APPH 4200
Physics of Fluids

Cartesian Tensors (Ch. 2)

1. Geometric Identities
2. Vector Calculus

Vladimir Zakharov

(Aug 1, 1939 –)

Mathematical theory of solitons – the Inverse Scattering Method (ISM). Integrable nonlinear wave equations, development of criteria for integrability. Asymptotic behavior of integrable systems. Reductions in integrable systems and their classification. The dressing method as a generator of new integrable equations and their exact solutions.

Vladimir Zakharov was born in Kazan, Russian SFSR in 1939, to Evgeniy and Elena Zakharov, an engineer and a schoolteacher. He studied at the Moscow Power Engineering Institute and at the Novosibirsk State University, where he received his specialist degree in physics in 1963 and his Candidate of Sciences degree in 1966, studying under Roald Sagdeev.

Awarded the Dirac Medal in 2003 for his work on turbulence.

Physics
spotlighting exceptional research

American Physical Society APS physics

Log in | Create Account (what's this?

Home About Browse APS Journals SEARCH

Focus: New Type of Turbulence on North Carolina's Coast

Published September 2, 2014 | Physics **7**, 89 (2014) | DOI: 10.1103/Physics.7.89

Analysis of sea surface height measurements during a storm in North Carolina's Outer Banks has led to the first observation of an unusual form of turbulence.

The chaotic fluid motion known as turbulence is notoriously difficult to describe precisely and to understand theoretically. In Physical Review Letters researchers report the first detection of a type of turbulence posited theoretically several decades ago, in which fluid flow can be modeled as a collection of the individual wave motions known as solitons. Such clean observations of predicted phenomena are rare in the study of turbulence, so experts see it as an important step. But some theoretical puzzles remain.

A soliton in water is a single "hump" that propagates without changing shape and can appear when the water depth is not much greater than the wave amplitude (see Focus Landmark from 2013). Solitons are solutions of the Korteweg-deVries (KdV) equation, a nonlinear equation that governs wave motion in this situation.

In general, turbulent flow consists of waves and eddies co-existing on many length scales, forming a constantly changing pattern that can't be represented as a sum of simpler motions. Over 40 years ago, Vladimir Zakharov, working at what is now the Budker Institute for Nuclear Physics in Novosibirsk, Russia, conceived of a novel kind of turbulence

Soliton Turbulence in Shallow Water Ocean Surface Waves
Andrea Costa, Alfred R. Osborne, Donald T. Resio, Silvia Alessio, Elisabetta Chrivì, Enrica Saggese, Katinka Bellomo, and Chuck E. Long
Phys. Rev. Lett. **113**, 108501 (2014)
Published September 2, 2014

+Enlarge image

Article Options

- Printable Version
- Share/Email This
- Export Citation (BibTeX)
- Export Citation (RIS)

Subject Areas

- **Fluid Dynamics**

Related Articles

More Fluid Dynamics

Turbulent Times
Synopsis | Jul 29, 2014

What Makes a Droplet Splash?
Focus | Jul 11, 2014

All Fluid Dynamics »

New in Physics

PRL **113,** 108501 (2014) PHYSICAL REVIEW LETTERS week ending 5 SEPTEMBER 2014

Soliton Turbulence in Shallow Water Ocean Surface Waves

Andrea Costa,[1,*] Alfred R. Osborne,[2,†] Donald T. Resio,[3] Silvia Alessio,[4] Elisabetta Chrivì,[4]
Enrica Saggese,[5] Katinka Bellomo,[6] and Chuck E. Long[7,‡]
[1]*Aix-Marseille Université, CNRS/INSU, IRD, MIO, UM 110, 13288 Marseille Cedex 9, France and Université de Toulon, CNRS/INSU, IRD, MIO, UM 110, 83957 La Garde, France*
[2]*Nonlinear Waves Research Corporation, Arlington, Virginia 22203, USA*
[3]*Department of Ocean Engineering, University of North Florida, Jacksonville, Florida 32224-7699, USA*
[4]*Dipartimento di Fisica, Universitá di Torino, Torino 10125, Italy*
[5]*Université Nice Sophia Antipolis, LPMC, UMR 7336, 06100 Nice, France*
[6]*Rosenstiel School of Marine and Atmospheric Science, University of Miami, Miami, Florida 33149, USA*
[7]*U.S. Army Engineer Research and Development Center, Vicksburg, Mississippi, USA*
(Received 3 March 2014; revised manuscript received 30 June 2014; published 2 September 2014)

We analyze shallow water wind waves in Currituck Sound, North Carolina and experimentally confirm, for the first time, the presence of soliton turbulence in ocean waves. Soliton turbulence is an exotic form of nonlinear wave motion where low frequency energy may also be viewed as a dense soliton gas, described theoretically by the soliton limit of the Korteweg–deVries equation, a completely integrable soliton system: Hence the phrase "soliton turbulence" is synonymous with "integrable soliton turbulence." For periodic-quasiperiodic boundary conditions the ergodic solutions of Korteweg–deVries are exactly solvable by finite gap theory (FGT), the basis of our data analysis. We find that large amplitude measured wave trains near the energetic peak of a storm have low frequency power spectra that behave as $\sim\omega^{-1}$. We use the linear Fourier transform to estimate this power law from the power spectrum and to filter densely packed soliton wave trains from the data. We apply FGT to determine the soliton spectrum and find that the low frequency $\sim\omega^{-1}$ region is soliton dominated. The solitons have random FGT phases, a soliton random phase approximation, which supports our interpretation of the data as soliton turbulence. From the probability density of the solitons we are able to demonstrate that the solitons are dense in time and highly non-Gaussian.

DOI: 10.1103/PhysRevLett.113.108501 PACS numbers: 92.10.Hm, 92.10.Lq, 92.10.Sx

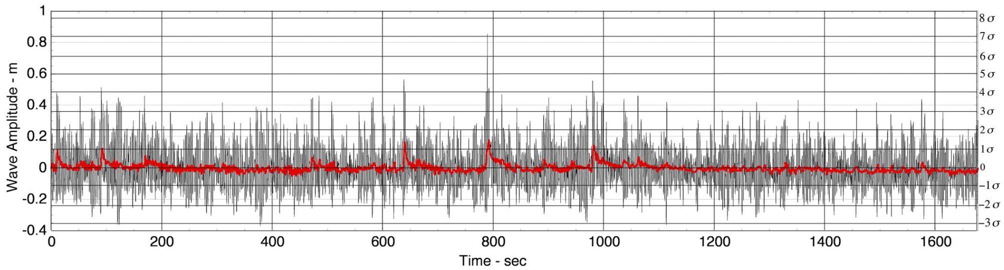


FIG. 1 (color online). Measured surface wave time series of 8192 points (27.96 min, sampling interval 0.2048 s, black curve) from Currituck Sound beginning at 21:00 h on 4 February 2002. The significant wave height was 0.52 m in a depth of 2.63 m. The red curve is the low frequency soliton signal obtained by low pass filtering the (black) measured time series.

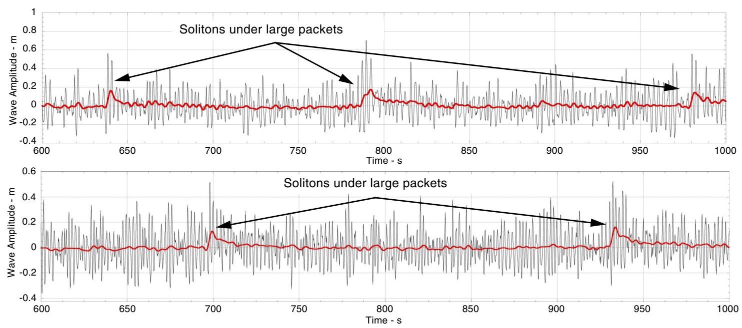


FIG. 6 (color online). Two measured wave trains (black) together with the underlying soliton trains obtained by low pass filtering of the data (red). The results show how large solitons tend to occur under large packets.

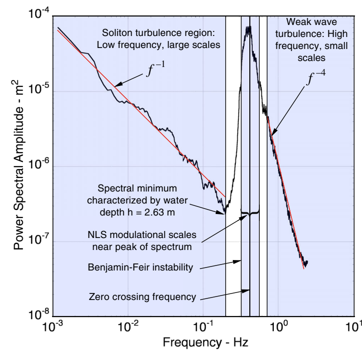


FIG. 2 (color online). Power spectrum of the measured time series in Fig. 1. Validity intervals for KdV ($f < 0.22$ Hz) and NLS (0.34 Hz $< f < 0.56$ Hz) are shown. Exact power laws (red lines) are shown in the low-frequency soliton turbulent region ($\sim f^{-1}$) and high-frequency cascade region ($\sim f^{-4}$).

Surface Tension and Contact Angle

In 1804, Young developed the theory of capillary phenomena on the principle of surface tension. He also observed the constancy of the angle of contact of a liquid surface with a solid, and showed how from these two principles to deduce the phenomena of capillary action. In 1805, Pierre-Simon Laplace, the French philosopher, discovered the significance of meniscus radii with respect to capillary action.

In 1830, Carl Friedrich Gauss, the German mathematician, unified the work of these two scientists to derive the Young–Laplace equation, the formula that describes the capillary pressure difference sustained across the interface between two static fluids.

Thomas Young
1773 – 1829

https://en.wikipedia.org/wiki/Thomas_Young_(scientist)

PHYSICAL REVIEW LETTERS **121,** 048002 (2018)

Editors' Suggestion

Detaching Microparticles from a Liquid Surface

Frank Schellenberger,[1] Periklis Papadopoulos,[2] Michael Kappl,[1] Stefan A. L. Weber,[1,3] Doris Vollmer,[1] and Hans-Jürgen Butt[1,*]

[1]*Max Planck Institute for Polymer Research, Ackermannweg 10, 55128 Mainz, Germany*
[2]*University of Ioannina, Department of Physics, P.O. Box 1186, 45110 Ioannina, Greece*
[3]*Johannes Gutenberg University, Department of Physics, Staudingerweg 10, 55128 Mainz, Germany*

(Received 23 February 2018; revised manuscript received 30 May 2018; published 27 July 2018)

The work required to detach microparticles from fluid interfaces depends on the shape of the liquid meniscus. However, measuring the capillary force on a single microparticle and simultaneously imaging the shape of the liquid meniscus has not yet been accomplished. To correlate force and shape, we combined a laser scanning confocal microscope with a colloidal probe setup. While moving a hydrophobic microsphere (radius 5–10 μm) in and out of a 2–5 μm thick glycerol film, we simultaneously measured the force and imaged the shape of the liquid meniscus. In this way we verified the fundamental equations [D. F. James, J. Fluid Mech. **63**, 657 (1974); A. D. Scheludko, A. D. Nikolov, Colloid Polymer Sci. **253**, 396 (1975)] that describe the adhesion of particles in flotation, deinking of paper, the stability of Pickering emulsions and particle-stabilized foams. Comparing experimental results with theory showed, however, that the receding contact angle has to be applied, which can be much lower than the static contact angle obtained right after jump in of the particle.

DOI: 10.1103/PhysRevLett.121.048002

$$F = 2\pi\gamma R \sin\beta \sin\alpha = 2\pi\gamma R \sin\beta \sin(\Theta + \beta). \quad (1)$$

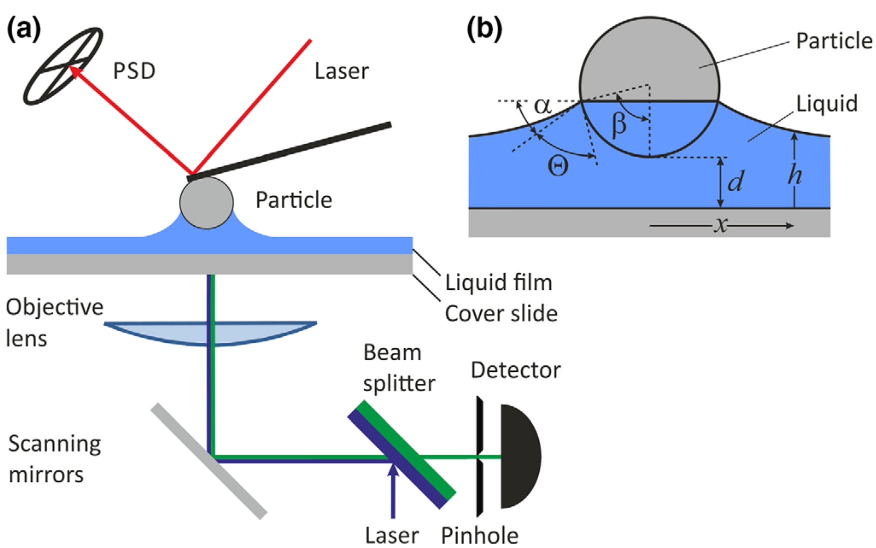


FIG. 1. (a) Schematic setup combining a colloidal probe setup with a confocal microscope. Position sensitive device (PSD). (b) Schematic of a particle in contact with a liquid film on a solid substrate.

Scalars, Vectors, &Tensors

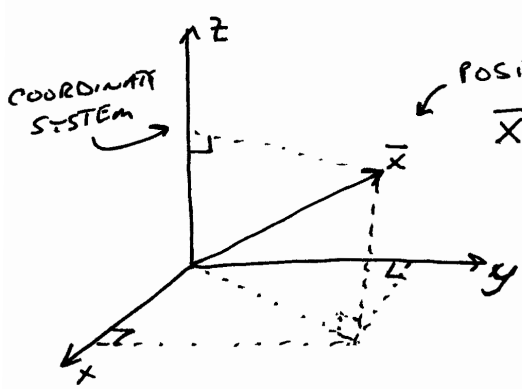
- Scalars: mass density (ρ), temperature (T), concentration (S), charge density (ρ_q)
- Vectors: flow (U), force (F), magnetic field (B), current density (J), vorticity (Ω)
- Tensors: stress (τ), strain rate (ε), rotation (R), identity (I)

How to work and operate with tensors…

What is a Tensor?

DEFINITION: A TENSOR (VECTOR) IS A ENTITY THAT TRANSFORMS LIKE A TENSOR (VECTOR) UNDER COORDINATE TRANSFORMATION DUE TO ROTATION. — THAT IS LIKE A POSITION VECTOR, $\overline{x}$.

In other words: a vector has a direction and magnitude …

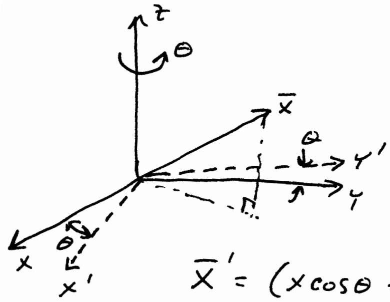


POSITION VECTOR

$$\overline{x} = (x, y, z) = (x_1, x_2, x_3) = x_i$$

$i = (1, 2, 3)$

Rotation about the z-Axis



$$\bar{X}' = (X\cos\theta + Y\sin\theta,\ -X\sin\theta + Y\cos\theta,\ z)$$

OR $$\bar{X}' = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$$

$$X'_j = C_{ij} X_i \quad \leftarrow \text{CARTESIAN SUMMATION CONVENTION}$$

$$= \sum_{i=1,2,3} C_{ij} X_i$$

DEFINITION: IF $U'_j = C_{ij} U_i$, THEN $\bar{U}$ IS A VECTOR.

Ch 2: Problem 4

4. Show that

$$\mathbf{C} \cdot \mathbf{C}^{\mathrm{T}} = \mathbf{C}^{\mathrm{T}} \cdot \mathbf{C} = \boldsymbol{\delta},$$

where **C** is the direction cosine matrix and **δ** is the matrix of the Kronecker delta. Any matrix obeying such a relationship is called an *orthogonal matrix* because it represents transformation of one set of orthogonal axes into another.

Rotation Matrix is Orthogonal

TRANSPOSE: $(C_{ij})^T = C_{ji}$

NOTE: $(C_{ij})^T \cdot C_{ij} = \delta_{ij}$

$$\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \cos^2\theta + \sin^2\theta & \cos\theta\sin\theta - \cos\theta\sin\theta & 0 \\ \cos\theta\sin\theta - \cos\theta\sin\theta & \cos^2\theta + \sin^2\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \delta_{ij}$$ IDENTITY TENSOR

SO $x_i = (C_{ij})^T x'_j = C_{ji} x'_j$

AND $x'_j = C_{ij} x_i$

Tensors

$$A_{mn} = C_{im} C_{jn} A_{ij}$$

DEFINES A TENSOR.

WHILE 3 COMPONENTS / VALUES ARE NEEDED TO DEFINE A VECTOR, 9 COMPONENTS / VALUES ARE NEEDED TO DEFINE A 2ND-ORDER TENSOR.

ISOTROPIC 2ND-ORDER TENSOR IS

$$\delta_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$ SAME IN ALL COORDINATE SYSTEMS

$$\begin{aligned} \delta_{\ell m} &= C_{i\ell} C_{jm} \delta_{ij} \\ &= C^T_{\ell i} \delta_{ij} C_{jm} \\ &= C^T_{\ell j} C_{jm} \\ &= \delta_{\ell m} \end{aligned}$$

Vector Identities

Notation: f, g, are scalars; $\mathbf{A}$, $\mathbf{B}$, etc., are vectors; T is a tensor; I is the unit dyad.

(1) $\mathbf{A}\cdot\mathbf{B}\times\mathbf{C} = \mathbf{A}\times\mathbf{B}\cdot\mathbf{C} = \mathbf{B}\cdot\mathbf{C}\times\mathbf{A} = \mathbf{B}\times\mathbf{C}\cdot\mathbf{A} = \mathbf{C}\cdot\mathbf{A}\times\mathbf{B} = \mathbf{C}\times\mathbf{A}\cdot\mathbf{B}$

(2) $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = (\mathbf{C}\times\mathbf{B})\times\mathbf{A} = (\mathbf{A}\cdot\mathbf{C})\mathbf{B} - (\mathbf{A}\cdot\mathbf{B})\mathbf{C}$

(3) $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) + \mathbf{B}\times(\mathbf{C}\times\mathbf{A}) + \mathbf{C}\times(\mathbf{A}\times\mathbf{B}) = 0$

(4) $(\mathbf{A}\times\mathbf{B})\cdot(\mathbf{C}\times\mathbf{D}) = (\mathbf{A}\cdot\mathbf{C})(\mathbf{B}\cdot\mathbf{D}) - (\mathbf{A}\cdot\mathbf{D})(\mathbf{B}\cdot\mathbf{C})$

(5) $(\mathbf{A}\times\mathbf{B})\times(\mathbf{C}\times\mathbf{D}) = (\mathbf{A}\times\mathbf{B}\cdot\mathbf{D})\mathbf{C} - (\mathbf{A}\times\mathbf{B}\cdot\mathbf{C})\mathbf{D}$

(6) $\nabla(fg) = \nabla(gf) = f\nabla g + g\nabla f$

(7) $\nabla\cdot(f\mathbf{A}) = f\nabla\cdot\mathbf{A} + \mathbf{A}\cdot\nabla f$

(8) $\nabla\times(f\mathbf{A}) = f\nabla\times\mathbf{A} + \nabla f\times\mathbf{A}$

(9) $\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot\nabla\times\mathbf{A} - \mathbf{A}\cdot\nabla\times\mathbf{B}$

(10) $\nabla\times(\mathbf{A}\times\mathbf{B}) = \mathbf{A}(\nabla\cdot\mathbf{B}) - \mathbf{B}(\nabla\cdot\mathbf{A}) + (\mathbf{B}\cdot\nabla)\mathbf{A} - (\mathbf{A}\cdot\nabla)\mathbf{B}$

(11) $\mathbf{A}\times(\nabla\times\mathbf{B}) = (\nabla\mathbf{B})\cdot\mathbf{A} - (\mathbf{A}\cdot\nabla)\mathbf{B}$

(12) $\nabla(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A}) + (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A}$

(13) $\nabla^2 f = \nabla\cdot\nabla f$

(14) $\nabla^2\mathbf{A} = \nabla(\nabla\cdot\mathbf{A}) - \nabla\times\nabla\times\mathbf{A}$

(15) $\nabla\times\nabla f = 0$

(16) $\nabla\cdot\nabla\times\mathbf{A} = 0$

Vector Examples

- IF $\bar{a}$ IS A VECTOR, THEN $\alpha\bar{a}$ IS A VECTOR IF α IS A SCALAR.
- IF $\bar{a}$, $\bar{b}$, AND $\bar{c}$ ARE VECTORS, THEN

  $\bar{a} + \bar{b}$ IS A VECTOR

  $\bar{a} - \bar{b}$ IS A VECTOR

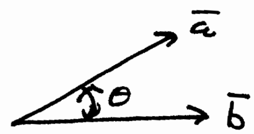
  $\bar{a} + (\bar{b} + \bar{c}) = (\bar{a} + \bar{b}) + \bar{c}$ IS A VECTOR

Scalar Product

$$\bar{a}\cdot\bar{b} = \bar{b}\cdot\bar{a} = |\bar{a}||\bar{b}|\cos\theta$$

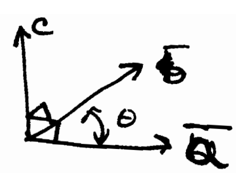
= A SCALAR INVARIANT UNDER ROTATION.

IF $\bar{a}\cdot\bar{b}=0$, THEN $\bar{a}$ AND $\bar{b}$ ARE MUTUALLY ORTHOGONAL $(\theta = \pm 90^\circ)$



Vector Product

$$\bar{c} = \bar{a}\times\bar{b}$$
$$= |\bar{a}||\bar{b}|\sin\theta$$



NOTE: RIGHT HANDED DEFINITION!

$$(\bar{a}\times\bar{b})_k = \epsilon_{ijk}\, a_i\, b_j\, \hat{e}_k = \begin{vmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

ϵ_{ijk} = PERMUTATION TENSOR

= ISOTROPIC 3RD-ORDER TENSOR

Permutation Tensor

$$\epsilon_{ijk} = \begin{cases} 1 & \text{IF } ijk \text{ ARE CYCLIC} \\ 0 & \text{IF ANY } ijk \text{ ARE REPEATED} \\ -1 & \text{IF } ijk \text{ ARE ANTI-CYCLIC} \end{cases}$$

$$\epsilon_{123} = \epsilon_{312} = \epsilon_{231}$$

$$\epsilon_{123} = -\epsilon_{132}$$

$$\epsilon_{113} = 0$$

ϵ_{ijk} IS AN ISOTROPIC, 3RD-ORDER TENSOR

$$\epsilon_{lmn} = C_{il}\, C_{jm}\, C_{kn}\, \epsilon_{ijk}$$

(ϵ_{ijk} TRANSFORMS TO ITSELF.)

ϵ_{ijk} Identity

$$\epsilon_{ijk}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$$

CHECK:

- IF ANY $i=j$, $j=k$, OR $l=m$, THEN ϕ
- IF BOTH CYCLIC

  $ijk \quad klm$

  $123 \quad 312 \qquad$ THEN 1
- IF BOTH ANTI-CYCLIC

  $ijk \quad klm$

  $132 \quad 213 \qquad$ THEN 1
- IF ONE IS ANTI-CYCLIC

  $ijk \quad klm$

  $123 \quad 321 \qquad$ THEN -1

PROVE: $(\bar{a}\times\bar{b})\cdot(\bar{a}\times\bar{b}) = |\bar{a}|^2|\bar{b}|^2 - (\bar{a}\cdot\bar{b})^2$

$$\epsilon_{ijk}\, a_i b_j\, \epsilon_{lmk}\, a_l b_m = (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\, a_i b_j a_l b_m$$

$$= a^2 b^2 - (a\cdot b)^2$$

Gradient Operator (Scalar)

$\varphi(x)$ is a scalar, then

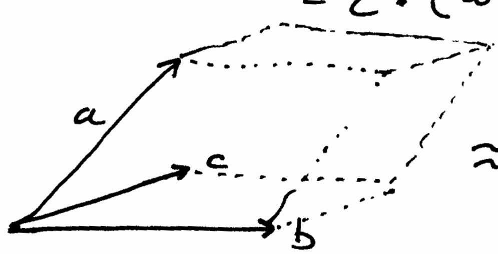
$\frac{\partial \varphi}{\partial x_i} = \nabla \varphi$ is a vector

$$\frac{\partial \varphi}{\partial x_j'} = \frac{\partial \varphi}{\partial x_i} \frac{\partial x_i}{\partial x_j'} = C_{ij} \frac{\partial \varphi}{\partial x_i}$$

so $\bar{\nabla} \varphi$ transforms like a vector.

Triple Scalar Product

$$\bar{a} \cdot (\bar{b} \times \bar{c}) = \epsilon_{ijk} a_i b_j c_k$$
$$= \bar{b} \cdot (\bar{c} \times \bar{a})$$
$$= \bar{c} \cdot (\bar{a} \times \bar{b})$$



$\approx$ Volume of parallelepiped with sides $\bar{a}, \bar{b}, \bar{c}$

If $\bar{a} \cdot (\bar{b} \times \bar{c}) = 0$, the vectors are co-planar

Co-Planar and Not

IF $\bar a$, $\bar b$, AND $\bar c$ ARE CO-PLANAR, THEN

$$\bar c = \alpha \bar a + \beta \bar b$$

$$c_i = \alpha a_i + \beta b_i$$

OR

$$\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} \cdot \begin{pmatrix} \alpha \\ \beta \\ -1 \end{pmatrix} = 0$$

THIS HOMOGENEOUS EQUATION HAS A NON-TRIVIAL SOLUTION ONLY IF

$$\mathrm{DET} \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0 \qquad \left(\text{SAME AS } \bar c \cdot (\bar a \times \bar b)\right)$$

IF $\bar a$, $\bar b$, $\bar c$ ARE <u>NOT</u> CO-PLANAR, THEN

$$\alpha \bar a + \beta \bar b + \gamma \bar c = \bar d$$

AND $\bar a$, $\bar b$, $\bar c$ CAN SERVE AS A "BASIS" TO REPRESENT A VECTOR ($\bar d$).

Triple Vector Product

$$\bar a \times (\bar b \times \bar c) = \epsilon_{ijk} \epsilon_{k\ell m} a_j b_\ell c_m$$

$$= b_i (a_j c_j) - c_i (a_j b_j)$$

$$= (\bar a \cdot \bar c) \bar b - (\bar a \cdot \bar b) \bar c$$

IF $\hat n$ IS A UNIT VECTOR, AND $\bar a$ A VECTOR, THEN

$$\bar a = (\bar a \cdot \hat n) \hat n + \hat n \times (\bar a \times \hat n)$$

PROOF: $\hat n \times (\bar a \times \hat n) = \bar a \underbrace{(\hat n \cdot \hat n)}_{=1} - \hat n (\hat n \cdot \bar a)$

THUS, $\bar a$ CAN BE RESOLVED INTO A COMPONENT ALONG $\hat n$ AND ONE PERPENDICULAR TO $\hat n$.

In class Problem

Ch. 2 Question 10

PROVE $\nabla \cdot \nabla \times \bar{u} = 0$ FOR ANY VECTOR FIELD.

Ch. 2 Question 10

PROVE $\nabla \cdot \nabla \times \bar{u} = 0$ FOR ANY VECTOR FIELD.

METHOD #1

$$\nabla \cdot \nabla \times \bar{u} = \sum_{i,j,k} \epsilon_{ijk} \frac{\partial^2 u_j}{\partial x_i \partial x_k}$$

BUT $\epsilon_{ijk} = -\epsilon_{ikj}$ WHILE $\frac{\partial^2 u_j}{\partial x_i \partial x_k} = \frac{\partial^2 u_j}{\partial x_k \partial x_i}$

SO THE SUM MUST VANISH

Ch. 2 Question 10

PROVE $\nabla \cdot \nabla \times \bar{u} = 0$ FOR ANY VECTOR FIELD.

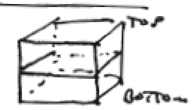
METHOD #1

$$\nabla \cdot \nabla \times \bar{u} = \sum_{i,j,k} \epsilon_{ijk} \frac{\partial^2 u_j}{\partial x_i \partial x_k}$$

BUT $\epsilon_{ijk} = -\epsilon_{ikj}$ WHILE $\frac{\partial^2 u_j}{\partial x_i \partial x_k} = \frac{\partial^2 u_j}{\partial x_k \partial x_i}$

SO THE SUM MUST VANISH

METHOD #2



$$\iiint dV\, \nabla \cdot \nabla \times \bar{u} = \iint \nabla \times \bar{u} \cdot d\bar{S}$$

$$= \iint_{TOP} \nabla \times \bar{u} \cdot dS + \iint_{BOTTOM} \nabla \times \bar{u} \cdot d\bar{S}$$

$$= \oint_{TOP} d\bar{l} \cdot \bar{u} + \oint_{BOT} d\bar{l} \cdot \bar{u}$$

BUT THE SENSE OF THESE TWO LINE INTEGRALS ARE OPPOSITE, SO THE SUM MUST VANISH.

Ch. 2 Question 11

PROVE $\nabla \times \nabla \varphi = 0$ FOR ANY WELL-BEHAVED FUNCTION φ.

Ch. 2 Question 11

PROVE $\nabla \times \nabla \varphi = 0$ FOR ANY WELL-BEHAVED FUNCTION φ.

METHOD #1

$$(\nabla \times \nabla \varphi)_i = \sum_{j,k} \epsilon_{ijk} \frac{\partial^2 \varphi}{\partial x_j \partial x_k}$$

BUT $\epsilon_{ijk} = -\epsilon_{ikj}$ AND $\frac{\partial^2 \varphi}{\partial x_j \partial x_k}$ IS SYMMETRIC;

SO SUM MUST VANISH.

Ch. 2 Question 11

PROVE $\nabla \times \nabla \varphi = 0$ FOR ANY WELL-BEHAVED FUNCTION φ.

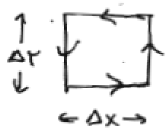
METHOD #1

$$(\nabla \times \nabla \varphi)_i = \sum_{j,k} \epsilon_{ijk} \frac{\partial^2 \varphi}{\partial x_j \partial x_k}$$

BUT $\epsilon_{ijk} = -\epsilon_{ikj}$ AND $\dfrac{\partial^2 \varphi}{\partial x_j \partial x_k}$ IS SYMMETRIC;

SO SUM MUST VANISH.

METHOD #2



$$\iint d\mathbf{s} \cdot \nabla \times \nabla \varphi = \oint_{\partial S} d\bar{l} \cdot \nabla \varphi$$

$$= \Delta x \left(\frac{\partial \varphi}{\partial x}\Big|_y - \frac{\partial \varphi}{\partial x}\Big|_{y+\Delta y} \right)$$

$$+ \Delta y \left(\frac{\partial \varphi}{\partial y}\Big|_{x+\Delta x} - \frac{\partial \varphi}{\partial y}\Big|_x \right)$$

$$= -\Delta x \Delta y \frac{\partial^2 \varphi}{\partial x \partial y} + \Delta x \Delta y \frac{\partial^2 \varphi}{\partial y \partial x}$$

$$= 0$$

SINCE LINE SEGMENTS ADDED IN OPPOSITION.

Ch 2: Problem 5

5. Show that for a second-order tensor **A**, the following three quantities are invariant under the rotation of axes:

$$I_1 = A_{ii}$$

$$I_2 = \begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix} + \begin{vmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{vmatrix} + \begin{vmatrix} A_{11} & A_{13} \\ A_{31} & A_{33} \end{vmatrix}$$

$$I_3 = \det(A_{ij}).$$

[*Hint*: Use the result of Exercise 4 and the transformation rule (2.12) to show that $I_1' = A_{ii}' = A_{ii} = I_1$. Then show that $A_{ij}A_{ji}$ and $A_{ij}A_{jk}A_{ki}$ are also invariants. In fact, *all* contracted scalars of the form $A_{ij}A_{jk}\cdots A_{mi}$ are invariants. Finally, verify that

$$I_2 = \tfrac{1}{2}[I_1^2 - A_{ij}A_{ji}]$$

$$I_3 = A_{ij}A_{jk}A_{ki} - I_1 A_{ij}A_{ji} + I_2 A_{ii}.$$

Because the right-hand sides are invariant, so are I_2 and I_3.]

Ch 2: Problem 5

$$I_1 = A_{ii}$$

$$I_2 = \begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix} + \begin{vmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{vmatrix} + \begin{vmatrix} A_{11} & A_{13} \\ A_{31} & A_{33} \end{vmatrix}$$

$$I_3 = \det(A_{ij}).$$

Chapter 2 : Problem 5

Kundu & Cohen, *Fluid Dynamics*

- **Part a**

```
In[1]:= i1 = Sum[a[i, i], {i, 3}]

Out[1]= a[1, 1] + a[2, 2] + a[3, 3]
```

- **Part b**

```
In[2]:= Expand[Sum[a[i, j] a[j, i], {i, 3}, {j, 3}]]

Out[2]= a[1, 1]^2 + 2 a[1, 2] a[2, 1] + a[2, 2]^2 + 2 a[1, 3] a[3, 1] + 2 a[2, 3] a[3, 2] + a[3, 3]^2

In[3]:= i2 = Expand[(i1^2 - Sum[a[i, j] a[j, i], {i, 3}, {j, 3}]) / 2]

Out[3]= -a[1, 2] a[2, 1] + a[1, 1] a[2, 2] - a[1, 3] a[3, 1] - a[2, 3] a[3, 2] + a[1, 1] a[3, 3] + a[2, 2] a[3, 3]
```

- **Part c**

```
In[4]:= Array[a, {3, 3}] // MatrixForm

Out[4]//MatrixForm=
 ( a[1, 1]  a[1, 2]  a[1, 3] )
 ( a[2, 1]  a[2, 2]  a[2, 3] )
 ( a[3, 1]  a[3, 2]  a[3, 3] )

In[5]:= detA = Det[Array[a, {3, 3}]]

Out[5]= -a[1, 3] a[2, 2] a[3, 1] + a[1, 2] a[2, 3] a[3, 1] + a[1, 3] a[2, 1] a[3, 2] -
  a[1, 1] a[2, 3] a[3, 2] - a[1, 2] a[2, 1] a[3, 3] + a[1, 1] a[2, 2] a[3, 3]

In[6]:= Expand[Sum[a[i, j] a[j, k] a[k, i], {i, 3}, {j, 3}, {k, 3}]]

Out[6]= a[1, 1]^3 + 3 a[1, 1] a[1, 2] a[2, 1] + 3 a[1, 2] a[2, 1] a[2, 2] + a[2, 2]^3 +
  3 a[1, 1] a[1, 3] a[3, 1] + 3 a[1, 2] a[2, 3] a[3, 1] + 3 a[1, 3] a[2, 1] a[3, 2] +
  3 a[2, 2] a[2, 3] a[3, 2] + 3 a[1, 3] a[3, 1] a[3, 3] + 3 a[2, 3] a[3, 2] a[3, 3] + a[3, 3]^3

In[7]:= Expand[i1 Sum[a[i, j] a[j, i], {i, 3}, {j, 3}]]

Out[7]= a[1, 1]^3 + 2 a[1, 1] a[1, 2] a[2, 1] + a[1, 1]^2 a[2, 2] + 2 a[1, 2] a[2, 1] a[2, 2] + a[1, 1] a[2, 2]^2 +
  a[2, 2]^3 + 2 a[1, 1] a[1, 3] a[3, 1] + 2 a[1, 3] a[2, 2] a[3, 1] + 2 a[1, 1] a[2, 3] a[3, 2] +
  2 a[2, 2] a[2, 3] a[3, 2] + a[1, 1]^2 a[3, 3] + 2 a[1, 2] a[2, 1] a[3, 3] + a[2, 2]^2 a[3, 3] +
  2 a[1, 3] a[3, 1] a[3, 3] + 2 a[2, 3] a[3, 2] a[3, 3] + a[1, 1] a[3, 3]^2 + a[2, 2] a[3, 3]^2 + a[3, 3]^3

In[8]:= Expand[ i2 i1]

Out[8]= -a[1, 1] a[1, 2] a[2, 1] + a[1, 1]^2 a[2, 2] - a[1, 2] a[2, 1] a[2, 2] + a[1, 1] a[2, 2]^2 -
  a[1, 1] a[1, 3] a[3, 1] - a[1, 3] a[2, 2] a[3, 1] - a[1, 1] a[2, 3] a[3, 2] - a[2, 2] a[2, 3] a[3, 2] +
  a[1, 1]^2 a[3, 3] - a[1, 2] a[2, 1] a[3, 3] + 3 a[1, 1] a[2, 2] a[3, 3] + a[2, 2]^2 a[3, 3] -
  a[1, 3] a[3, 1] a[3, 3] - a[2, 3] a[3, 2] a[3, 3] + a[1, 1] a[3, 3]^2 + a[2, 2] a[3, 3]^2
```

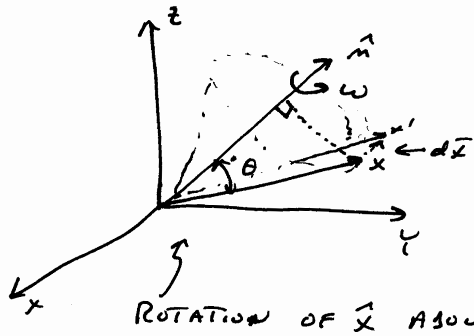
```
In[9]:= rhs = Expand[(Sum[a[i, j] a[j, k] a[k, i], {i, 3}, {j, 3}, {k, 3}] -
        i1 Sum[a[i, j] a[j, i], {i, 3}, {j, 3}] + i2 i1) / 3]

Out[9]= -a[1, 3] a[2, 2] a[3, 1] + a[1, 2] a[2, 3] a[3, 1] + a[1, 3] a[2, 1] a[3, 2] -
  a[1, 1] a[2, 3] a[3, 2] - a[1, 2] a[2, 1] a[3, 3] + a[1, 1] a[2, 2] a[3, 3]

In[10]:= detA - rhs // Simplify

Out[10]= 0
```

Rigid Rotation



ROTATION OF $\hat{x}$ ABOUT VECTOR $\hat{n}$.

$$|d\bar{x}| = |x \sin\theta \, \omega \, dt|$$

$$\frac{d\bar{x}}{dt} = \bar{u} = \bar{\omega} \times \bar{x} = \text{VELOCITY OF ROTATION}$$

$$\bar{\omega} = \hat{n}\,\omega$$

Symmetric and Antisymmetric

IF $A_{ij} = A_{ji}$, THEN $\bar{\bar{A}}$ IS SYMMETRIC.

IF $A_{ij} = -A_{ji}$, THEN $\bar{\bar{A}}$ IS ANTI-SYMMETRIC.

ALL TENSORS CAN BE DECOMPOSED AS...

$$A_{ij} = \underbrace{\tfrac{1}{2}\left(A_{ij} + A_{ji}\right)}_{\text{SYMMETRIC PART}} + \underbrace{\tfrac{1}{2}\left(A_{ij} - A_{ji}\right)}_{\text{ANTI-SYMMETRIC PART}}$$

SYMMETRIC TENSOR HAS 6 INDEPENDENT VALUES

ANTI-SYMMETRIC TENSOR HAS 3 INDEPENDENT VALUES

$$S_{ij} = \begin{pmatrix} 1 & 4 & 5 \\ 4 & 2 & 6 \\ 5 & 6 & 3 \end{pmatrix} \qquad A_{ij} = \begin{pmatrix} 0 & -1 & 2 \\ 1 & 0 & -3 \\ -2 & 3 & 0 \end{pmatrix}$$

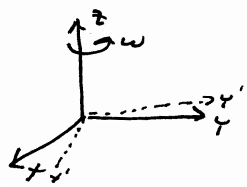
Antisymmetric Tensor

EVERY ANTI SYMMETRIC TENSOR IS ASSOCIATED WITH A VECTOR...

$$\bar{\bar{R}} = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix} \qquad \bar{\omega} = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix}$$

$$R_{ij} = -\epsilon_{ijk}\,\omega_k \qquad \omega_k = -\tfrac{1}{2}\epsilon_{ijk}\,R_{ij}$$

EXAMPLE: ROTATION ABOUT Z-AXIS



$$\bar{\omega} = \begin{pmatrix} 0 \\ 0 \\ \omega_0 \end{pmatrix}$$

$$\bar{\bar{R}} = \begin{pmatrix} 0 & -\omega_0 & 0 \\ \omega_0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\bar{x}(t+\delta t) = \bar{x}(t) + \delta t\,\bar{\bar{R}}\cdot\bar{x}(t)$$

Time Derivative

IF $A_{lm}(t)$ IS A TENSOR, THEN SO ARE ALL TIME DERIVATIVES...

$$\frac{d^n}{dt^n} A_{lm}(t) = C_{il}\,C_{jm}\,\frac{d^n}{dt^n} A_{ij}(t)$$

ALSO

$\bar{u}(t) = \dfrac{d\bar{x}(t)}{dt}$ IS A VECTOR

$$\frac{d}{dt}(\bar{a}\cdot\bar{b}) = \frac{d\bar{a}}{dt}\cdot\bar{b} + \bar{a}\cdot\frac{d\bar{b}}{dt}$$

PROVE: IF ACCELERATION IS PERPENDICULAR TO VELOCITY, THEN $|\bar{u}|$ IS A CONSTANT

$$\frac{d}{dt}(u^2) = \frac{d}{dt}(\bar{u}\cdot\bar{u}) = 2\,\bar{u}\cdot\frac{d\bar{u}}{dt} = 0$$

Vector Fields and Trajectory Lines

$\bar{U}(x_1, x_2, x_3, t)$ IS A VECTOR FIELD.

AT EVERY POINT IN SPACE, THERE IS A VECTOR (AND IT MAY CHANGE IN TIME.)

ASSOCIATED WITH ANY VECTOR FIELD ARE "TRAJECTORIES" (LIKE STREAMLINES FOR FLUID FLOW) WHICH ARE THE CURVES EVERYWHERE TANGENT TO LOCAL FIELD.

EXAMPLE:

$$\frac{d\bar{x}}{ds} = \bar{a}(\bar{x}) \quad \text{OR} \quad \frac{dx_i}{ds} = a_i(x_1(s), x_2(s), x_3(s))$$

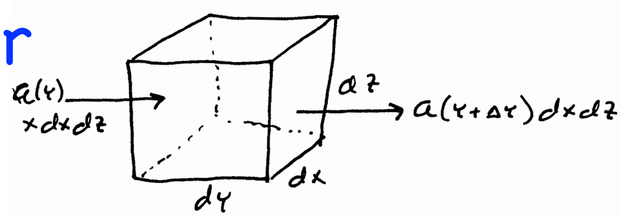
WHERE s IS A LENGTH PARAMETER ALONG THE CURVE.

$$\frac{dx_i}{a_i} = ds \quad \text{FOR ALL } i$$

Divergence of a Vector Field

$$\bar{\nabla}\cdot\bar{a} = \frac{\partial a_i}{\partial x_i} = \frac{\partial a_1}{\partial x_1} + \frac{\partial a_2}{\partial x_2} + \frac{\partial a_3}{\partial x_3} \equiv a_{i,i}$$

COMMA MEANS DERIVATIVE



NET FLOW INTO VOLUME ALONG Y IS

$$[a_y(y+\Delta y) - a_y(y)]\,dx\,dz$$

$$= \frac{\partial a_y}{\partial y}\,dx\,dy\,dz \quad (+\ 4 \text{ OTHER SIDES})$$

$$\text{SO } \nabla\cdot\bar{a} = \lim_{dx\,dy\,dz \to 0} \frac{1}{\Delta x \Delta y \Delta z} \iint_{\Delta x \Delta y \Delta z} \bar{a}\cdot\hat{n}\, dS$$

$\hat{n}$: UNIT SURFACE NORMAL

IF $\nabla\cdot\bar{a} = 0$, THEN VECTOR FIELD IS SOLENOIDAL (LIKE $\bar{B}$)

Laplacian

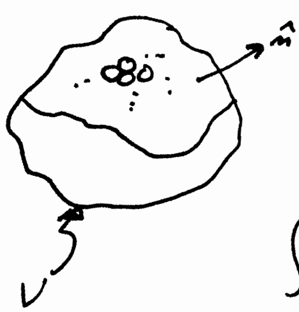
IF $\nabla \cdot \bar{a} = 0$, THE A POTENTIAL FUNCTION EXISTS WITH $\bar{a} = \nabla\varphi$ AND

$$\nabla^2\varphi = \bar{\nabla}\cdot\bar{\nabla}\varphi = \frac{\partial^2\varphi}{\partial x_i\,\partial x_i} = 0$$

$$\bar{\nabla}\cdot\bar{\nabla}\varphi = \lim_{dx\,dy\,dz \to 0} \frac{1}{\Delta x\Delta y\Delta z}\iint_{\Delta x\Delta y\Delta z} \nabla\varphi\cdot\hat{n}\,dS$$

= SUM OF "GRADIENT FLUX" FROM VOLUME

Green's (or Gauss') Theorem



SUBDIVIDE VOLUME INTO A COLLECTION OF INFINTESIMAL VOLUME ELEMENTS. SUM.

$$\iiint_V \bar{\nabla}\cdot\bar{a}\,dV = \iint_S \bar{a}\cdot\hat{n}\,dS$$

Green's Theorem Variants

If V is a volume enclosed by a surface S and $d\mathbf{S} = \mathbf{n}dS$, where $\mathbf{n}$ is the unit normal outward from V,

$$(27) \quad \int_V dV \nabla f = \int_S d\mathbf{S} f$$

$$(28) \quad \int_V dV \nabla \cdot \mathbf{A} = \int_S d\mathbf{S} \cdot \mathbf{A}$$

$$(29) \quad \int_V dV \nabla \cdot \mathcal{T} = \int_S d\mathbf{S} \cdot \mathcal{T}$$

$$(30) \quad \int_V dV \nabla \times \mathbf{A} = \int_S d\mathbf{S} \times \mathbf{A}$$

$$(31) \quad \int_V dV (f\nabla^2 g - g\nabla^2 f) = \int_S d\mathbf{S} \cdot (f\nabla g - g\nabla f)$$

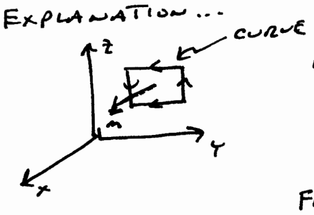
$$(32) \quad \int_V dV (\mathbf{A} \cdot \nabla \times \nabla \times \mathbf{B} - \mathbf{B} \cdot \nabla \times \nabla \times \mathbf{A}) = \int_S d\mathbf{S} \cdot (\mathbf{B} \times \nabla \times \mathbf{A} - \mathbf{A} \times \nabla \times \mathbf{B})$$

Curl of a Vector Field

$$\nabla \times \bar{a} = \epsilon_{ijk} \frac{\partial a_k}{\partial x_j} \hat{e}_i$$

$$= \left(\frac{\partial a_3}{\partial x_2} - \frac{\partial a_2}{\partial x_3},\ \frac{\partial a_1}{\partial x_3} - \frac{\partial a_3}{\partial x_1},\ \frac{\partial a_2}{\partial x_1} - \frac{\partial a_1}{\partial x_2} \right)$$

EXPLANATION...



LET $\bar{t}$ = TANGENT UNIT VECTOR TO CURVE

s = LENGTH PARAMETER ALONG CURVE

FIND $\oint \bar{a} \cdot \bar{t}\, ds$ AS $\Delta y, \Delta z \to 0$

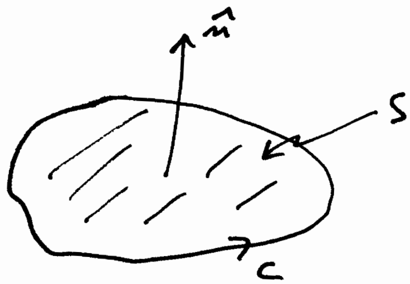
RIGHT + LEFT $= \left[a_z(y+\Delta y) - a_z(y) \right] dz = \frac{\partial a_z}{\partial y} dy\, dz$

TOP + BOTTOM $= \left[-a_y(z+\Delta z) + a_y(z) \right] dy = \frac{\partial a_y}{\partial z} dy\, dz$

SO $\nabla \times \bar{A} = \lim_{AREA \to 0} \frac{1}{AREA} \oint \bar{a} \cdot \bar{t}\, ds$

A VECTOR FIELD WITH $\nabla \times \bar{a} = 0$ IS IRROTATIONAL.

Stokes' Theorem



$$\iint_S \nabla\times\bar{a}\cdot\hat{n}\,ds = \oint_C \bar{a}\cdot\hat{t}\,ds$$

(BY SUMMING INFINITESIMAL SURFACE ELEMENTS)

Stokes' Theorem Variants

If S is an open surface bounded by the contour C, of which the line element is $d\mathbf{l}$,

$$\text{(33)} \quad \int_S d\mathbf{S}\times\nabla f = \oint_C d\mathbf{l}f$$

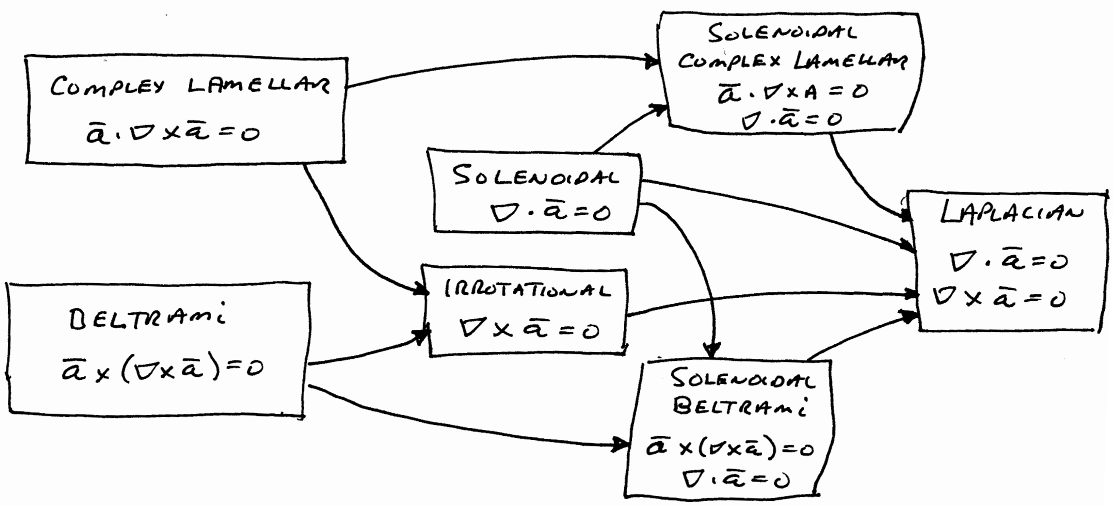
$$\text{(34)} \quad \int_S d\mathbf{S}\cdot\nabla\times\mathbf{A} = \oint_C d\mathbf{l}\cdot\mathbf{A}$$

$$\text{(35)} \quad \int_S (d\mathbf{S}\times\nabla)\times\mathbf{A} = \oint_C d\mathbf{l}\times\mathbf{A}$$

$$\text{(36)} \quad \int_S d\mathbf{S}\cdot(\nabla f\times\nabla g) = \oint_C f dg = -\oint_C g df$$

Classification of Vector Fields

NAMES ....



Irrotational Field

$$\nabla \times \bar{a} = 0$$

LET $\bar{a} = \nabla \varphi$, $\varphi =$ POTENTIAL

THEN $\nabla \times \nabla \varphi = 0 = \epsilon_{ijk} \dfrac{\partial^2 \varphi}{\partial x_i \partial x_j} \hat{e}_k$

Solenoidal Field

$$\nabla \cdot \bar{a} = 0$$

LET $\bar{a} = \nabla\varphi \times \nabla\psi$ OR

$$= \nabla \times (\varphi \nabla \psi)$$

THEN, SINCE

$$\nabla \cdot (\nabla \times \bar{a}) = \epsilon_{ijk} \frac{\partial^2 a_j}{\partial x_i \partial x_i} = 0$$

OR

$$\nabla \cdot (\nabla\varphi \times \nabla\psi) = \epsilon_{ijk} \left(\frac{\partial^2 \varphi}{\partial x_i \partial x_k} \frac{\partial \psi}{\partial x_j} + \frac{\partial \varphi}{\partial x_i} \frac{\partial^2 \psi}{\partial x_k \partial x_i} \right)$$

$$= 0$$

WHERE ψ = STREAM FUNCTION IF φ IS A SYMMETRY DIRECTION.

Summary

- Vectors & tensors transform under coordinate rotation like position vector
- Vector operators: scalar product, vector product, triple scalar product, triple vector product
- Tensors: isotropic, symmetric, antisymmetric, orthogonal
- Calculus of vectors: derivative, gradient, divergence, curl
- Gauss' & Stokes' Theorems
- Classification of Vector Fields: irrotational, solenoidal, ...
- Next Lecture: Kinematics of fluids