APPH 4200
Physics of Fluids

Cartesian Tensors (Ch. 2)
Lecture 2a

$$\text{Part 1}: \begin{cases} \bullet \text{ Last Lecture: Review} \\ \bullet \text{ Example Fluid Flow Problems} \end{cases}$$

$$\text{Part 2}: \begin{cases} \bullet \text{ Geometric Identities} \\ \bullet \text{ Vector Calculus} \end{cases}$$

Hydrostatics of a Sphere

Gradient of a scalar

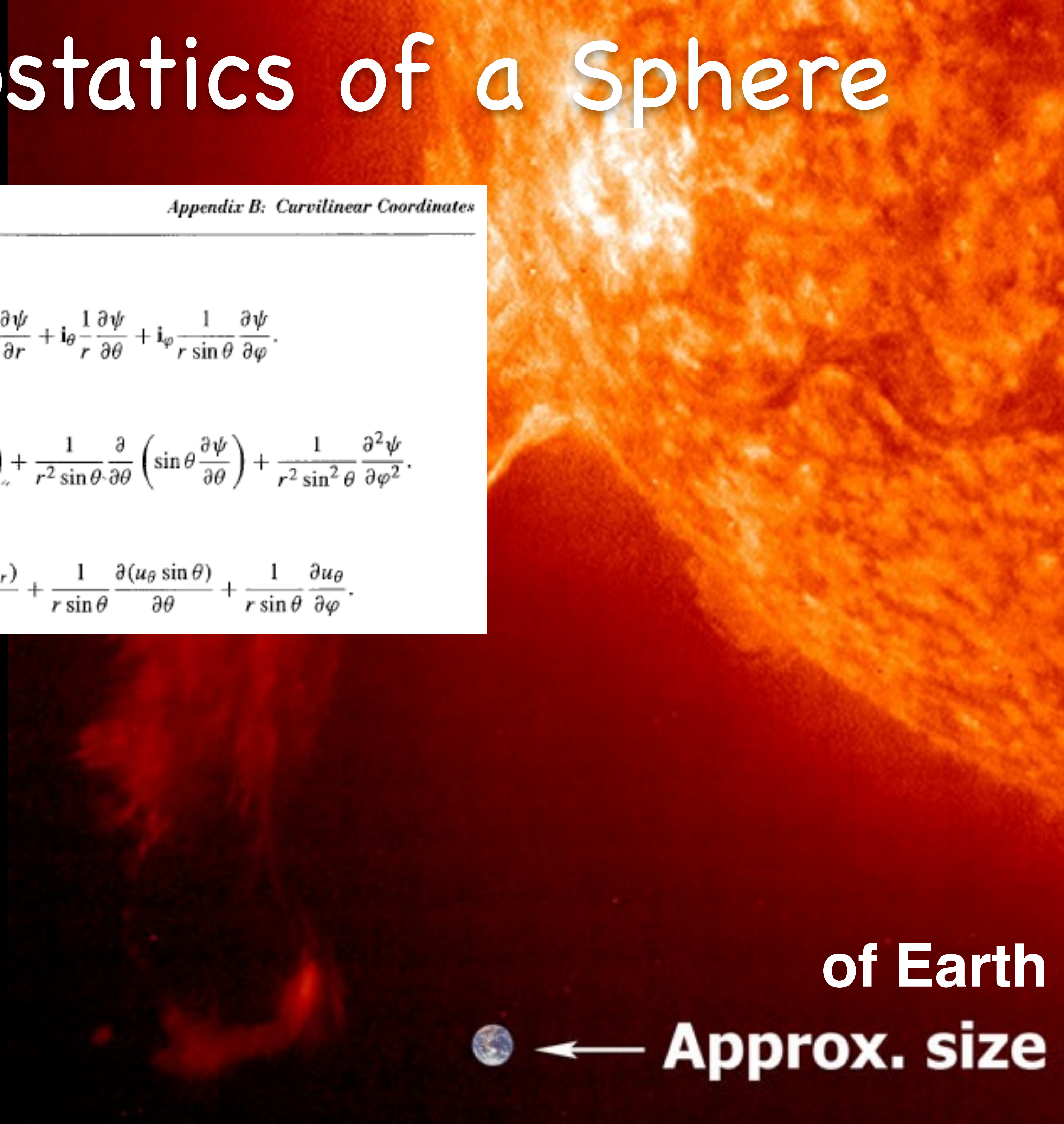
$$\nabla \psi = \mathbf{i}_r \frac{\partial \psi}{\partial r} + \mathbf{i}_\theta \frac{1}{r}\frac{\partial \psi}{\partial \theta} + \mathbf{i}_\varphi \frac{1}{r \sin\theta}\frac{\partial \psi}{\partial \varphi}.$$

Laplacian of a scalar

$$\nabla^2 \psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \psi}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \psi}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 \psi}{\partial \varphi^2}.$$

Divergence of a vector

$$\nabla \cdot \mathbf{u} = \frac{1}{r^2}\frac{\partial (r^2 u_r)}{\partial r} + \frac{1}{r \sin\theta}\frac{\partial (u_\theta \sin\theta)}{\partial \theta} + \frac{1}{r \sin\theta}\frac{\partial u_\theta}{\partial \varphi}.$$



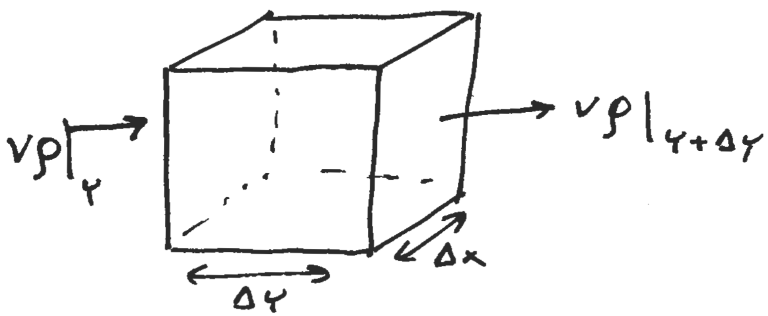
Examples: Fluid Flow

- Continuity (mass conservation)
- Euler equation (force and acceleration)
- Vorticity ($\nabla \times U$, swirls!!)
- Bernoulli's Principle (Lift and pressure)
- Visualizing flow (tracing lines...)
- Flow at corner
- Flow around cylinder

Fluid Continuity

(Conservation of Mass)

$\bar{u} = (u, v, w)$



$$\left.\frac{\Delta \rho}{\Delta t}\right|_y \sim - \frac{\left(v\rho|_{y+\Delta y} - v\rho|_y\right)}{\Delta y}$$

$$\sim - \frac{\partial}{\partial y}(v\rho)$$

FOR ALL 3 DIRECTIONS

$$\frac{\partial \rho}{\partial t} = - \nabla \cdot (\rho \bar{u}) = - \sum_i \frac{\partial}{\partial x_i}(\rho u_i)$$

$$= - \frac{\partial}{\partial x}(\rho u_x) - \frac{\partial}{\partial y}(\rho u_y) - \frac{\partial}{\partial z}(\rho u_z)$$

IF ρ = CONSTANT (BOUSSINESQ) THEN

$\nabla \cdot (\rho \bar{u}) = \nabla \cdot \bar{u} = 0$ "INCOMPRESSIBLE FLOW"

Newton's Law for a Fluid Element

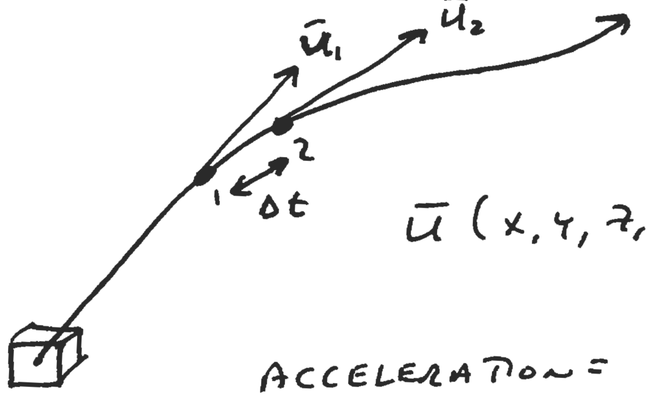
$\rho \times (\text{ACCELERATION}) = \text{FORCE}$

$$= -\nabla p + \rho \bar{g} + \cdots$$

$$= -\nabla (p + \rho \varphi) + \cdots$$

↑ GRAVITATIONAL POTENTIAL

WHAT IS ACCELERATION?



$\bar{u}(x, y, z, t)$

$$\text{ACCELERATION} = \frac{\bar{u}_2 - \bar{u}_1}{\Delta t} \quad \text{AS } \Delta t \to 0$$

$$\bar{u}_2 = \bar{u}(x + u_x dt,\ y + u_y dt,\ z + u_z dt,\ t + \delta t)$$

$$\approx \bar{u}(x, y, z, t) + \frac{\partial \bar{u}}{\partial t}\delta t + \frac{\partial \bar{u}}{\partial x} u_x dt + \frac{\partial \bar{u}}{\partial y} u_y dt + \frac{\partial \bar{u}}{\partial z} u_z dt + \cdots$$

CONVECTIVE OR ADVECTIVE DERIVATIVE

SO

$$\text{ACCELERATION} = \frac{\partial \bar{u}}{\partial t} + (\bar{u} \cdot \bar{\nabla})\bar{u}$$ AND NEWTON'S LAW IS

$$\boxed{\frac{\partial \bar{u}}{\partial t} + (\bar{u} \cdot \bar{\nabla})\bar{u} = -\frac{1}{\rho}\bar{\nabla} p - \nabla \varphi}$$

Vorticity and Definitions

FROM PROBLEM #3 IN 1ST PROBLEM SET

$$(\bar{u}\cdot\bar{\nabla})\bar{u} = (\nabla\times\bar{u})\times\bar{u} + \tfrac{1}{2}\bar{\nabla}(\bar{u}\cdot\bar{u})$$

SO THAT

$\bar{\Omega} = \nabla\times\bar{u} \equiv$ VORTICITY

$$\frac{\partial\bar{u}}{\partial t} + \bar{\Omega}\times\bar{u} + \tfrac{1}{2}\bar{\nabla}u^2 = -\frac{\nabla P}{\rho} - \nabla\varphi$$

IF $\bar{\Omega} = 0$, THEN FLOW IS "IRROTIONAL"

IRROTATIONAL AND INCOMPRESSIBLE FLOWS SATISFY

$$\nabla\cdot\bar{u} = 0 \qquad \nabla\times\bar{u} = 0$$

JUST LIKE EQUATIONS FOR ELECTROSTATICS AND MAGNETOSTATICS IN A CHARGE-FREE OR CURRENT-FREE REGION.

Bernoulli's Principle

FOR STEADY, INCOMPRESSIBLE FLOW...

$$\bar{u} \cdot \Big[\underbrace{\frac{\partial \varphi}{\partial t}}_{\to 0} + \underbrace{\bar{\Omega}}_{\to 0 \text{ why?}} \times \bar{u} = -\bar{\nabla}\left(\frac{1}{2}u^2 + \frac{p}{\rho} + \varphi\right)\Big]$$

OR

$$\bar{u} \cdot \bar{\nabla}\left(\frac{1}{2}u^2 + \frac{p}{\rho} + \varphi\right) = 0$$

BERNOULLI'S LAW STATES THAT

$$\frac{1}{2}u^2 + \frac{p}{\rho} + \varphi = \text{CONSTANT ALONG A STREAMLINE}$$

FURTHERMORE, IF $\bar{\Omega} = 0$, THEN

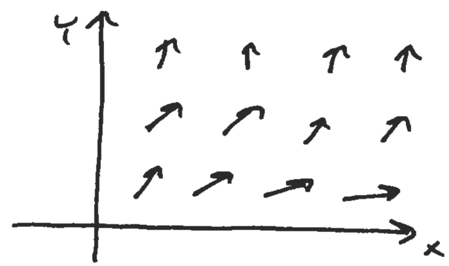
$$\frac{1}{2}u^2 + \frac{p}{\rho} + \varphi = \text{CONSTANT EVERYWHERE!}$$

VERY IMPORTANT: AIRPLANE WING LIFT, VERY USEFUL PRESSURE-FLOW RELATIONSHIP!

(CAN BE SHOWN TO BE EQUIVALENT TO CONSERVATION OF ENERGY.)

Visualizing Fluid Flow

- STREAMLINES = INSTANTANEOUS CURVES EVERYWHERE TANGENT TO FLOW



$\bar{u}(x, y, t)$

- PATH LINES = TRAJECTORY OF A FLUID ELEMENT AS IT TRAVELS. (TRACER)

PATHLINES $\Longleftrightarrow$ STREAMLINES IN STEADY FLOW

- STREAKLINES = "SMOKE LINES"

= THE LOCATION OF ALL OF THE PATH ENDS THAT INITIATED AT A FIXED POINT

TWO EXAMPLES ....

(1) FLOW AT A CORNER

(2) FLOW AROUND A CYLINDER

$\frac{\partial}{\partial t} \to 0$ STEADY FLOW | 2D $\bar{u} = (u_x, u_y, 0)$ | $\nabla \cdot \bar{u} = 0$, $\nabla \times \bar{u} = 0$

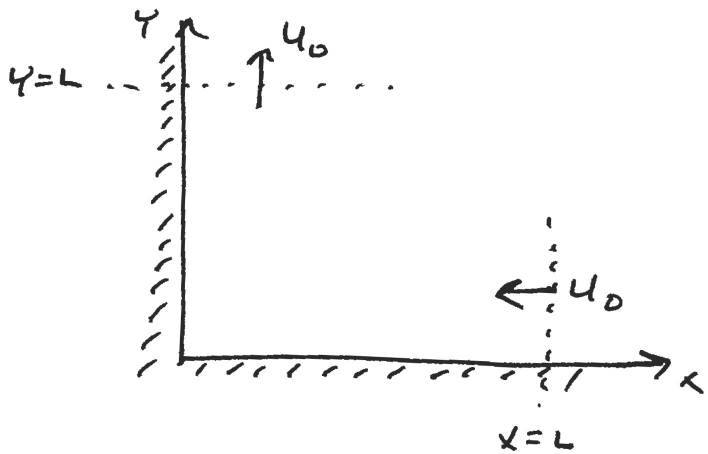
Continuity, Vorticity, and Acceleration

$$\nabla \cdot \mathbf{U} = 0$$

$$\mathbf{\Omega} = \nabla \times \mathbf{U} = 0$$

$$\frac{1}{2}U^2 + \frac{P}{\rho} = \text{Constant}$$

Flow at a Corner



$\nabla \cdot \bar{u} = 0$

$\nabla \times \bar{u} = 0$

(LIKE ELECTRO- OR MAGNETOSTATICS)

TWO SOLUTION METHODS (REALLY EQUIVALENT)

- VELOCITY POTENTIAL

$$\bar{u} = \bar{\nabla} \varphi \quad \text{so} \quad \nabla \times \nabla \varphi = 0$$

$$\nabla \cdot \bar{u} \Rightarrow \boxed{\nabla^2 \varphi = 0}$$

LAPLACES EQUATION

- STREAM FUNCTION

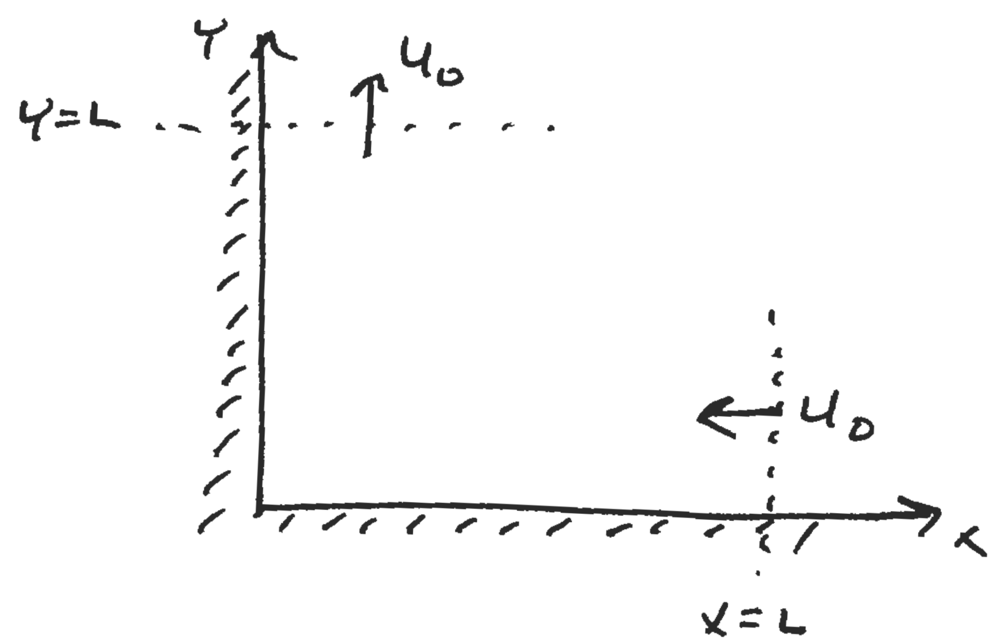
$$\bar{u} = \hat{z} \times \nabla \psi = \left(-\frac{\partial \psi}{\partial y}, \frac{\partial \psi}{\partial x}, 0 \right)$$

$$\nabla \cdot (\hat{z} \times \nabla \psi) = 0$$

$$\nabla \times (\hat{z} \times \nabla \psi) = \boxed{\nabla^2 \psi = 0}$$

Boundary Conditions

- NORMAL FLOW TO WALL MUST VANISH
- FLOW AT $X = L$ ~ $\bar{u} = (-u_0, 0, 0)$
- FLOW AT $Y = L$ ~ $\bar{u} = (0, u_0, 0)$



Solution Using Velocity Potential

(and Streamlines)

SOLUTION USING VELOCITY POTENTIAL...

$$\nabla^2 \varphi = \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0$$

$\frac{\partial \varphi}{\partial y} = 0$ AT $y = 0$

$\frac{\partial \varphi}{\partial x} = 0$ AT $x = 0$

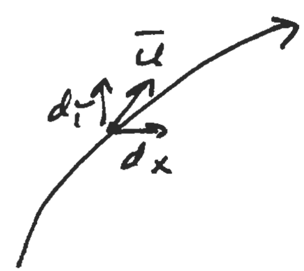
$$\varphi(x, y) = c_1 \left(x^2 - y^2 \right)$$

$$u_x = 2 c_1 x \qquad u_y = -2 c_1 y$$

BOUNDARY CONDITIONS AT $x, y = L$ ...

SET $c_1 = -u_0/2L$, SO $\varphi(x, y) = -\frac{u_0}{2L}\left(x^2 - y^2 \right)$

STREAMLINES



$$\frac{dx}{u_x\, dt} = \frac{dy}{u_y\, dt}$$

$$\frac{dx}{x} = \frac{dy}{y} \Longrightarrow \ln y = -\ln x + c \quad \text{OR}$$

$$xy = \text{CONSTANT}$$

Solution Using Streamfunction

$$\bar{U} = \hat{z} \times \nabla\psi = \left(-\frac{\partial\psi}{\partial y}, \frac{\partial\psi}{\partial x}, 0\right)$$

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial\psi}{\partial y^2} = 0$$ AND $\frac{\partial\psi}{\partial x} = 0$ AT $y = 0$

$\frac{\partial\psi}{\partial y} = 0$ AT $x = 0$

SO $\psi(x,y) = C_1 x y$

$U_x = -U_0$ AT $x = L \Rightarrow C_1 = \frac{U_0}{L}$

$$\boxed{\psi(x,y) = \frac{U_0}{L} x y}$$

$\psi =$ CONSTANT IS A STREAMLINE

NOTE: LINES OF CONSTANT φ, ψ ARE ORTHOGONAL

$\nabla\varphi \cdot \nabla\psi = 0$ ← CAN YOU PROVE THIS?

$\nabla\varphi \cdot (\hat{z} \times \nabla\psi) = u^2$

What is the Pressure Along the Corner?

AT $x = L$, FLUID IS CHARACTERIZED BY
u_0, P_0

USE BERNOULLI'S LAW...

$$\frac{1}{2}\rho u_0^2 + P_0 = \frac{1}{2}\rho u^2 + P(x,y)$$

BUT $u^2 = \bar{u}\cdot\bar{u} = \frac{u_0^2}{L^2}(x^2+y^2)$ AND

ALONG LOWER SIDE ($y=0$)

$$P(x,y) = P_0 + \frac{1}{2}\rho\left(u_0^2 - u^2\right)$$

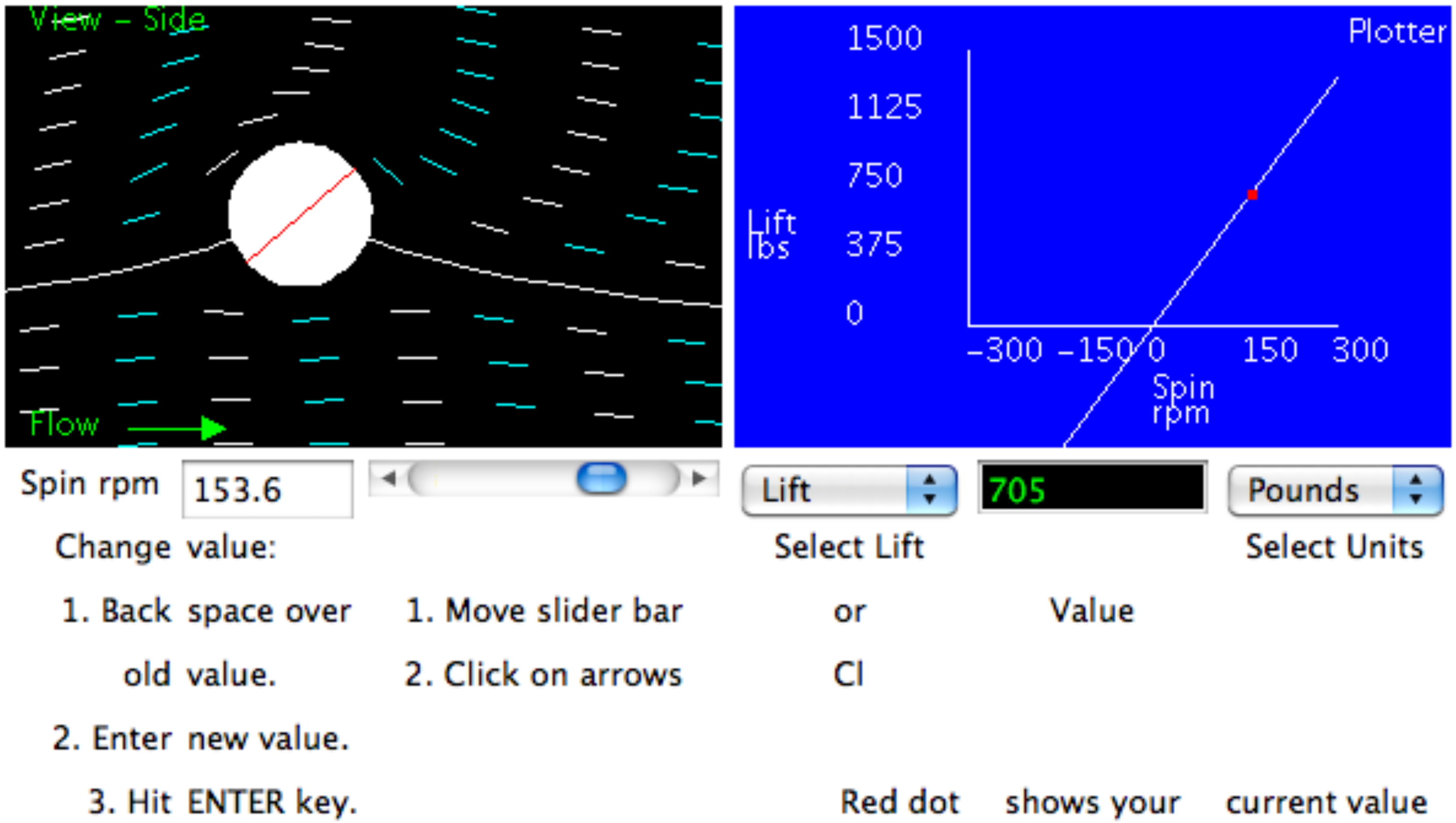
$$= P_0 + \frac{1}{2}\rho u_0^2\left(1 - \frac{x^2}{L^2}\right)$$

$\therefore$ PRESSURE <u>INCREASES</u> INTO CORNER

$$P(0,0) = P_0 + \frac{1}{2}\rho u_0^2$$

Rotating Cylinder

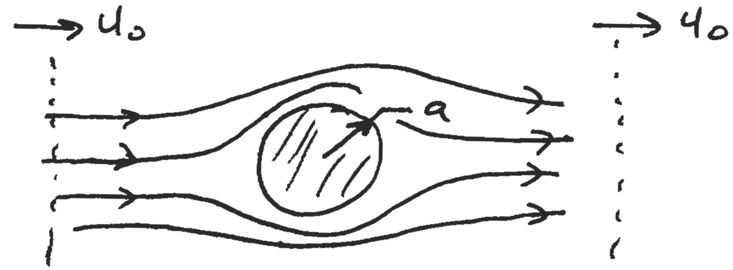
Let's investigate the lift of a rotating cylinder by using a Java simulator.



http://www.grc.nasa.gov/WWW/K-12/airplane/cyl.html

Flow Around a Cylinder

(See Ch. 6-9)



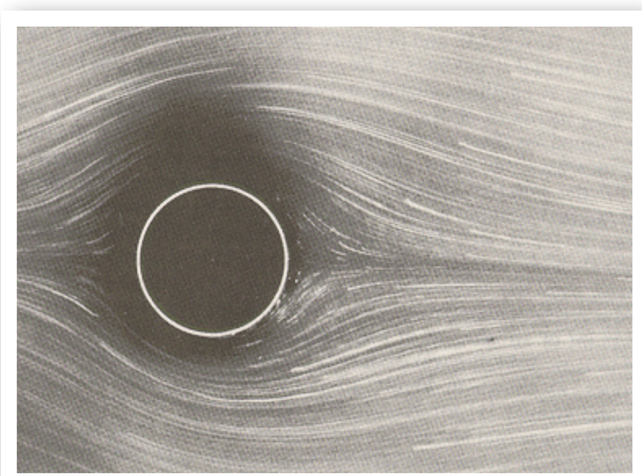
USE VELOCITY POTENTIAL

CYLINDRICAL COORDINATES (SEE APPENDIX B)

$$\nabla^2 \varphi = 0 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \varphi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \varphi}{\partial \theta^2}$$

BOUNDARY CONDITIONS:

$$\left.\frac{\partial \varphi}{\partial r}\right|_{r=a} = 0 \qquad \begin{aligned} u_r &= u_0 \cos\theta \text{ AS } r \to \infty \\ u_\theta &= -u_0 \sin\theta \text{ AS } r \to \infty \end{aligned}$$



SOLUTION:

$$\varphi(r,\theta) = f(r)\cos\theta$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) - \frac{f}{r^2} = 0$$

LET $f \propto r^\alpha$ $\quad f' \propto \alpha r^{\alpha-1}$ $\quad f'' \propto \alpha(\alpha-1) r^{\alpha-2}$

SUBSTITUTING

$$\alpha(\alpha-1) + \alpha - 1 = 0 \Rightarrow \underline{\underline{\alpha = \pm 1}}$$

Velocity Potential Solution

$$\varphi(r,\theta) = \left(C_1 r + \frac{C_2}{r}\right)\cos\theta$$

$$\left.\frac{\partial\varphi}{\partial r}\right|_{r=a} = 0 = \left(C_1 - \frac{C_2}{a^2}\right)\cos\theta \Rightarrow \boxed{C_2 = C_1 a^2}$$

As $r \to \infty$

$$u_r = \frac{\partial\varphi}{\partial r} = C_1\cos\theta = U_0\cos\theta \qquad \boxed{C_1 = U_0}$$

THUS,

$$\varphi(r,\theta) = U_0\left(r + \frac{a^2}{r}\right)\cos\theta$$

$$u_r(r,\theta) = U_0\left(1 - \frac{a^2}{r^2}\right)\cos\theta$$

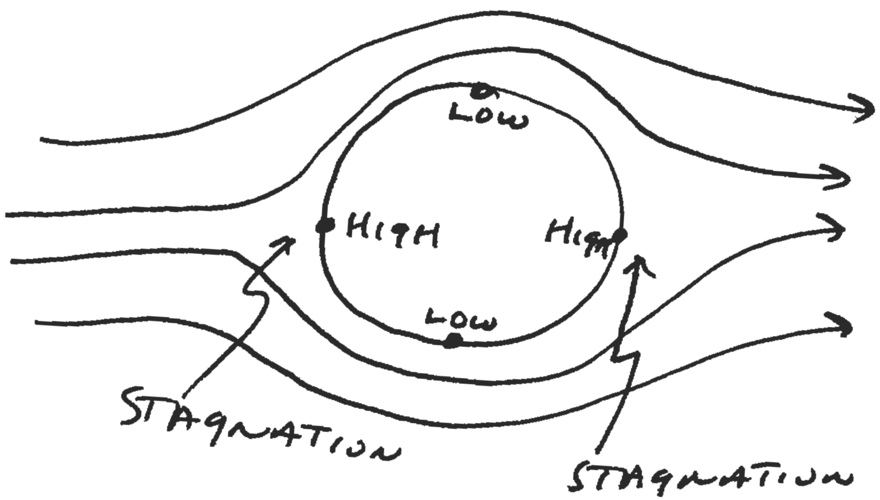
$$u_\theta(r,\theta) = -U_0\left(1 + \frac{a^2}{r^2}\right)\sin\theta$$

What is the Pressure at the Surface of the Cylinder?

$$u^2(r=a) = u_\theta^2 = 4u_0^2 \sin^2\theta$$

BERNOULLI'S LAW GIVES

$$P(r=a, \theta) = P_\infty + \frac{1}{2}\rho\left(u_0^2 - 4u_0^2 \sin^2\theta\right)$$



LOWEST PRESSURE AT $\theta = 90°, 270°$

HIGHEST PRESSURE AT $\theta = 0, 180°$

$$\text{LOW PRESSURE} = P_\infty - 2\rho u_0^2$$

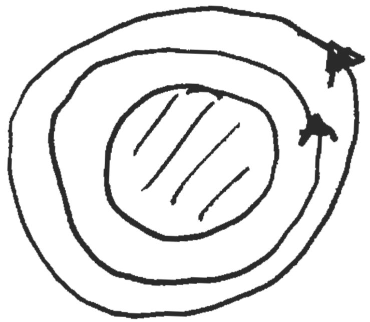
$$\text{HIGH PRESSURE} = P_\infty + \frac{1}{2}\rho u_0^2$$

Equations for Incompressible and Irrotational Flow are Linear

$\nabla \cdot \bar{u} = 0 \qquad \nabla \times \bar{u} = 0$

SO IF $\bar{u}_1$ AND $\bar{u}_2$ ARE SOLUTIONS THEN ALSO $\bar{u}_3 = \bar{u}_1 + \bar{u}_2$ IS A SOLUTION.

A Solution with Circulation



ROTATING, CIRCULATING FLOW ABOUT THE CYLINDER

AXISYMMETRIC FLOW

$$\bar{u} = (0, u_\theta, 0)$$

$$u_\theta(r) = \frac{1}{r}\frac{\partial \varphi}{\partial \theta} \quad \text{AND} \quad \nabla^2 \phi = 0 = \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial \varphi}{\partial r}\right)$$

SOLUTION: $u_\theta(r) = \dfrac{c_1}{r} \qquad \varphi = c_1 \theta$

DEFINE CIRCULATION ABOUT CYLINDER AS

$$\Gamma \equiv \oint_{C_1} \bar{u} \cdot d\bar{s} = \int_0^{2\pi} u_\theta \, a \, d\theta = 2\pi c_1$$

SO $u_\theta(r) = \dfrac{\Gamma}{2\pi r}$ WHERE Γ = CIRCULATION

Add Circulation to Flow

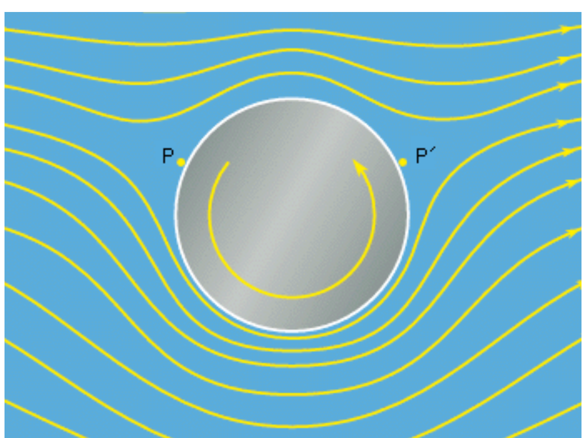
ADD CIRCULATION TO FLOW PAST CYLINDER...,

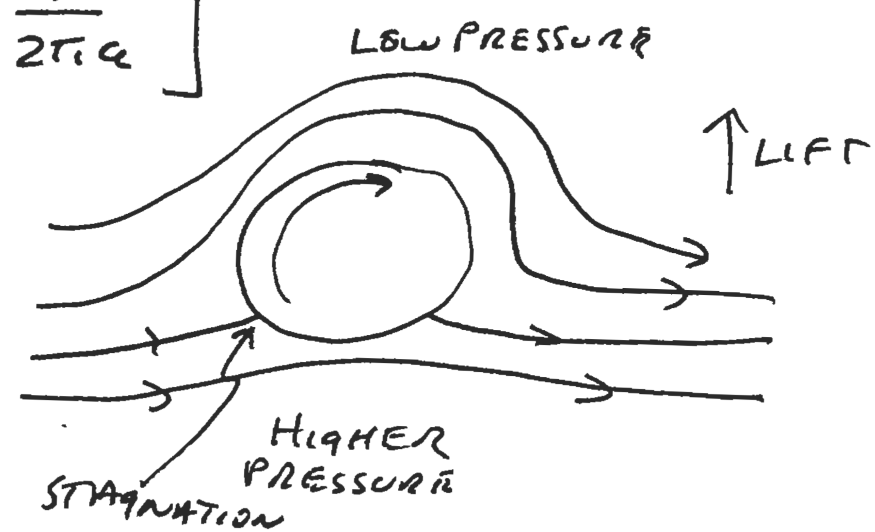
$$u_r(r,\theta) = U_0\left(1 - \frac{a^2}{r^2}\right)\cos\theta$$

$$u_\theta(r,\theta) = -U_0\left(1 + \frac{a^2}{r^2}\right)\sin\theta + \frac{\Gamma}{2\pi r}$$

BERNOULLI'S LAW AT SURFACE...

$$\frac{1}{2}\rho u^2 = \frac{1}{2}\rho\left[2U_0\sin\theta - \frac{\Gamma}{2\pi a}\right]^2$$





Summary

- Continuity (incompressible flow and the Boussinesq approximation)

$$\nabla \cdot \mathbf{U} \approx 0$$

- Force and acceleration (Euler equation)

$$\frac{\partial \mathbf{U}}{\partial t} + (\mathbf{U} \cdot \nabla)\mathbf{U} = -\frac{1}{\rho}\nabla P - \nabla \phi_g$$

- Vorticity ($\nabla\times$U, swirls!!)

$$\frac{d\mathbf{U}}{dt} = \frac{\partial \mathbf{U}}{\partial t} + \mathbf{\Omega} \times \mathbf{U} + \frac{1}{2}\nabla U^2$$

- Bernoulli's Principle (Lift and pressure)

$$\mathbf{U} \cdot \nabla \left(\frac{1}{2}U^2 + \frac{P}{\rho} + \phi_g \right) \approx 0$$

- Visualizing flow (tracing lines...)

- 2D Euler flow (inviscid)

Important: B.P. can be used to find pressure from flow!