Examples: Fluid Flow

- Continuity (mass conservation)
- Euler equation (force and acceleration)
- Vorticity ($\nabla \times \mathbf{U}$, swirls!!)
- Bernoulli’s Principle (Lift and pressure)
- Visualizing flow (tracing lines...)
- Flow at corner
- Flow around cylinder
Fluid Continuity

(Conservation of Mass)

\[ \vec{u} = (u, v, w) \]

\[ \frac{\Delta P}{\Delta t} \rho \sim \left( \frac{v P_{x+\Delta x} - v P_{x}}{\Delta y} \right) \sim \frac{2}{\Delta y} (v \rho) \]

For all 3 directions

\[ \frac{\partial \rho}{\partial t} = - \nabla \cdot (\rho \vec{u}) = - \sum_{i} \frac{2}{\partial x_i} (\rho u_i) \]

\[ = \frac{2}{\partial x} (\rho u_x) - \frac{2}{\partial y} (\rho u_y) - \frac{2}{\partial z} (\rho u_z) \]

If \( \rho = \text{constant} \) (incompressible) then

\[ \nabla \cdot (\rho \vec{u}) = \nabla \cdot \vec{u} = 0 \quad \text{"incompressible flow"} \]

Newton's Law for a Fluid Element

\[ \rho \times \text{(ACCELERATION)} = \text{FORCE} \]

\[ = - \nabla \rho + \rho \vec{g} + \ldots \]

\[ = - \nabla (\rho + \rho \varphi) + \ldots \]

What is acceleration?

\[ \text{Gravitational Potential} \]

\[ \text{Accelerations} = \frac{\vec{u}_2 - \vec{u}_1}{\Delta t} \quad \text{as } \Delta t \to 0 \]

\[ \vec{u}_2 = \vec{u} (x + u dt, y + v dt, z + w dt, t + \Delta t) \]

\[ \approx \vec{u} (x, y, z, t) + \frac{2u}{\partial x} \Delta t + \frac{2v}{\partial y} \Delta t \]

\[ + \frac{2w}{\partial z} \Delta t + \frac{2u}{\partial x} u dt + \frac{2v}{\partial y} v dt + \ldots \]

So

\[ \text{ACCELERATION} = \frac{2u}{\partial x} + (\vec{u} \cdot \nabla) \vec{u} \quad \text{and Newton's Law is} \]

\[ \frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla) \vec{u} = - \frac{1}{\rho} \nabla P - \nabla \varphi \]
**Vorticity and Definitions**

From Problem #3 in 1st Problem Set

\[(\nabla \times \vec{u}) \cdot \vec{u} = (\nabla \times \vec{u}) \times \vec{u} + \frac{1}{2} \nabla \cdot (\vec{u} \times \vec{u})\]

So that

\[\vec{\omega} = \nabla \times \vec{u} \equiv \text{vorticity}\]

\[\frac{\partial \vec{u}}{\partial t} + \vec{\omega} \times \vec{u} + \frac{1}{2} \nabla \cdot \vec{u}^2 = -\frac{\nabla \rho}{\rho} - \nabla \phi\]

If \[\vec{\omega} = 0\], then flow is "IRROTATIONAL".

IRROTATIONAL AND INCOMPRESSIBLE FLOWS SATISFY

\[\nabla \cdot \vec{u} = 0 \quad \nabla \times \vec{u} = 0\]

Just like equations for electrostatics and magnetostatics in a charge-free or current-free region.

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**Bernoulli’s Principle**

For STEADY, INCOMPRESSIBLE FLOW...

\[\vec{u} \cdot \left[ \frac{1}{2} \frac{\partial \rho}{\partial t} + \nabla \times \vec{u} \right] = -\nabla \left( \frac{1}{2} \rho \vec{u}^2 + \frac{p}{\rho} + \phi \right)\]

or

\[\vec{u} \cdot \left( \frac{1}{2} \rho \vec{u}^2 + \frac{p}{\rho} + \phi \right) = 0\]

**Bernoulli’s Law** states that

\[\frac{1}{2} \rho \vec{u}^2 + \frac{p}{\rho} + \phi = \text{constant along a streamline}\]

Furthermore, if \[\vec{\omega} = 0\], then

\[\frac{1}{2} \rho \vec{u}^2 + \frac{p}{\rho} + \phi = \text{constant everywhere!}\]

Very important!  Airplane wing lift, very useful pressure-flow relationship!

(Can be shown to be equivalent to conservation of energy.)
Visualizing Fluid Flow

- **STREAMLINES** = instantaneous curves everywhere tangent to flow
  \[ \mathbf{u}(x, y, t) \]

- **PATHLINES** = trajectory of a fluid element as it travels. (tracer)
  Pathlines \( \cong \) streamlines in steady flow

- **STREAKLINES** = "smoke lines"
  = the location of all of the path ends that initiated at a fixed point

Two examples:
1. Flow at a corner
2. Flow around a cylinder

Two examples...
1. Flow at a corner
2. Flow around a cylinder

\[ \frac{\partial \mathbf{u}}{\partial t} = 0 \]

2D steady flow \( \mathbf{u} = (u_x, u_y, 0) \)

\[ \nabla \cdot \mathbf{u} = 0 \]

\[ \nabla \times \mathbf{u} = 0 \]

Flow at a Corner

Two solutions methods (really equivalent)

- **Velocity Potential**
  \[ \mathbf{u} = \mathbf{\nabla} \Phi \]
  \[ \nabla \cdot \mathbf{u} = 0 \]
  \[ \nabla \times \mathbf{u} = 0 \]

  (like electro- or magnetostatics)

- **Stream Function**
  \[ \mathbf{u} = \mathbf{\nabla} \times \Psi = (\frac{-2y}{2^2}, \frac{2x}{2^2}, 0) \]
  \[ \nabla \cdot (\mathbf{\nabla} \times \Psi) = 0 \]
  \[ \nabla \times (\mathbf{\nabla} \times \Psi) = \nabla^2 \Psi = 0 \]
Boundary Conditions

- Normal flow to wall must vanish
- Flow at $x=L \rightarrow \mathbf{u} = (-u_0, 0, 0)$
- Flow at $y=L \rightarrow \mathbf{u} = (0, u_0, 0)$

Solution Using Velocity Potential
(and Streamlines)

\[
\nabla^2 \psi = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0
\]

\[
\frac{\partial \psi}{\partial y} = 0 \text{ at } y = 0
\]

\[
\frac{\partial \psi}{\partial x} = 0 \text{ at } x = 0
\]

\[
\psi(x, y) = c_1 (x^2 - y^2)
\]

\[
u_x = 2c_1 x \quad \nu_y = -2c_1 y
\]

Boundary conditions at $x, y = L$ ...

Set $c_1 = -u_0/2L$, so $\psi(x, y) = -\frac{u_0}{2L} (x^2 - y^2)$

Streamlines

\[
\frac{dx}{u_x dt} = \frac{dy}{u_y dt}
\]

\[
\frac{dx}{x} = \frac{dy}{y} \Rightarrow \ln y = -\ln x + c \quad \text{or} \quad xy = \text{constant}
\]
Solution Using Streamfunction

\[ \vec{U} = \nabla \psi = \left( -\frac{\partial \psi}{\partial x}, \frac{\partial \psi}{\partial y}, 0 \right) \]

\[ \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0 \quad \text{AND} \quad \frac{\partial \psi}{\partial x} = 0 \quad \text{AT} \quad x = 0 \]

\[ \frac{\partial \psi}{\partial y} = 0 \quad \text{AT} \quad y = 0 \]

So \[ \psi(x, y) = C_1 x y \]

\[ U_x = -U_0 \quad \text{AT} \quad x = L \quad \Rightarrow \quad C_1 = \frac{U_0}{L} \]

\[ \psi(x, y) = \frac{U_0}{L} x y \]

\[ \psi = \text{constant is a streamline} \]

NOTE: Lines of constant \( \psi \), \( \psi \) are orthogonal.

\[ \nabla \psi \cdot \nabla \psi = 0 \quad \text{--- can you prove this?} \]

\[ \nabla \psi \cdot (\nabla \times \nabla \psi) = \nabla^2 \psi \]

What is the Pressure Along the Corner?

At \( x = L \), fluid is characterized by \( U_0, P_0 \).

Use Bernoulli’s Law...

\[ \frac{1}{2} \rho U_0^2 + P_0 = \frac{1}{2} \rho U^2 + P(x, y) \]

But \( U = \vec{U} \cdot \vec{U} = \frac{U_0^2}{L^2} (x^2 + y^2) \) AND

Along lower side \( (y = 0) \)

\[ P(x, y) = P_0 + \frac{1}{2} \rho \left( U_0^2 - U^2 \right) \]

\[ = P_0 + \frac{1}{2} \rho U_0^2 \left( 1 - \frac{x^2}{L^2} \right) \]

Pressure increases into corner

\[ P(0, 0) = P_0 + \frac{1}{2} \rho U_0^2 \]
Rotating Cylinder

Let's investigate the lift of a rotating cylinder by using a Java simulator.

http://www.grc.nasa.gov/WWW/K-12/airplane/cyl.html

Flow Around a Cylinder

(See Ch. 6-9)
Velocity Potential Solution

\[ \varphi(r, \theta) = (c_1 r + \frac{c_2}{r}) \cos \theta \]

\[ \frac{2 \varphi}{r} \bigg|_{r=a} = 0 = (c_1 - \frac{c_2}{a^2}) \cos \theta \Rightarrow c_2 = c_1 a^2 \]

As \( r \to \infty \)

\[ U_a = \frac{2 \varphi}{2a} = c_1 \cos \theta = u_0 \cos \theta \quad \Rightarrow c_1 = u_0 \]

Thus,

\[ \varphi(r, \theta) = U_0 \left( r + \frac{a^2}{r} \right) \cos \theta \]

\[ U_a(r, \theta) = U_0 \left( 1 - \frac{a^2}{r^2} \right) \cos \theta \]

\[ U_b(r, \theta) = -U_0 \left( 1 + \frac{a^2}{r^2} \right) \sin \theta \]

What is the Pressure at the Surface of the Cylinder?

\[ U^2(\lambda=a) = U_0^2 = 4 U_0^2 \sin^2 \theta \]

Bernoulli’s Law gives

\[ p(\lambda=a, \theta) = p_\infty + \frac{1}{2} p \left( U_0^2 - 4 U_0^2 \sin^2 \theta \right) \]

Low Pressure: \( p_\infty - 2 p U_0^2 \)

High Pressure: \( p_\infty + \frac{1}{2} p U_0^2 \)

Low Pressure at \( \theta = 90^\circ, 270^\circ \)

Highest Pressure at \( \theta = 0, 180^\circ \)
A Solution with Circulation

\[ \overline{U} = (0, \theta, 0) \]

\[ U_\theta(\eta) = \frac{1}{\eta} \cdot \frac{2}{2} \frac{\eta}{\eta} \quad \text{and} \quad \nabla^2 \phi = 0 = \frac{1}{\eta} \cdot \frac{2}{2} \left( \frac{\eta}{\eta} \right) \]

Solution: \[ U_\theta(\eta) = \frac{C_i}{\eta} \quad \phi = C_i \theta \]

**Define Circulation About Cylinder As**

\[ \Gamma = \int_{\Gamma} \overline{U} \cdot d\overline{s} = \int_0^{2\pi} U_\theta \eta d\theta = 2\pi C_i \]

So \[ U_\theta(\eta) = \frac{\Gamma}{2\pi \eta} \quad \text{where} \quad \Gamma = \text{circulation} \]

Equations for Incompressible and Irrotational Flow are Linear

\[ \nabla \cdot \overline{U} = 0 \quad \nabla \times \overline{U} = 0 \]

So if \( \overline{U}_1 \) and \( \overline{U}_2 \) are solutions then also \( \overline{U}_3 = \overline{U}_1 + \overline{U}_2 \) is a solution.
Add Circulation to Flow

*Add Circulation to Flow past Cylinder...*

\[
U_a (r, \theta) = U_0 \left(1 - \frac{r^2}{a^2}\right) \cos \theta \\
U_b (r, \theta) = -U_0 \left(1 + \frac{r^2}{a^2}\right) \sin \theta + \frac{P}{2 \pi a}
\]

*Boussinesq's Law at Surface...*

\[
\frac{1}{2} \rho u^2 = \frac{1}{2} \rho \left[2 U_0 \sin \theta - \frac{P}{2 \pi a}\right]^2
\]

Summary

- Continuity (incompressible flow and the Boussinesq approximation)
  \[\nabla \cdot \mathbf{U} \approx 0\]

- Force and acceleration (Euler equation)
  \[
  \frac{\partial \mathbf{U}}{\partial t} + (\mathbf{U} \cdot \nabla) \mathbf{U} = -\frac{1}{\rho} \nabla P - \nabla \phi_g
  \]

- Vorticity (\(\nabla \times \mathbf{U}\), swirls!)
  \[
  \frac{d\mathbf{U}}{dt} = \frac{\partial \mathbf{U}}{\partial t} + \Omega \times \mathbf{U} + \frac{1}{2} \nabla \mathbf{U}^2
  \]

- Bernoulli's Principle (Lift and pressure)
  \[
  \mathbf{U} \cdot \nabla \left(\frac{1}{2} \mathbf{U}^2 + \frac{P}{\rho} + \phi_g\right) \approx 0
  \]

- Visualizing flow (tracing lines...)

- 2D Euler flow (inviscid)

*Important: B.P. can be used to find pressure from flow!*
