Vladimir Zakharov
(Aug 1, 1939 - )


Vladimir Zakharov was born in Kazan, Russian SFSR in 1939, to Evgeniy and Elena Zakharov, an engineer and a schoolteacher. He studied at the Moscow Power Engineering Institute and at the Novosibirsk State University, where he received his specialist degree in physics in 1963 and his Candidate of Sciences degree in 1966, studying under Roald Sagdeev.

Awarded the Dirac Medal in 2003 for his work on turbulence.
Soliton Turbulence in Shallow Water Ocean Surface Waves

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We analyze shallow water wind waves in Currituck Sound, North Carolina and experimentally confirm, for the first time, the presence of soliton turbulence in ocean waves. Soliton turbulence is an exotic form of nonlinear wave motion where low frequency energy may also be viewed as a dense soliton gas, described theoretically by the soliton limit of the Korteweg–deVries equation, a completely integrable soliton system: Hence the phrase “soliton turbulence” is synonymous with “integrable soliton turbulence.” For periodic-quasiperiodic boundary conditions the ergodic solutions of Korteweg–deVries are exactly solvable by finite gap theory (FGT), the basis of our data analysis. We find that large amplitude measured wave trains near the energetic peak of a storm have low frequency power spectra that behave as $\sim \alpha^{-1}$. We use the linear Fourier transform to estimate this power law from the power spectrum and to filter densely packed soliton wave trains from the data. We apply FGT to determine the soliton spectrum and find that the low frequency $\sim \alpha^{-1}$ region is soliton dominated. The solitons have random FGT phases, a soliton random phase approximation, which supports our interpretation of the data as soliton turbulence. From the probability density of the solitons we are able to demonstrate that the solitons are dense in time and highly non-Gaussian.
Surface Tension and Contact Angle

In 1804, Young developed the theory of capillary phenomena on the principle of surface tension. He also observed the constancy of the angle of contact of a liquid surface with a solid, and showed how from these two principles to deduce the phenomena of capillary action. In 1805, Pierre-Simon Laplace, the French philosopher, discovered the significance of meniscus radii with respect to capillary action.

In 1830, Carl Friedrich Gauss, the German mathematician, unified the work of these two scientists to derive the Young–Laplace equation, the formula that describes the capillary pressure difference sustained across the interface between two static fluids.

https://en.wikipedia.org/wiki/Thomas_Young_(scientist)

Thomas Young
1773 – 1829
Detaching Microparticles from a Liquid Surface

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The work required to detach microparticles from fluid interfaces depends on the shape of the liquid meniscus. However, measuring the capillary force on a single microparticle and simultaneously imaging the shape of the liquid meniscus has not yet been accomplished. To correlate force and shape, we combined a laser scanning confocal microscope with a colloidal probe setup. While moving a hydrophobic microsphere (radius 5–10 μm) in and out of a 2–5 μm thick glycerol film, we simultaneously measured the force and imaged the shape of the liquid meniscus. In this way we verified the fundamental equations [D. F. James, J. Fluid Mech. 63, 657 (1974); A. D. Scheludko, A. D. Nikolov, Colloid Polymer Sci. 253, 396 (1975)] that describe the adhesion of particles in flotation, deinking of paper, the stability of Pickering emulsions and particle-stabilized foams. Comparing experimental results with theory showed, however, that the receding contact angle has to be applied, which can be much lower than the static contact angle obtained right after jump in of the particle.

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\[ F = 2\pi\gamma R \sin \beta \sin \alpha = 2\pi\gamma R \sin \beta \sin(\Theta + \beta). \quad (1) \]

FIG. 1. (a) Schematic setup combining a colloidal probe setup with a confocal microscope. Position sensitive device (PSD).
(b) Schematic of a particle in contact with a liquid film on a solid substrate.
Scalars, Vectors, & Tensors

- Scalars: mass density ($\rho$), temperature (T), concentration (S), charge density ($\rho_q$)
- Vectors: flow (U), force (F), magnetic field (B), current density (J), vorticity ($\Omega$)
- Tensors: stress ($\tau$), strain rate ($\varepsilon$), rotation (R), identity (I)

How to work and operate with tensors...

What is a Tensor?

DEFINITION: A TENSOR (VECTOR) IS A ENTITY THAT TRANSFORMS LIKE A TENSOR (VECTOR) UNDER COORDINATE TRANSFORMATION DUE TO ROTATION — THAT IS LIKE A POSITION VECTOR, $\mathbf{x}$.

In other words: a vector has a direction and magnitude...
Rotation about the z-Axis

\[
\mathbf{x}' = (x \cos \theta + y \sin \theta, -x \sin \theta + y \cos \theta, z) \]

\[
\mathbf{x}' = \begin{pmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}
\]

\[
X'_j = C_{ij} X'_i \quad \text{cARTESIAN SUMMATION COnVENTION}
\]

\[
= \sum_{i=1,2,3} C_{ij} X'_i
\]

**DEFINITION:** If \( U'_j = C_{ij} U'_i \), then \( \mathbf{U} \) is a vector.

---

**Ch 2: Problem 4**

4. Show that

\[
\mathbf{C} \cdot \mathbf{C}^T = \mathbf{C}^T \cdot \mathbf{C} = \delta,
\]

where \( \mathbf{C} \) is the direction cosine matrix and \( \delta \) is the matrix of the Kronecker delta. Any matrix obeying such a relationship is called an *orthogonal matrix* because it represents transformation of one set of orthogonal axes into another.
Rotation Matrix is Orthogonal

\[ \text{TRANSPOSE: } \begin{pmatrix} c_{i,j} \end{pmatrix}^T = c_{j,i} \]

\[ \text{NOTE: } (c_{i,j})^T \cdot c_{i,j} = \delta_{i,j} \]

\[
\begin{pmatrix}
\cos \theta & -\sin \theta & 0 \\
\sin \theta & \cos \theta & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
\cos \theta & \sin \theta & 0 \\
-\sin \theta & \cos \theta & 0 \\
0 & 0 & 1
\end{pmatrix}
= \begin{pmatrix}
\cos^2 \theta + \sin^2 \theta & \cos \theta \sin \theta - \cos \theta \sin \theta & 0 \\
\cos \theta \sin \theta - \cos \theta \sin \theta & \cos^2 \theta + \sin^2 \theta & 0 \\
0 & 0 & 1
\end{pmatrix}
= \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix} = \delta_{i,j} \text{ IDENTITY TENSOR}
\]

SO \[ x'_i = (c_{i,j})^T x_j = c_{j,i} x'_j \]
and \[ x'_j = c_{i,j} x_i \]

---

Tensors

\[ A_{mn} = c_{im} c_{jm} A_{ij} \]

DEFINES A TENSOR.

While 3 components/values are needed to define a vector, 9 components/values are needed to define a 2nd-order tensor.

AN ISOTROPIC 2ND-ORDER TENSOR IS

\[ \delta_{i,j} = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}, \text{ SAME IN ALL COORDINATE SYSTEMS} \]

\[ \delta_{lm} = c_{il} c_{jm} \delta_{i,j} = c_{il} \delta_{i,j} c_{jm} = c_{il}^T \delta_{i,j} c_{jm} = \delta_{lm} \]
Vector Identities

Notation: \( f, g \), are scalars; \( \vec{A}, \vec{B}, \) etc., are vectors; \( T \) is a tensor; \( I \) is the unit dyad.

1. \( \vec{A} \cdot \vec{B} \times \vec{C} = \vec{A} \times \vec{B} \cdot \vec{C} = \vec{B} \cdot \vec{C} \times \vec{A} = \vec{B} \times \vec{C} \cdot \vec{A} = \vec{C} \cdot \vec{A} \times \vec{B} = \vec{C} \times \vec{A} \cdot \vec{B} \)

2. \( \vec{A} \times (\vec{B} \times \vec{C}) = (\vec{C} \times \vec{B}) \times \vec{A} = (\vec{A} \cdot \vec{C})\vec{B} - (\vec{A} \cdot \vec{B})\vec{C} \)

3. \( \vec{A} \times (\vec{B} \times \vec{C}) + \vec{B} \times (\vec{C} \times \vec{A}) + \vec{C} \times (\vec{A} \times \vec{B}) = 0 \)

4. \( (\vec{A} \times \vec{B}) \cdot (\vec{C} \times \vec{D}) = (\vec{A} \cdot \vec{C})(\vec{B} \cdot \vec{D}) - (\vec{A} \cdot \vec{D})(\vec{B} \cdot \vec{C}) \)

5. \( (\vec{A} \times \vec{B}) \times (\vec{C} \times \vec{D}) = (\vec{A} \times \vec{B} \cdot \vec{D})\vec{C} - (\vec{A} \times \vec{B} \cdot \vec{C})\vec{D} \)

6. \( \nabla(fg) = \nabla(gf) = f\nabla g + g\nabla f \)

7. \( \nabla \cdot (f\vec{A}) = f\nabla \cdot \vec{A} + \vec{A} \cdot \nabla f \)

8. \( \nabla \times (f\vec{A}) = f\nabla \times \vec{A} + \nabla f \times \vec{A} \)

9. \( \nabla \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot \nabla \times \vec{A} - \vec{A} \cdot \nabla \times \vec{B} \)

10. \( \nabla \times (\vec{A} \times \vec{B}) = \vec{A}(\nabla \cdot \vec{B}) - \vec{B}(\nabla \cdot \vec{A}) + (\vec{B} \cdot \nabla)\vec{A} - (\vec{A} \cdot \nabla)\vec{B} \)

11. \( \vec{A} \times (\nabla \times \vec{B}) = (\nabla \vec{B}) \cdot \vec{A} - (\vec{A} \cdot \nabla)\vec{B} \)

12. \( \nabla(\vec{A} \cdot \vec{B}) = \vec{A} \times (\nabla \times \vec{B}) + \vec{B} \times (\nabla \times \vec{A}) + (\vec{A} \cdot \nabla)\vec{B} + (\vec{B} \cdot \nabla)\vec{A} \)

13. \( \nabla^2 f = \nabla \cdot \nabla f \)

14. \( \nabla^2 \vec{A} = \nabla(\nabla \cdot \vec{A}) - \nabla \times \nabla \times \vec{A} \)

15. \( \nabla \times \nabla f = 0 \)

16. \( \nabla \cdot \nabla \times \vec{A} = 0 \)

Vector Examples

• If \( \vec{a} \) is a vector, then \( \alpha \vec{a} \) is a vector.

• If \( \alpha \) is a scalar.

• If \( \vec{a}, \vec{b}, \) and \( \vec{c} \) are vectors, then

\[ \vec{a} + \vec{b} \] is a vector,

\[ \vec{a} - \vec{b} \] is a vector,

\[ \vec{a} + (\vec{b} + \vec{c}) = (\vec{a} + \vec{b}) + \vec{c} \] is a vector.
Scalar Product

\[ \vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a} = |\vec{a}| |\vec{b}| \cos \theta \]

= A SCALAR INVARIANT UNDER ROTATION.

IF \( \vec{a} \cdot \vec{b} = 0 \), THEN \( \vec{a} \) AND \( \vec{b} \) ARE MUTUALLY ORTHOGONAL \((\theta = \pm 90^\circ)\)

\[ \vec{a} \]
\[ \theta \]
\[ \vec{b} \]

Vector Product

\[ \vec{c} = \vec{a} \times \vec{b} \]

= \( |\vec{a}| |\vec{b}| \sin \theta \)

NOTE: RIGHT HANDS DEFINITION!

\( (\vec{a} \times \vec{b})_{i} = \delta_{ij} \epsilon_{jkl} a_{k} b_{l} \hat{e}_{j} = \left| \begin{array}{ccc} \hat{e}_{1} & \hat{e}_{2} & \hat{e}_{3} \\ a_{1} & a_{2} & a_{3} \\ b_{1} & b_{2} & b_{3} \end{array} \right| \)

\( \epsilon_{ijk} = \) PERMUTATION TENSOR

= ISOTROPIC 3\(^{rd}\) ORDER TENSOR
Permutation Tensor

\[ \varepsilon_{ijk} = \begin{cases} 
1 & \text{if } ijk \text{ are cyclic} \\
0 & \text{if any } i,j,k \text{ are repeated} \\
-1 & \text{if } ijk \text{ are anti-cyclic} 
\end{cases} \]

\[ \varepsilon_{123} = \varepsilon_{312} = \varepsilon_{231} \]
\[ \varepsilon_{123} = -\varepsilon_{132} \]
\[ \varepsilon_{113} = 0 \]
\[ \varepsilon_{ijk} \text{ is an isotropic, } 3 \times 3 \text{ order tensor} \]

\[ \varepsilon_{ijk} \varepsilon_{klm} = \delta_{il} \delta_{jm} - \delta_{im} \delta_{jl} \]

CHECK:
- If any \( i=j, j=k \) or \( i=m \), then 0
- If both cyclic
  \[ \varepsilon_{ijk} \varepsilon_{klm} \]
  \[ 1 \ 2 \ 3 \ \ 2 \ 1 \ 3 \ \text{ then } 1 \]
- If both anti-cyclic
  \[ \varepsilon_{ijk} \varepsilon_{klm} \]
  \[ 1 \ 3 \ 2 \ \ 2 \ 1 \ 3 \ \text{ then } 1 \]
- If one is anti-cyclic
  \[ \varepsilon_{ijk} \varepsilon_{klm} \]
  \[ 1 \ 2 \ 3 \ \ 3 \ 2 \ 1 \ \text{ then } -1 \]

Proof: \( (\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{a} \times \mathbf{b}) = |\mathbf{a}|^2 |\mathbf{b}|^2 - (\mathbf{a} \cdot \mathbf{b})^2 \)

\[ \varepsilon_{ijk} \mathbf{a}_i \mathbf{b}_j \mathbf{c}_k \mathbf{d}_m = (\delta_{im} \delta_{jn} - \delta_{in} \delta_{jm}) \mathbf{a}_i \mathbf{b}_j \mathbf{c}_k \mathbf{d}_m \]
\[ = a^2 b^2 - (\mathbf{a} \cdot \mathbf{b})^2 \]
Gradient Operator (Scalar)

\( \phi(x) \) is a scalar, then

\[
\frac{2\phi}{2x_i} = \nabla \phi \text{ is a vector}
\]

\[
\frac{2\phi}{2x'_j} = \frac{2\phi}{2x_i} \frac{2x_i}{2x'_j} = c_{ij} \frac{2\phi}{2x_i}
\]

so \( \nabla \phi \) transforms like a vector.

Triple Scalar Product

\[
\vec{a} \cdot (\vec{b} \times \vec{c}) = e_{ijk} a_i b_j c_k
\]

\[
= \vec{b} \cdot (\vec{c} \times \vec{a})
\]

\[
= \vec{c} \cdot (\vec{a} \times \vec{b})
\]

\( a \times (b \times c) \approx \text{Volume of parallelipiped with sides } \vec{a}, \vec{b}, \vec{c} \)

If \( a \cdot (b \times c) = 0 \), the vectors are co-planar.
Co-Planar and Not

If \( \overrightarrow{a}, \overrightarrow{b}, \text{ and } \overrightarrow{c} \) are co-planar, then

\[
\overrightarrow{e} = \alpha \overrightarrow{a} + \beta \overrightarrow{b} \quad \text{and} \quad \overrightarrow{c} = \gamma \overrightarrow{c} + \beta \overrightarrow{c}
\]

or

\[
\begin{pmatrix}
\alpha \\
\beta \\
\gamma
\end{pmatrix} = \begin{pmatrix}
a_1 & b_1 & c_1 \\
a_2 & b_2 & c_2 \\
a_3 & b_3 & c_3
\end{pmatrix} \begin{pmatrix}
\alpha \\
\beta \\
\gamma
\end{pmatrix}
\]

This homogeneous equation has a non-trivial solution only if

\[\text{DET} \begin{pmatrix}
a_1 & b_1 & c_1 \\
a_2 & b_2 & c_2 \\
a_3 & b_3 & c_3
\end{pmatrix} = 0 \quad \text{(same as } \overrightarrow{e} \cdot (\overrightarrow{a} \times \overrightarrow{b}))\]

If \( \overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c} \) are not co-planar, then

\[\alpha \overrightarrow{a} + \beta \overrightarrow{b} + \gamma \overrightarrow{c} = \overrightarrow{d}\]

and \( \overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c} \) can serve as a “basis” to represent a vector \( \overrightarrow{d} \).

Triple Vector Product

\[
\overrightarrow{a} \times (\overrightarrow{b} \times \overrightarrow{c}) = \epsilon_{ijk} \epsilon_{klm} a_j b_k c_l
\]

\[= b_i (a_i c_j) - c_i (a_i b_j)
\]

\[= (\overrightarrow{a} \cdot \overrightarrow{c}) \overrightarrow{b} - (\overrightarrow{a} \cdot \overrightarrow{b}) \overrightarrow{c}
\]

If \( \overrightarrow{a} \) is a unit vector, and \( \overrightarrow{\hat{a}} \) a vector, then

\[\overrightarrow{\hat{a}} = (\overrightarrow{\hat{a}} \cdot \overrightarrow{a}) \overrightarrow{a} + \overrightarrow{a} \times (\overrightarrow{\hat{a}} \times \overrightarrow{a})
\]

Proof: \[\overrightarrow{a} \times (\overrightarrow{\hat{a}} \times \overrightarrow{a}) = \overrightarrow{\hat{a}} (\overrightarrow{\hat{a}} \cdot \overrightarrow{a}) - \overrightarrow{a} (\overrightarrow{\hat{a}} \cdot \overrightarrow{a}) \equiv 1
\]

Thus, \( \overrightarrow{\hat{a}} \) can be resolved into a component along \( \overrightarrow{a} \) and one perpendicular to \( \overrightarrow{a} \).
In class Problem

Ch. 2 Question 10

Prove \( \nabla \times \mathbf{A} = \mathbf{0} \) for any vector field.
Ch. 2 Question 10

Prove \( \nabla \cdot \mathbf{U} = 0 \) for any vector field.

Method #1

\[
\nabla \cdot \mathbf{U} = \sum_{i,j,k} E_{i,j,k} \frac{\partial U_j}{\partial x_i} \frac{\partial U_k}{\partial x_j}
\]

But \( E_{i,j,k} = -E_{j,i,k} \) while \( \frac{\partial U_j}{\partial x_i} \frac{\partial U_k}{\partial x_j} = \frac{\partial^2 U_j}{\partial x_i \partial x_j} \).

So the sum must vanish.

Method #2

\[
\begin{align*}
\int \int \int \int U \cdot d\mathbf{V} &= \int \int U \cdot d\mathbf{S} \\
&= \int \int U \cdot d\mathbf{S} + \int \int U \cdot d\mathbf{S} \\
&= \int \int \mathbf{U} \cdot d\mathbf{S} + \int \int \mathbf{U} \cdot d\mathbf{S} \\
&= \int \int \mathbf{U} \cdot d\mathbf{S} + \int \int \mathbf{U} \cdot d\mathbf{S}
\end{align*}
\]

But the sense of these two line integrals are opposite, so the sum must vanish.
Ch. 2 Question 11

Prove \( \nabla \times \nabla \phi = 0 \) for any well-behaved function \( \phi \).

**Method #1**

\[
(\nabla \times \nabla \phi)_i = \sum_{i,j,k} \epsilon_{ijk} \frac{\partial^2 \phi}{\partial x_j \partial x_k}
\]

But \( \epsilon_{ijk} = -\epsilon_{ikj} \) and \( \frac{\partial^2 \phi}{\partial x_j \partial x_k} \) is symmetric, so sum must vanish.
Ch. 2 Question 11

Prove $\nabla \times \nabla \phi = 0$ for any well-behaved function $\phi$.

Method #1

$$(\nabla \times \nabla \phi)_i = \sum \epsilon_{ijk} \frac{\partial^2 \phi}{\partial x_j \partial x_k}$$

But $\epsilon_{ijk} = -\epsilon_{ikj}$ and $\frac{\partial^2 \phi}{\partial x_j \partial x_k}$ is symmetric, so $\sum$ must vanish.

Method #2

$$\int \int \int \check{dS} \nabla \times \nabla \phi = \int \frac{\partial^2 \phi}{\partial x_i \partial x_j} \check{dS} \nabla \phi$$

$$= \Delta x \left( \frac{\partial^2 \phi}{\partial x_i \partial x_j} \right)_{x_j} \left( \frac{\partial^2 \phi}{\partial x_i \partial x_j} \right)_{x_j}$$

$$+ \Delta y \left( \frac{\partial^2 \phi}{\partial x_i \partial x_j} \right)_{x_j} \left( \frac{\partial^2 \phi}{\partial x_i \partial x_j} \right)_{x_j}$$

$$- \Delta x \Delta y \frac{\partial^2 \phi}{\partial x_i \partial x_j} \left( \frac{\partial^2 \phi}{\partial x_i \partial x_j} \right)_{x_j}$$

Since line segments $\check {dS}$ are in opposite.

Ch. 2: Problem 5

5. Show that for a second-order tensor $A$, the following three quantities are invariant under the rotation of axes:

$$I_1 = A_{ii}$$

$$I_2 = A_{11} A_{22} + A_{22} A_{33} + A_{33} A_{11}$$

$$I_3 = \det(A_{ij})$$

[Hint: Use the result of Exercise 4 and the transformation rule (2.12) to show that $I'_i = A'_{ii} = A_{ii} = I_1$. Then show that $A_{ij} A_{ji}$ and $A_{ij} A_{jk} A_{ki}$ are also invariants. In fact, all contracted scalars of the form $A_{ij} A_{jk} \cdots A_{mi}$ are invariants. Finally, verify that

$$I_2 = \frac{1}{2} [I_1^2 - A_{ij} A_{ji}]$$

$$I_3 = A_{ij} A_{jk} A_{ki} - I_1 A_{ij} A_{ji} + I_2 A_{ii}.$$}

Because the right-hand sides are invariant, so are $I_2$ and $I_3.$]
\[ I_1 = A_{ii} \]
\[ I_2 = \begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix} + \begin{vmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{vmatrix} + \begin{vmatrix} A_{11} & A_{13} \\ A_{31} & A_{33} \end{vmatrix} \]
\[ I_3 = \text{det}(A_{ij}). \]
Rigid Rotation

\[
\begin{align*}
\frac{d\mathbf{\bar{r}}}{dt} &= \mathbf{\bar{u}} = \mathbf{\bar{w}} \times \mathbf{\bar{r}} = \text{velocity of rotation} \\
\mathbf{\bar{w}} &= \hat{\mathbf{\omega}}
\end{align*}
\]

Symmetric and Antisymmetric

If \( A_{ij} = A_{ji} \), then \( \mathbf{\bar{A}} \) is symmetric.

If \( A_{ij} = -A_{ji} \), then \( \mathbf{\bar{A}} \) is anti-symmetric.

All tensors can be decomposed as...

\[
A_{ij} = \frac{1}{2} \left( A_{ij} + A_{ji} \right) + \frac{1}{2} \left( A_{ij} - A_{ji} \right)
\]

Symmetric tensor has 6 independent values.

Anti-symmetric tensor has 3 independent values.

\[
S_{ij} = \begin{pmatrix}
1 & 4 & 5 \\
4 & 2 & 5 \\
5 & 6 & 3
\end{pmatrix} \quad A_{ij} = \begin{pmatrix}
0 & -1 & 2 \\
1 & 0 & -3 \\
-2 & 3 & 0
\end{pmatrix}
\]
Antisymmetric Tensor

Every antisymmetric tensor is associated with a vector...

\[ \mathbf{R} = \begin{pmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{pmatrix} \quad \mathbf{\omega} = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} \]

\[ R_{ij} = -\varepsilon_{ijk} \omega_k \quad \omega_j = -\frac{1}{2} \varepsilon_{ijk} R_{ij} \]

Example: Rotation about z-axis

\[ \mathbf{\omega} = \begin{pmatrix} 0 \\ 0 \\ \omega_0 \end{pmatrix} \]

\[ \mathbf{R} = \begin{pmatrix} 0 & -\omega_0 & 0 \\ \omega_0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \]

\[ \mathbf{x}(t + \Delta t) = \mathbf{x}(t) + \Delta t \mathbf{\dot{R}} \mathbf{x}(t) \]

Time Derivative

If \( A^m_{\alpha}(t) \) is a tensor, then so are all time derivatives...

\[ \frac{d^m}{d\tau^m} A^m_{\alpha}(t) = C_{\beta \gamma} C_{\delta \alpha} \frac{d^m}{d\tau^m} A_{ij}(t) \]

Also

\[ \mathbf{U}(t) = \frac{d\mathbf{x}(t)}{dt} \text{ is a vector} \]

\[ \frac{d}{dt} (\mathbf{a} \cdot \mathbf{b}) = \frac{d\mathbf{a}}{dt} \cdot \mathbf{b} + \mathbf{a} \cdot \frac{d\mathbf{b}}{dt} \]

Prove: If acceleration is perpendicular to velocity, then \( |\mathbf{U}| \) is a constant

\[ \frac{d}{dt} (U^2) = \frac{d}{dt} (\mathbf{U} \cdot \mathbf{U}) = 2 \mathbf{U} \cdot \frac{d\mathbf{U}}{dt} = 0 \]
Vector Fields and Trajectory Lines

\( \mathbf{U}(x_1, x_2, x_3, t) \) is a vector field.

At every point in space, there is a vector (and it may change in time.)

Associated with any vector field are "trajectories" (like streamlines for fluid flow) which are the curves everywhere tangent to the field.

Example:

\[
\frac{dX^i}{ds} = \mathbf{a}(x) \quad \text{or} \quad \frac{dX_i}{ds} = a^i(x_1(s), x_2(s), x_3(s))
\]

where \( s \) is a length parameter along the curve.

\[
\frac{dX_i}{ds} = ds \quad \text{for all } i
\]

Divergence of a Vector Field

The divergence of a vector field \( \mathbf{a} \) is

\[
\nabla \cdot \mathbf{a} = \frac{2a^1}{2x^1} + \frac{2a^2}{2x^2} + \frac{2a^3}{2x^3}
\]

Comma means derivative

Net flow into volume along \( y \) is

\[
\left[ a^y(y+\Delta y) - a^y(y) \right] \, dx \, dy \, dz
\]

= \( \frac{2a^y}{2y} \) \, dx \, dy \, dz (x \text{ and other sides})

So \( \nabla \cdot \mathbf{a} = \lim_{\Delta y \to 0} \frac{1}{\Delta y} \int_{x \Delta x \Delta z} a^y \, n \, ds \)

\[ \text{Unit surface normal} \]

If \( \nabla \cdot \mathbf{a} = 0 \), then vector field is solenoidal (like \( \mathbf{B} \) )
Laplacian

If $\nabla \cdot \vec{a} = 0$, the potential function exists with $\vec{a} = \nabla \phi$ and

$$\nabla^2 \phi = \nabla \cdot \nabla \phi = \frac{\partial^2 \phi}{\partial x_i \partial x_i} = 0$$

$$\nabla \cdot \nabla \phi = \lim_{\Delta x \Delta y \Delta z \to 0} \frac{1}{\Delta x \Delta y \Delta z} \iint \phi \cdot \vec{A} dS$$

is the sum of "gradient flux" from volume.

Green's (or Gauss') Theorem

Subdivide volume into a collection of infinitesmal volume elements. Sum.

$$\iiint \nabla \cdot \vec{a} \, dV = \iiint \vec{a} \cdot dS$$
Green's Theorem Variants

If $V$ is a volume enclosed by a surface $S$ and $dS = ndS$, where $n$ is the unit normal outward from $V$,

\begin{align*}
(27) \quad & \int_V dV \nabla f = \int_S dS f \\
(28) \quad & \int_V dV \nabla \cdot A = \int_S dS \cdot A \\
(29) \quad & \int_V dV \nabla \cdot T = \int_S dS \cdot T \\
(30) \quad & \int_V dV \nabla \times A = \int_S dS \times A \\
(31) \quad & \int_V dV (f \nabla^2 g - g \nabla^2 f) = \int_S dS \cdot (f \nabla g - g \nabla f) \\
(32) \quad & \int_V dV (A \cdot \nabla \times \nabla \times B - B \cdot \nabla \times \nabla \times A) \\
\quad & \quad \quad \quad = \int_S dS \cdot (B \times \nabla \times A - A \times \nabla \times B)
\end{align*}

Curl of a Vector Field

\[ \nabla \times \vec{A} = \varepsilon_{ijk} \frac{2a_k}{2x_j} \hat{e}_i = \left( \frac{2a_3}{2x_2}, \frac{2a_2}{2x_3}, \frac{2a_1}{2x_2}, \frac{2a_2}{2x_3}, \frac{2a_3}{2x_1}, \frac{2a_1}{2x_2}, \frac{2a_2}{2x_3}, \frac{2a_3}{2x_1} \right) \]

\[ \text{Explanation...} \]

\[ \text{Let} \quad \vec{e} = \text{tangent unit vector to curve} \]
\[ S = \text{length parameter along curve} \]
\[ \text{Find} \quad \int \vec{A} \cdot d\vec{s} \quad \text{as} \quad dy, dz \to 0 \]
\[ \text{Right + Left} = \left[ a_x(t+0) - a_x(t) \right] dt = \frac{d^2 a_x}{dt^2} \Delta t \Delta z \]
\[ \text{Top + Bottom} = \left[ - a_y(t+0) + a_y(t) \right] dt = \frac{d^2 a_y}{dt^2} \Delta t \Delta z \]

\[ \text{So} \quad \nabla \times \vec{A} = \lim_{\text{Area} \to 0} \frac{1}{\text{Area}} \int \vec{A} \cdot d\vec{s} \]

A vector field with $\nabla \times \vec{A} = 0$ is irrotational.
Stokes' Theorem

\[ \int_S \nabla \times \mathbf{a} \cdot d\mathbf{s} = \oint_C \mathbf{a} \cdot d\mathbf{t} \]

(by summing infinitesimal surface elements)

Stokes' Theorem Variants

If \( S \) is an open surface bounded by the contour \( C \), of which the line element is \( dl \),

(33) \[ \int_S dS \times \nabla f = \oint_C df \]

(34) \[ \int_S dS \cdot \nabla \times A = \oint_C dl \cdot A \]

(35) \[ \int_S (dS \times \nabla) \times A = \oint_C dl \times A \]

(36) \[ \int_S dS \cdot (\nabla f \times \nabla g) = \oint_C fdg = - \oint_C gdf \]
Classification of Vector Fields

Irrotational Field

\[ \nabla \times \vec{a} = 0 \]

Let \( \vec{a} = \nabla \varphi \), \( \varphi = \text{potential} \)

Then \( \nabla \times \nabla \varphi = 0 = e_i \cdot \frac{\partial^2 \varphi}{\partial x_i \partial x_j} \hat{e}_k \)
Solenoidal Field

\[ \nabla \cdot \vec{a} = 0 \]

Let \( \vec{a} = \nabla \varphi \times \nabla \psi \) or
\[ = \nabla \times (\varphi \nabla \psi) \]

Then, since
\[ \nabla \cdot (\nabla \times \vec{a}) = \epsilon_{ijk} \frac{\partial^2 \varphi_j}{\partial x_i \partial x_k} = 0 \]
on or
\[ \nabla \cdot (\nabla \varphi \times \nabla \psi) = \epsilon_{ijk} \left( \frac{\partial \varphi_i}{\partial x_j} \frac{\partial \varphi_j}{\partial x_k} + \frac{\partial \varphi_j}{\partial x_k} \frac{\partial \varphi_k}{\partial x_i} + \frac{\partial \varphi_k}{\partial x_i} \frac{\partial \varphi_i}{\partial x_j} \right) \]
\[ = 0 \]

Where \( \varphi = \text{stream function if } \psi \text{ is a symmetric direction.} \)

Summary

- Vectors & tensors transform under coordinate rotation like position vector
- Vector operators: scalar product, vector product, triple scalar product, triple vector product
- Tensors: isotropic, symmetric, antisymmetric, orthogonal
- Calculus of vectors: derivative, gradient, divergence, curl
- Gauss’ & Stokes’ Theorems
- Classification of Vector Fields: irrotational, solenoidal, ...
- Next Lecture: Kinematics of fluids