APPH 4200
Physics of Fluids

Cartesian Tensors (Ch. 2)
Lecture 2b

1. Geometric Identities

2. Vector Calculus

Scalars, Vectors, &Tensors

- Scalars: mass density (ρ), temperature (T), concentration (S), charge density (ρ_q)
- Vectors: flow (U), force (F), magnetic field (B), current density (J), vorticity (Ω)
- Tensors: stress (τ), strain rate (ε), rotation (R), identity (I)

How to work and operate with tensors...

Vector Identities

Notation: f, g, are scalars; $\mathbf{A}$, $\mathbf{B}$, etc., are vectors; T is a tensor; I is the unit dyad.

(1) $\mathbf{A}\cdot\mathbf{B}\times\mathbf{C} = \mathbf{A}\times\mathbf{B}\cdot\mathbf{C} = \mathbf{B}\cdot\mathbf{C}\times\mathbf{A} = \mathbf{B}\times\mathbf{C}\cdot\mathbf{A} = \mathbf{C}\cdot\mathbf{A}\times\mathbf{B} = \mathbf{C}\times\mathbf{A}\cdot\mathbf{B}$

(2) $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = (\mathbf{C}\times\mathbf{B})\times\mathbf{A} = (\mathbf{A}\cdot\mathbf{C})\mathbf{B} - (\mathbf{A}\cdot\mathbf{B})\mathbf{C}$

(3) $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) + \mathbf{B}\times(\mathbf{C}\times\mathbf{A}) + \mathbf{C}\times(\mathbf{A}\times\mathbf{B}) = 0$

(4) $(\mathbf{A}\times\mathbf{B})\cdot(\mathbf{C}\times\mathbf{D}) = (\mathbf{A}\cdot\mathbf{C})(\mathbf{B}\cdot\mathbf{D}) - (\mathbf{A}\cdot\mathbf{D})(\mathbf{B}\cdot\mathbf{C})$

(5) $(\mathbf{A}\times\mathbf{B})\times(\mathbf{C}\times\mathbf{D}) = (\mathbf{A}\times\mathbf{B}\cdot\mathbf{D})\mathbf{C} - (\mathbf{A}\times\mathbf{B}\cdot\mathbf{C})\mathbf{D}$

(6) $\nabla(fg) = \nabla(gf) = f\nabla g + g\nabla f$

(7) $\nabla\cdot(f\mathbf{A}) = f\nabla\cdot\mathbf{A} + \mathbf{A}\cdot\nabla f$

(8) $\nabla\times(f\mathbf{A}) = f\nabla\times\mathbf{A} + \nabla f\times\mathbf{A}$

(9) $\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot\nabla\times\mathbf{A} - \mathbf{A}\cdot\nabla\times\mathbf{B}$

(10) $\nabla\times(\mathbf{A}\times\mathbf{B}) = \mathbf{A}(\nabla\cdot\mathbf{B}) - \mathbf{B}(\nabla\cdot\mathbf{A}) + (\mathbf{B}\cdot\nabla)\mathbf{A} - (\mathbf{A}\cdot\nabla)\mathbf{B}$

(11) $\mathbf{A}\times(\nabla\times\mathbf{B}) = (\nabla\mathbf{B})\cdot\mathbf{A} - (\mathbf{A}\cdot\nabla)\mathbf{B}$

(12) $\nabla(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A}) + (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A}$

(13) $\nabla^2 f = \nabla\cdot\nabla f$

(14) $\nabla^2\mathbf{A} = \nabla(\nabla\cdot\mathbf{A}) - \nabla\times\nabla\times\mathbf{A}$

(15) $\nabla\times\nabla f = 0$

(16) $\nabla\cdot\nabla\times\mathbf{A} = 0$

Green's Theorem Variants

If V is a volume enclosed by a surface S and $d\mathbf{S} = \mathbf{n}dS$, where $\mathbf{n}$ is the unit normal outward from V,

$$(27)\quad \int_V dV \nabla f = \int_S d\mathbf{S} f$$

$$(28)\quad \int_V dV \nabla \cdot \mathbf{A} = \int_S d\mathbf{S} \cdot \mathbf{A}$$

$$(29)\quad \int_V dV \nabla \cdot \mathsf{T} = \int_S d\mathbf{S} \cdot \mathsf{T}$$

$$(30)\quad \int_V dV \nabla \times \mathbf{A} = \int_S d\mathbf{S} \times \mathbf{A}$$

$$(31)\quad \int_V dV (f\nabla^2 g - g\nabla^2 f) = \int_S d\mathbf{S} \cdot (f\nabla g - g\nabla f)$$

$$(32)\quad \int_V dV (\mathbf{A} \cdot \nabla \times \nabla \times \mathbf{B} - \mathbf{B} \cdot \nabla \times \nabla \times \mathbf{A})$$

$$= \int_S d\mathbf{S} \cdot (\mathbf{B} \times \nabla \times \mathbf{A} - \mathbf{A} \times \nabla \times \mathbf{B})$$

Stokes' Theorem Variants

If S is an open surface bounded by the contour C, of which the line element is $d\mathbf{l}$,

$$(33)\quad \int_S d\mathbf{S} \times \nabla f = \oint_C d\mathbf{l} f$$

$$(34)\quad \int_S d\mathbf{S} \cdot \nabla \times \mathbf{A} = \oint_C d\mathbf{l} \cdot \mathbf{A}$$

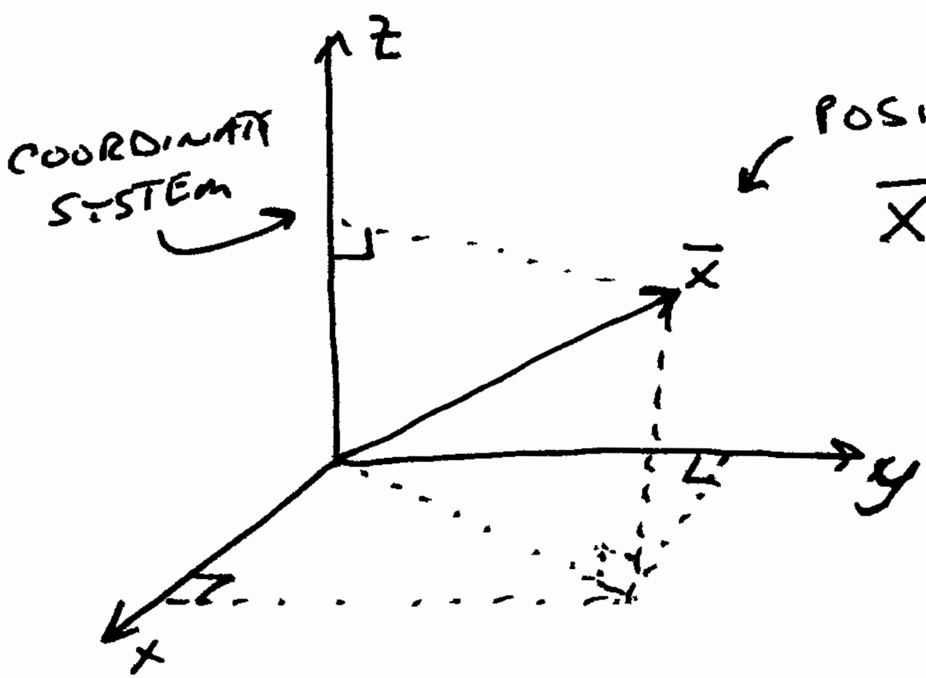
$$(35)\quad \int_S (d\mathbf{S} \times \nabla) \times \mathbf{A} = \oint_C d\mathbf{l} \times \mathbf{A}$$

$$(36)\quad \int_S d\mathbf{S} \cdot (\nabla f \times \nabla g) = \oint_C f dg = -\oint_C g df$$

What is a Tensor?

DEFINITION: A TENSOR (VECTOR) IS A ENTITY THAT TRANSFORMS LIKE A TENSOR (VECTOR) UNDER COORDINATE TRANSFORMATION DUE TO ROTATION. — THAT IS LIKE A POSITION VECTOR, $\bar{X}$.

In other words: a vector has a direction and magnitude …

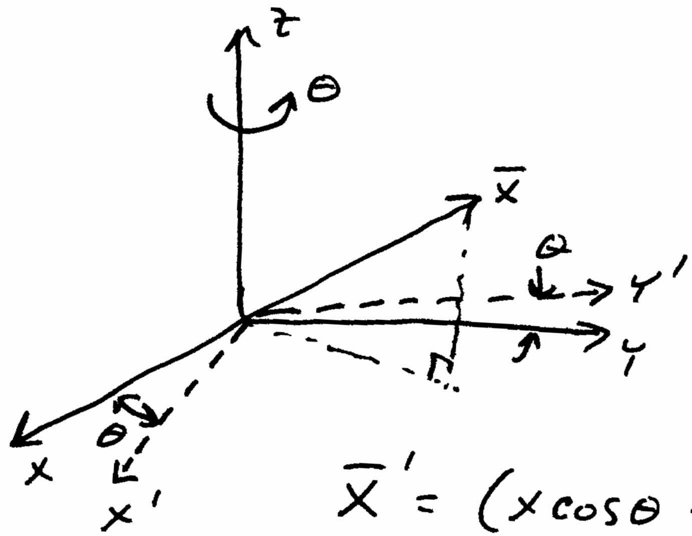


POSITION VECTOR

$$\bar{X} = (x, y, z) = (x_1, x_2, x_3)$$
$$= x_i$$

$i = (1, 2, 3)$

Rotation about the z-Axis



$$\bar{x}' = (x\cos\theta + y\sin\theta,\ -x\sin\theta + y\cos\theta,\ z)$$

OR

$$\bar{x}' = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$x'_j = C_{ij}\, x_i$$ ← CARTESIAN SUMMATION CONVENTION

$$= \sum_{i=1,2,3} C_{ij} x_i$$

DEFINITION: IF $u'_j = C_{ij} u_i$, THEN $\bar{u}$ IS A VECTOR.

Rotation Matrix is Orthogonal

TRANSPOSE: $(C_{ij})^T = C_{ji}$

NOTE: $(C_{ij})^T \cdot C_{ij} = \delta_{ij}$

$$\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \cos^2\theta + \sin^2\theta & \cos\theta\sin\theta - \cos\theta\sin\theta & 0 \\ \cos\theta\sin\theta - \cos\theta\sin\theta & \cos^2\theta + \sin^2\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \delta_{ij}$$ IDENTITY TENSOR

SO $X_i = (C_{ij})^T X'_j = C_{ji} X'_j$

AND $X'_j = C_{ij} X_i$

Vector Examples

- IF $\bar{a}$ IS A VECTOR, THEN $\alpha\bar{a}$ IS A VECTOR IF α IS A SCALAR.
- IF $\bar{a}$, $\bar{b}$, AND $\bar{c}$ ARE VECTORS, THEN

  $\bar{a} + \bar{b}$ IS A VECTOR

  $\bar{a} - \bar{b}$ IS A VECTOR

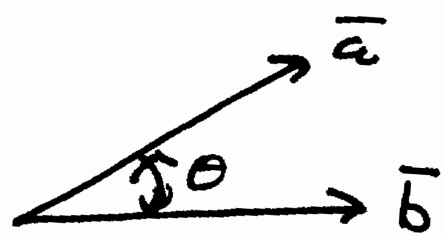
  $\bar{a} + (\bar{b} + \bar{c}) = (\bar{a} + \bar{b}) + \bar{c}$ IS A VECTOR

Scalar Product

$$\bar{a} \cdot \bar{b} = \bar{b} \cdot \bar{a} = |\bar{a}||\bar{b}| \cos\theta$$

= A SCALAR INVARIANT UNDER ROTATION.

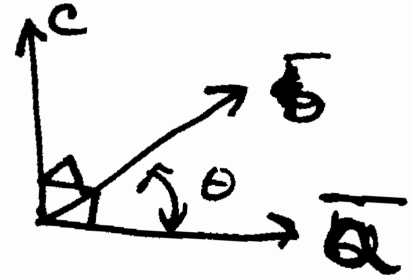
IF $\bar{a} \cdot \bar{b} = 0$, THEN $\bar{a}$ AND $\bar{b}$ ARE MUTUALLY ORTHOGONAL ($\theta = \pm 90°$)



Vector Product

$$\bar{c} = \bar{a} \times \bar{b}$$

$$= |\bar{a}|\,|\bar{b}| \sin\theta$$



NOTE: RIGHT HANDED DEFINITION!

$$(\bar{a} \times \bar{b})_k = \epsilon_{ijk}\, a_i\, b_j\, \hat{e}_k = \begin{vmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

ϵ_{ijk} = PERMUTATION TENSOR

= ISOTROPIC 3RD-ORDER TENSOR

Permutation Tensor

$$\epsilon_{ijk} = \begin{cases} 1 & \text{IF } ijk \text{ ARE CYCLIC} \\ 0 & \text{IF ANY } ijk \text{ ARE REPEATED} \\ -1 & \text{IF } ijk \text{ ARE ANTI-CYCLIC} \end{cases}$$

$$\epsilon_{123} = \epsilon_{312} = \epsilon_{231}$$

$$\epsilon_{123} = -\epsilon_{132}$$

$$\epsilon_{113} = 0$$

ϵ_{ijk} IS AN ISOTROPIC, 3RD-ORDER TENSOR

$$\epsilon_{lmn} = C_{il}\, C_{jm}\, C_{kn}\, \epsilon_{ijk}$$

(ϵ_{ijk} TRANSFORMS TO ITSELF.)

Co-Planar and Not

IF $\bar{a}$, $\bar{b}$, AND $\bar{c}$ ARE CO-PLANAR, THEN

$$\bar{c} = \alpha \bar{a} + \beta \bar{b}$$

$$c_i = \alpha a_i + \beta b_i$$

OR

$$\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} \cdot \begin{pmatrix} \alpha \\ \beta \\ -1 \end{pmatrix} = 0$$

THIS HOMOGENEOUS EQUATION HAS A NON-TRIVIAL SOLUTION ONLY IF

$$\mathrm{DET} \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0 \qquad \left(\text{SAME AS } \bar{c} \cdot (\bar{a} \times \bar{b})\right)$$

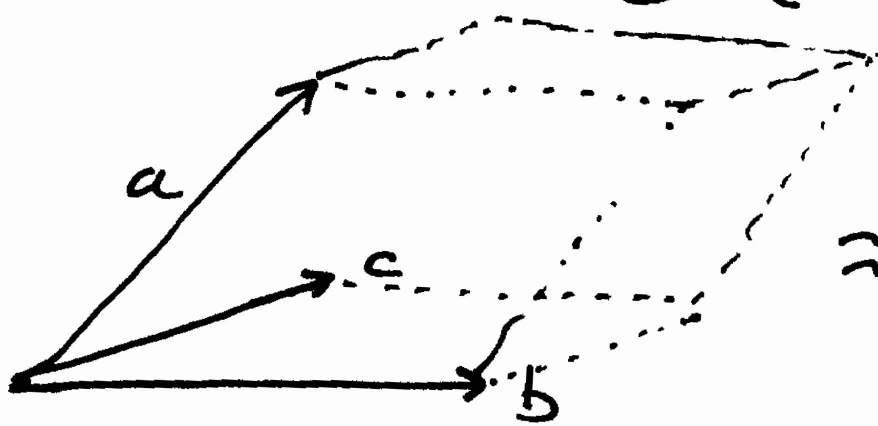
IF $\bar{a}$, $\bar{b}$, $\bar{c}$ ARE <u>NOT</u> CO-PLANAR, THEN

$$\alpha \bar{a} + \beta \bar{b} + \gamma \bar{c} = \bar{d}$$

AND $\bar{a}$, $\bar{b}$, $\bar{c}$ CAN SERVE AS A "BASIS" TO REPRESENT A VECTOR ($\bar{d}$).

Triple Scalar Product

$$\bar{a} \cdot (\bar{b} \times \bar{c}) = \epsilon_{ijk} a_i b_j c_k$$
$$= \bar{b} \cdot (\bar{c} \times \bar{a})$$
$$= \bar{c} \cdot (\bar{a} \times \bar{b})$$



$\approx$ VOLUME OF PARALLELEPIPED WITH SIDES $\bar{a}, \bar{b}, \bar{c}$

IF $\bar{a} \cdot (\bar{b} \times \bar{c}) = 0$, THE VECTORS ARE CO-PLANAR

ε_{ijk} Identity

$$\epsilon_{ijk}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$$

CHECK:

- IF ANY $i=j$, $j=k$, OR $l=m$, THEN ϕ
- IF BOTH CYCLIC

i	j	k	k	l	m	
1	2	3	3	1	2	THEN 1

- IF BOTH ANTI-CYCLIC

i	j	k	k	l	m	
1	3	2	2	1	3	THEN 1

- IF ONE IS ANT-CYCLIC

i	j	k	k	l	m	
1	2	3	3	2	1	THEN -1

PROVE: $(\bar{a}\times\bar{b})\cdot(\bar{a}\times\bar{b}) = |\bar{a}|^2|\bar{b}|^2 - (\bar{a}\cdot\bar{b})^2$

$$\epsilon_{ijk}a_i b_j \epsilon_{lmk}a_l b_m = (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})a_i b_j a_l b_m$$

$$= a^2 b^2 - (a\cdot b)^2$$

Triple Vector Product

$$\bar{a} \times (\bar{b} \times \bar{c}) = \epsilon_{ijk} \epsilon_{klm} a_j b_l c_m$$

$$= b_i (a_j c_j) - c_i (a_j b_j)$$

$$= (\bar{a} \cdot \bar{c}) \bar{b} - (\bar{a} \cdot \bar{b}) \bar{c}$$

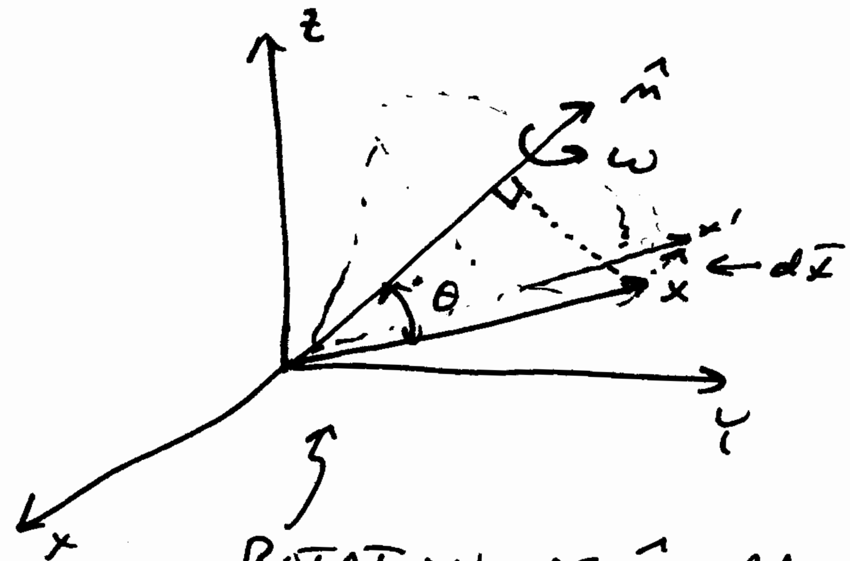
IF $\hat{n}$ IS A UNIT VECTOR, AND $\bar{a}$ A VECTOR, THEN

$$\bar{a} = (\bar{a} \cdot \hat{n}) \hat{n} + \hat{n} \times (\bar{a} \times \hat{n})$$

PROOF: $\hat{n} \times (\bar{a} \times \hat{n}) = \bar{a} \underbrace{(\hat{n} \cdot \hat{n})}_{=1} - \hat{n} (\hat{n} \cdot \bar{a})$

THUS, $\bar{a}$ CAN BE RESOLVED INTO A COMPONENT ALONG $\hat{n}$ AND ONE PERPENDICULAR TO $\hat{n}$.

Rigid Rotation



ROTATION OF $\hat{x}$ ABOUT VECTOR $\hat{n}$.

$$|d\bar{x}| = |x \sin\theta \, \omega \, dt|$$

$$\frac{d\bar{x}}{dt} = \bar{u} = \bar{\omega} \times \bar{x} = \text{VELOCITY OF ROTATION}$$

$$\bar{\omega} = \hat{n}\,\omega$$

Tensors

$$A_{mn} = C_{im} C_{jn} A_{ij}$$

DEFINES A TENSOR.

WHILE 3 COMPONENTS/VALUES ARE NEEDED TO DEFINE A VECTOR, 9 COMPONENTS/VALUES ARE NEEDED TO DEFINE A 2ND-ORDER TENSOR.

ISOTROPIC 2ND-ORDER TENSOR IS

$$\delta_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$ SAME IN ALL COORDINATE SYSTEMS

$$\begin{aligned}
\delta_{lm} &= C_{il} C_{jm} \delta_{ij} \\
&= C^T_{li} \delta_{ij} C_{jm} \\
&= C^T_{lj} C_{jm} \\
&= \delta_{lm}
\end{aligned}$$

Symmetric and Antisymmetric

IF $A_{ij} = A_{ji}$, THEN $\bar{\bar{A}}$ IS SYMMETRIC.

IF $A_{ij} = -A_{ji}$, THEN $\bar{\bar{A}}$ IS ANTI-SYMMETRIC.

ALL TENSORS CAN BE DECOMPOSED AS...

$$A_{ij} = \underbrace{\tfrac{1}{2}\left(A_{ij} + A_{ji}\right)}_{\text{SYMMETRIC PART}} + \underbrace{\tfrac{1}{2}\left(A_{ij} - A_{ji}\right)}_{\text{ANTI-SYMMETRIC PART}}$$

SYMMETRIC TENSOR HAS 6 INDEPENDENT VALUES

ANTI-SYMMETRIC TENSOR HAS 3 INDEPENDENT VALUES

$$S_{ij} = \begin{pmatrix} 1 & 4 & 5 \\ 4 & 2 & 6 \\ 5 & 6 & 3 \end{pmatrix} \qquad A_{ij} = \begin{pmatrix} 0 & -1 & 2 \\ 1 & 0 & -3 \\ -2 & 3 & 0 \end{pmatrix}$$

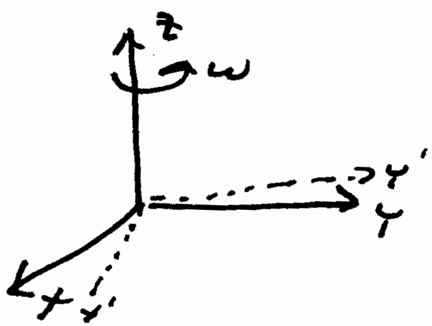
Antisymmetric Tensor

EVERY ANTISYMMETRIC TENSOR IS ASSOCIATED WITH A VECTOR...

$$\bar{\bar{R}} = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix} \qquad \bar{\omega} = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix}$$

$$R_{ij} = -\epsilon_{ijk}\,\omega_k \qquad \omega_k = -\tfrac{1}{2}\epsilon_{ijk} R_{ij}$$

EXAMPLE: ROTATION ABOUT Z-AXIS



$$\bar{\omega} = \begin{pmatrix} 0 \\ 0 \\ \omega_0 \end{pmatrix}$$

$$\bar{\bar{R}} = \begin{pmatrix} 0 & -\omega_0 & 0 \\ \omega_0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\bar{x}(t+\delta t) = \bar{x}(t) + \delta t\,\bar{\bar{R}}\cdot\bar{x}(t)$$

Time Derivative

IF $A_{\ell m}(t)$ IS A TENSOR, THEN SO ARE ALL TIME DERIVATIVES...

$$\frac{d^n}{dt^n} A_{\ell m}(t) = C_{i\ell}\, C_{jm} \frac{d^n}{dt^n} A_{ij}(t)$$

ALSO

$$\bar{u}(t) = \frac{d\bar{x}(t)}{dt} \quad \text{IS A VECTOR}$$

$$\frac{d}{dt}(\bar{a}\cdot\bar{b}) = \frac{d\bar{a}}{dt}\cdot\bar{b} + \bar{a}\cdot\frac{d\bar{b}}{dt}$$

PROVE: IF ACCELERATION IS PERPENDICULAR TO VELOCITY, THEN $|\bar{u}|$ IS A CONSTANT

$$\frac{d}{dt}(u^2) = \frac{d}{dt}(\bar{u}\cdot\bar{u}) = 2\,\bar{u}\cdot\frac{d\bar{u}}{dt} = 0$$

Vector Fields and Trajectory Lines

$\overline{U}(x_1, x_2, x_3, t)$ IS A VECTOR FIELD.

AT EVERY POINT IN SPACE, THERE IS A VECTOR (AND IT MAY CHANGE IN TIME.)

ASSOCIATED WITH ANY VECTOR FIELD ARE "TRAJECTORIES" (LIKE STREAMLINES FOR FLUID FLOW) WHICH ARE THE CURVES EVERYWHERE TANGENT TO LOCAL FIELD.

EXAMPLE:

$$\frac{d\overline{x}}{ds} = \overline{a}(\overline{x}) \quad \text{OR} \quad \frac{dx_i}{ds} = a_i(x_1(s), x_2(s), x_3(s))$$

WHERE s IS A LENGTH PARAMETER ALONG THE CURVE.

$$\frac{dx_i}{a_i} = ds \quad \text{FOR ALL } i$$

Gradient Operator (Scalar)

$\varphi(x)$ IS A SCALAR, THEN

$$\frac{\partial \varphi}{\partial x_i} = \nabla \varphi \quad \text{IS A VECTOR}$$

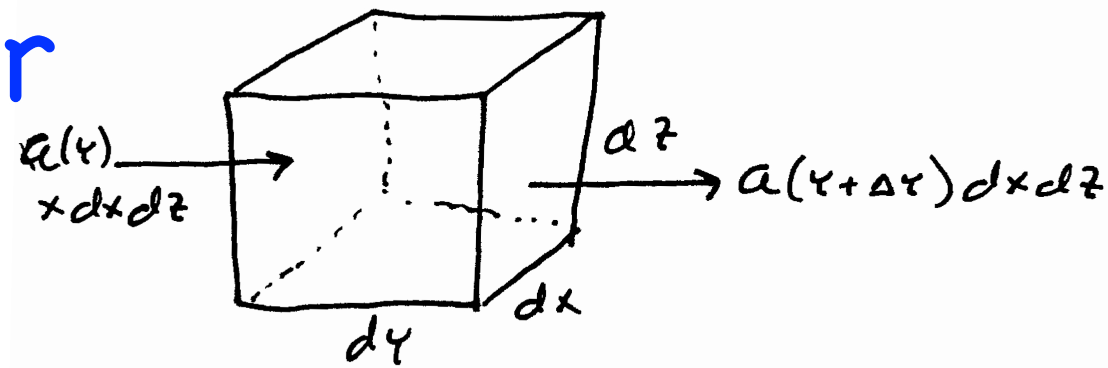
$$\frac{\partial \varphi}{\partial x_j'} = \frac{\partial \varphi}{\partial x_i} \frac{\partial x_i}{\partial x_j'} = C_{ij} \frac{\partial \varphi}{\partial x_i}$$

SO $\bar{\nabla} \varphi$ TRANSFORMS LIKE A VECTOR.

Divergence of a Vector Field

$$\bar{\nabla}\cdot\bar{a} = \frac{\partial a_i}{\partial x_i} = \frac{\partial a_1}{\partial x_1} + \frac{\partial a_2}{\partial x_2} + \frac{\partial a_3}{\partial x_3} \equiv a_{i,i}$$

COMMA MEANS DERIVATIVE



NET FLOW INTO VOLUME ALONG Y IS

$$\left[a_y(y+\Delta y) - a_y(y)\right] dx\,dz$$

$$= \frac{\partial a_y}{\partial y}\,dx\,dy\,dz \quad (+\ 4\ \text{OTHER SIDES})$$

$$\text{So } \nabla\cdot\bar{a} = \lim_{dx\,dy\,dz \to 0} \frac{1}{\Delta x \Delta y \Delta z} \iiint_{\Delta x\,dy\,dz} \bar{a}\cdot\hat{n}\,dS$$

UNIT SURFACE NORMAL

IF $\nabla\cdot\bar{a} = 0$, THEN VECTOR FIELD IS SOLENOIDAL (LIKE $\bar{B}$)

Laplacian

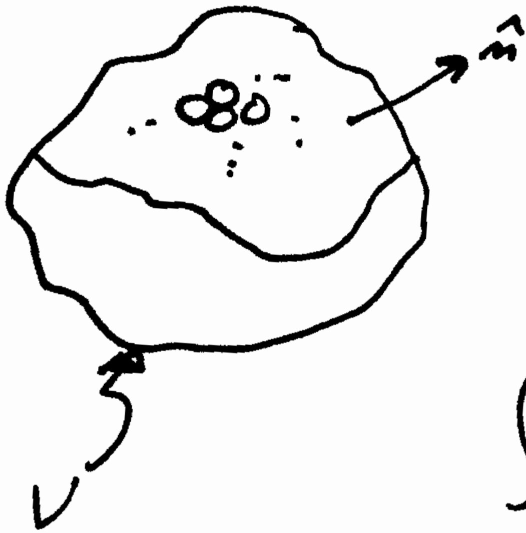
IF $\nabla \cdot \bar{a} = 0$, THEN A POTENTIAL FUNCTION EXISTS WITH $\bar{a} = \nabla \varphi$ AND

$$\nabla^2 \varphi = \bar{\nabla} \cdot \bar{\nabla} \varphi = \frac{\partial^2 \varphi}{\partial x_i \partial x_i} = 0$$

$$\bar{\nabla} \cdot \bar{\nabla} \varphi = \lim_{dx\,dy\,dz \to 0} \frac{1}{\Delta x \Delta y \Delta z} \iint_{\Delta x \Delta y \Delta z} \nabla \varphi \cdot \hat{n} \, dS$$

= SUM OF "GRADIENT FLUX" FROM VOLUME

Green's (or Gauss') Theorem



SUBDIVIDE VOLUME INTO A COLLECTION OF INFINITESIMAL VOLUME ELEMENTS. SUM.

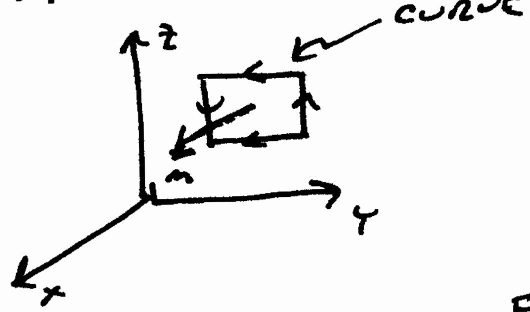
$$\iiint_V \bar{\nabla} \cdot \bar{a} \, dv = \iint_S \bar{a} \cdot \hat{n} \, dS$$

Curl of a Vector Field

$$\nabla \times \bar{a} = \epsilon_{ijk} \frac{\partial a_k}{\partial x_j} \hat{e}_i$$

$$= \left(\frac{\partial a_3}{\partial x_2} - \frac{\partial a_2}{\partial x_3} , \frac{\partial a_1}{\partial x_3} - \frac{\partial a_3}{\partial x_1} , \frac{\partial a_2}{\partial x_1} - \frac{\partial a_1}{\partial x_2} \right)$$

EXPLANATION...



LET $\bar{t}$ = TANGENT UNIT VECTOR TO CURVE

s = LENGTH PARAMETER ALONG CURVE

FIND $\oint \bar{a} \cdot \bar{t}\, ds$ AS $\Delta y, \Delta z \to 0$

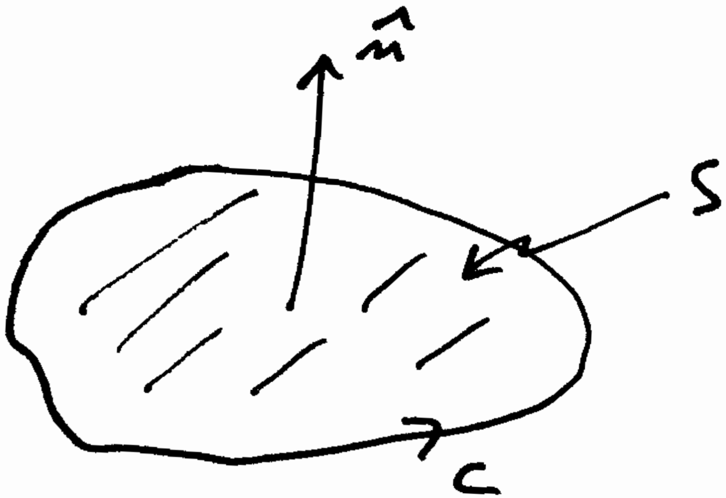
RIGHT + LEFT $= \left[a_z(y+\Delta y) - a_z(y) \right] dz = \frac{\partial a_z}{\partial y} dy\, dz$

TOP + BOTTOM $= \left[- a_y(z+\Delta z) + a_y(z) \right] dy = \frac{\partial a_y}{\partial z} dy\, dz$

SO $\nabla \times \bar{A} = \lim_{AREA \to 0} \frac{1}{AREA} \oint \bar{a} \cdot \bar{t}\, ds$

A VECTOR FIELD WITH $\nabla \times \bar{a} = 0$ IS IRROTATIONAL.

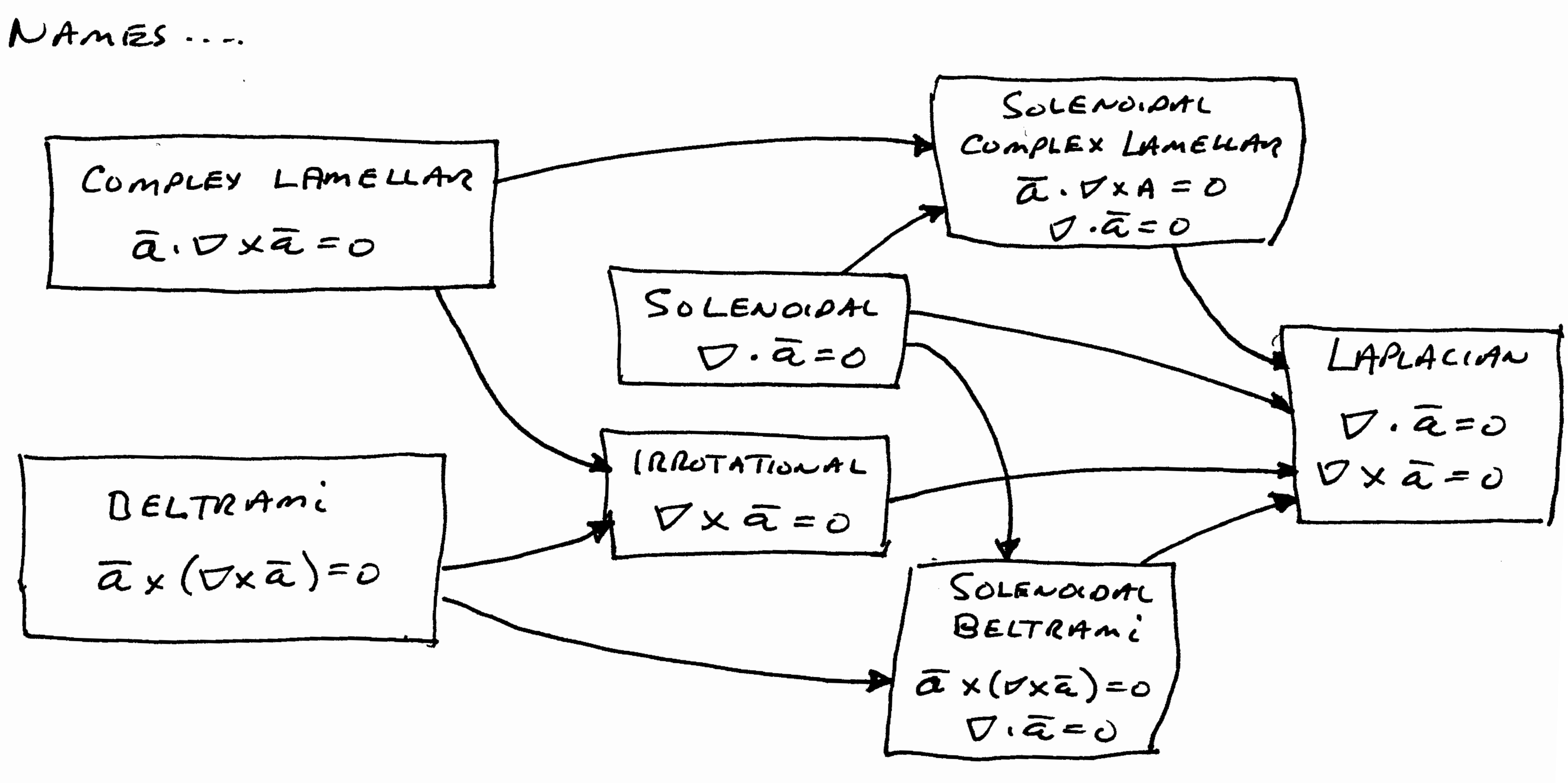
Stokes' Theorem



$$\iint_S \nabla \times \bar{a} \cdot \hat{n}\, ds = \oint_C \bar{a} \cdot \hat{t}\, ds$$

(BY SUMMING INFINTESIMAL SURFACE ELEMENTS)

Classification of Vector Fields



Irrotational Field

$$\nabla \times \bar{a} = 0$$

LET $\bar{a} = \nabla \varphi$, φ = POTENTIAL

THEN $\nabla \times \nabla \varphi = 0 = \epsilon_{ijk} \dfrac{\partial^2 \varphi}{\partial x_i \partial x_j} \hat{e}_k$

Solenoidal Field

$$\nabla \cdot \bar{a} = 0$$

LET $\bar{a} = \nabla\varphi \times \nabla\psi$ OR

$$= \nabla \times (\varphi \nabla \psi)$$

THEN, SINCE

$$\nabla \cdot (\nabla \times \bar{a}) = \epsilon_{ijk} \frac{\partial^2 a_j}{\partial x_i \partial x_i} = 0$$

OR

$$\nabla \cdot (\nabla\varphi \times \nabla\psi) = \epsilon_{ijk} \left(\frac{\partial^2 \varphi}{\partial x_i \partial x_k} \frac{\partial \psi}{\partial x_j} + \frac{\partial \varphi}{\partial x_i} \frac{\partial^2 \psi}{\partial x_k \partial x_i} \right)$$

$$= 0$$

WHERE ψ = STREAM FUNCTION IF φ IS A SYMMETRY DIRECTION.

Summary

- Vectors & tensors transform under coordinate rotation like position vector
- Vector operators: scalar product, vector product, triple scalar product, triple vector product
- Tensors: isotropic, symmetric, antisymmetric, orthogonal
- Calculus of vectors: derivative, gradient, divergence, curl
- Gauss' & Stokes' Theorems
- Classification of Vector Fields
- Next Lecture: Kinematics of fluids