) IS A VECTOR FIELD.

AT EVERY POINT IN SPACE, THERE IS A VECTOR (AND IT MAY CHANGE IN TIME.)

ASSOCIATED WITH ANY VECTOR FIELD ARE "TRAJECTORIES" (LIKE STREAMLINES FOR FLUID FLOW) WHICH ARE THE CURVES EVERYWHERE TANGENT TO LOCAL FIELD.

EXAMPLE:

\[
\frac{d\dddot{x}}{ds} = \ddot{a}(x) \quad \text{OR} \quad \frac{dx_i}{ds} = a_i(x_1(s), x_2(s), x_3(s))
\]

WHERE \( s \) IS A LENGTH PARAMETER ALONG THE CURVE.

\[
\frac{dx_i}{ds} = ds \quad \text{FOR ALL} \quad i
\]
Gradient Operator (Scalar)

\[ \phi(x) \text{ is a scalar, then} \]

\[ \frac{2\psi}{2x_i} = \nabla \psi \text{ is a vector} \]

\[ \frac{2\psi}{2x_j} = \frac{2\psi}{2x_i} \frac{2x_i}{2x_j} = C_{ij} \frac{2\psi}{2x_i} \]

so \( \nabla \psi \) transforms like a vector.

Divergence of a Vector Field

\[ \nabla \cdot \vec{a} = \frac{2a_i}{2x_i} = \frac{2a_1}{2x_1} + \frac{2a_2}{2x_2} + \frac{2a_3}{2x_3} = a_{i,j} \]

\[ \text{means derivative} \]

\[ \text{NET FLOW INTO VOLUME AROUND } \gamma \text{ IS} \]

\[ \left[ a(y+\delta y) - a(y) \right] dxdydz \]

\[ = \frac{2a_y}{2y} dxdydz \text{ (+ 4 other sides)} \]

So \( \nabla \cdot \vec{a} = \lim_{\delta x \to 0} \frac{1}{\delta x} \iint \vec{a} \cdot dS \text{ UNIT SURFACE normal} \]

\[ \text{IF } \nabla \cdot \vec{a} = 0, \text{ THEN VECTOR FIELD IS SOLENOIDAL (LIKE } \vec{a}) \]
Laplacian

If \( \nabla \cdot \vec{a} = 0 \), then a potential function exists with \( \vec{a} = \nabla \psi \) and

\[
\nabla^2 \psi = \nabla \cdot \nabla \psi = \frac{\partial^2 \psi}{\partial x_i \partial x_i} = 0
\]

\[
\nabla \psi = \lim_{\Delta x \rightarrow 0} \frac{1}{\Delta x^3} \iint_{\Delta x \Delta y \Delta z} \nabla \psi \cdot \hat{n} dS
\]

= sum of “gradient flux” from volume

Green’s (or Gauss’) Theorem

Subdivide volume into a collection of infinitesimal volume elements. Sum,

\[
\iiint \nabla \cdot \vec{a} \, dV = \iiint \vec{a} \cdot \hat{n} \, dS
\]
Curl of a Vector Field

\[ \nabla \times \vec{a} = \vec{\epsilon}_i \nabla \times \frac{\partial a_k}{\partial x_j} \vec{e}^i = \left( \frac{\partial a_3}{\partial x_2} - \frac{\partial a_2}{\partial x_3} , \frac{\partial a_1}{\partial x_3} - \frac{\partial a_3}{\partial x_1} , \frac{\partial a_2}{\partial x_1} - \frac{\partial a_1}{\partial x_2} \right) \]

Explanation:

\[ \vec{c} = \text{tangent unit vector to curve} \]
\[ S = \text{length parameter along curve} \]

Find \( \oint \vec{a} \cdot \vec{E} \, ds \) as \( \Delta y, \Delta z \to 0 \)

\[ \text{Right + Left} = \left[ a_x(y+\Delta y) - a_x(y) \right] \, dt = \frac{\partial a_x}{\partial x} \, d\Delta x \, d\Delta y \]
\[ \text{Top + Bottom} = \left[ -a_y(\dot{x}+\Delta x) + a_y(\dot{x}) \right] \, d\Delta x = \frac{\partial a_y}{\partial x} \, d\Delta x \, d\Delta y \]

So
\[ \nabla \times \vec{a} = \lim_{\text{Area} \to 0} \frac{1}{\text{Area}} \oint \vec{a} \cdot \vec{E} \, ds \]

A vector field with \( \nabla \times \vec{a} = 0 \) is irrotational.

Stokes' Theorem

\[ \iint_S (\nabla \times \vec{a} \cdot \vec{m}) \, dS = \oint_C \vec{a} \cdot \vec{E} \, ds \]

(by summing infinitesimal surface elements)
Classification of Vector Fields

Irrotational Field

$$\nabla \times \overline{a} = 0$$

Let $$\overline{a} = \nabla \psi$$, $$\psi = \text{Potential}$$

Then $$\nabla \times \nabla \psi = 0 = \varepsilon_{ijk} \frac{\partial^2 \psi}{\partial x_i \partial x_j} \hat{e}_k$$
Solenoidal Field

\[ \nabla \cdot \vec{a} = 0 \]

\[ \text{LET } \vec{a} = \nabla \phi \times \nabla \phi \text{ OR } \]
\[ = \nabla \times (\phi \nabla \phi) \]

\[ \text{THEN, since } \]
\[ \nabla \cdot (\nabla \times \vec{a}) = \varepsilon_{ijk} \frac{\partial^2 a_j}{2x_k 2x_i} = 0 \]
\[ \text{or } \]
\[ \nabla \cdot (\nabla \phi \times \nabla \phi) = \varepsilon_{ijk} \left( \frac{2 \phi^2}{2x_k 2x_i} + \frac{2 \phi \frac{\partial^2 \phi}{2x_k 2x_i}}{2x_k 2x_i} \right) \]
\[ = 0 \]

WHERE \( \phi = \text{STREAM FUNCTION IF } \phi \text{ IS} \)
\[ \text{A SYMMETRY DIRECTION.} \]

Summary

- Vectors & tensors transform under coordinate rotation like position vector
- Vector operators: scalar product, vector product, triple scalar product, triple vector product
- Tensors: isotropic, symmetric, antisymmetric, orthogonal
- Calculus of vectors: derivative, gradient, divergence, curl
- Gauss’ & Stokes’ Theorems
- Classification of Vector Fields
- Next Lecture: Kinematics of fluids