Part 1: 
- Last Lecture: Review
- Example Fluid Flow Problems

Part 2: 
- Geometric Identities
- Vector Calculus
Hydrostatics of a Sphere

Gradient of a scalar
\[ \nabla \psi = i_r \frac{\partial \psi}{\partial r} + i_\theta \frac{1}{r} \frac{\partial \psi}{\partial \theta} + i_\varphi \frac{1}{r \sin \theta} \frac{\partial \psi}{\partial \varphi}. \]

Laplacian of a scalar
\[ \nabla^2 \psi = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \psi}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \psi}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 \psi}{\partial \varphi^2}. \]

Divergence of a vector
\[ \nabla \cdot \mathbf{u} = \frac{1}{r^2} \frac{\partial (r^2 u_r)}{\partial r} + \frac{1}{r \sin \theta} \frac{\partial (u_\theta \sin \theta)}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial u_\varphi}{\partial \varphi}. \]
Examples: Fluid Flow

- Continuity (mass conservation)
- Euler equation (force and acceleration)
- Vorticity ($\nabla \times \mathbf{U}$, swirls!!)
- Bernoulli’s Principle (Lift and pressure)
- Visualizing flow (tracing lines...)
- Flow at corner
- Flow around cylinder
Fluid Continuity

(Conservation of Mass)

\[ \boldsymbol{\bar{u}} = (u, v, w) \]

\[ \frac{\Delta \rho}{\Delta t} |_{\gamma} \approx - \left( v \rho l_{\gamma+\Delta \gamma} - v \rho l_{\gamma} \right) \]

\[ \approx - \frac{2}{2 \gamma} (v \rho) \]

For all 3 directions:

\[ \frac{2 \rho}{2} = - \nabla \cdot (\rho \bar{u}) = - \sum_{i=1}^{3} \frac{2}{2x_i} (\rho u_i) \]

\[ = - \frac{2}{2x} (\rho u_x) - \frac{2}{2y} (\rho u_y) - \frac{2}{2z} (\rho u_z) \]

If \( \rho = \text{constant} \) (Boussinesq) then

\[ \nabla \cdot (\rho \bar{u}) = \nabla \cdot \bar{u} = 0 \quad \text{"incompressible flow"} \]
Newton's Law for a Fluid Element

\[ \rho x (\textit{ACCELERATION}) = \textit{FORCE} \]

\[ = -\nabla \rho + \rho \vec{g} + \ldots \]

\[ = -\nabla (\rho + \rho \psi) + \ldots \]

WHAT IS ACCELERATION?

\[ \text{ACCELERATION} = \frac{\vec{u}_2 - \vec{u}_1}{\Delta t} \quad \text{as} \quad \Delta t \to 0 \]

\[ \vec{u}_2 = \vec{u} (x + u_x \Delta t, y + u_y \Delta t, z + u_z \Delta t, t + \Delta t) \]

\[ = \vec{u} (x, y, z, t) + \frac{\partial \vec{u}}{\partial t} \Delta t + \frac{\partial \vec{u}}{\partial x} u_x \Delta t + \frac{\partial \vec{u}}{\partial y} u_y \Delta t + \frac{\partial \vec{u}}{\partial z} u_z \Delta t + \ldots \]

SO

\[ \text{ACCELERATION} = \frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla) \vec{u} \quad \text{and Newton's Law is} \]

\[ \frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla) \vec{u} = -\frac{1}{\rho} \nabla \rho - \nabla \psi \]
Vorticity and Definitions

From Problem #3 in 1st Problem Set

\[(\overline{U} \cdot \overline{D}) \overline{U} = (\nabla \times \overline{U}) \times \overline{U} + \frac{1}{2} \overline{D} \cdot (\overline{U} \cdot \overline{U})\]

so that

\[\overline{J} = \nabla \times \overline{U} \equiv \text{vorticity}\]

\[\frac{\partial \overline{U}}{\partial t} + \nabla \times \overline{U} + \frac{1}{2} \overline{D} \cdot U^2 = - \frac{\nabla \rho}{\rho} - \nabla \phi\]

If \(\overline{J} = 0\), then flow is "IRROTATIONAL"

IRROTATIONAL AND INCOMPRESSIBLE FLOWS SATISFY

\[\nabla \cdot \overline{U} = 0 \quad \nabla \times \overline{U} = 0\]

Just like equations for electrostatics and magnetostatics in a charge-free or current-free region.
Bernoulli's Principle

For steady, incompressible flow...

\[ \bar{u} \cdot \left[ \frac{2\gamma}{\gamma-1} + \bar{\omega} \times \bar{u} \right] = -\nabla \left( \frac{1}{2} u^2 + \frac{p}{\rho} + \phi \right) \]

or

\[ \bar{u} \cdot \nabla \left( \frac{1}{2} u^2 + \frac{p}{\rho} + \phi \right) = 0 \]

Bernoulli's Law states that

\[ \frac{1}{2} u^2 + \frac{p}{\rho} + \phi = \text{constant along a streamline} \]

Furthermore, if \( \bar{\omega} = 0 \), then

\[ \frac{1}{2} u^2 + \frac{p}{\rho} + \phi = \text{constant everywhere!} \]

Very important: Airplane wing lift, very useful pressure-flow relationship!

(Can be shown to be equivalent to conservation of energy.)
Visualizing Fluid Flow

- **Streamlines** = instantaneous curves everywhere tangent to flow

\[ \vec{u}(x, y, t) \]

- **Pathlines** = trajectory of a fluid element as it travels. (tracer)

Pathlines \( \cong \) Streamlines in steady flow

- **Streaklines** = "smoke lines"
  = the location of all of the path ends that initiated at a fixed point

Two examples:

1. Flow at a corner
2. Flow around a cylinder

\[ \frac{\partial^2 \vec{u}}{\partial t^2} \rightarrow \text{steady flow} \quad \begin{cases} 2D \quad \vec{u} = (u_x, u_y, 0) \quad \nabla \cdot \vec{u} = 0 \\ \nabla \times \vec{u} = 0 \end{cases} \]
Continuity, Vorticity, and Acceleration

\[ \nabla \cdot \mathbf{U} = 0 \]

\[ \Omega = \nabla \times \mathbf{U} = 0 \]

\[ \frac{1}{2} U^2 + \frac{P}{\rho} = \text{Constant} \]
Flow at a Corner

\[ \nabla \cdot \mathbf{u} = 0 \\
\nabla \times \mathbf{u} = 0 \\
\text{(like electro- or magneto-statics)} \\
\]

Two solution methods (really equivalent)

- Velocity potential

  \[ \mathbf{u} = \nabla 
\phi \quad \text{so} \quad \nabla \times \nabla \phi = 0 \]

  \[ \nabla \cdot \mathbf{u} \Rightarrow \nabla^2 \phi = 0 \quad \text{Laplace's equation} \]

- Stream function

  \[ \mathbf{u} = \hat{z} \times \nabla \psi = \left( -\frac{\partial \psi}{\partial x}, \frac{\partial \psi}{\partial z}, 0 \right) \]

  \[ \nabla \cdot (\hat{z} \times \nabla \psi) = 0 \]

  \[ \nabla \times (\hat{z} \times \nabla \psi) = \nabla^2 \psi = 0 \]
Boundary Conditions

- Normal flow to wall must vanish
- Flow at \( x = L \) \( \sim \mathbf{u} = (-u_0, 0, 0) \)
- Flow at \( y = L \) \( \sim \mathbf{u} = (0, u_0, 0) \)
Solution Using Velocity Potential
(and Streamlines)

\[ \nabla^2 \psi = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0 \]
\[ \frac{\partial \psi}{\partial y} = 0 \text{ at } y = 0 \]
\[ \frac{\partial \psi}{\partial x} = 0 \text{ at } x = 0 \]

\[ \psi(x, y) = c_1 (x^2 - y^2) \]

\[ u_x = 2c_1 x \quad u_y = -2c_1 \gamma \]

**Boundary Conditions** at \( x, y = 2L \)... set \( c_1 = -u_0 / 2L \), so \( \psi(x, y) = -\frac{u_0}{2L} (x^2 - y^2) \)

**Streamlines**

\[ \frac{dx}{u_x \, dt} = \frac{dy}{u_y \, dt} \]

\[ \frac{dx}{x} = \frac{dy}{y} \quad \Rightarrow \quad \ln y = -\ln x + c \quad \text{or} \quad xy = \text{constant} \]
Solution Using Streamfunction

\[ \overline{U} = \hat{\nabla} \psi = \left( -\frac{2\psi}{y^2}, \frac{2\psi}{x}, 0 \right) \]

\[ \frac{2^2 \psi}{2x^2} + \frac{2\psi}{2y^2} = 0 \quad \text{and} \quad \frac{2\psi}{2x} = 0 \quad \text{at} \quad y = 0 \]

\[ \frac{2\psi}{2y} = 0 \quad \text{at} \quad x = 0 \]

So \[ \psi(x, y) = C_1 \times y \]

\[ U_x = -U_0 \quad \text{at} \quad x = L \Rightarrow C_1 = \frac{U_0}{L} \]

\[ \boxed{\psi(x, y) = \frac{U_0}{L} \times y} \]

\[ \psi = \text{constant is a streamline} \]

**Note:** Lines of constant \( \psi \), \( \psi \) are orthogonal

\[ \nabla \psi \cdot \nabla \psi = 0 \quad \text{--- can you prove this?} \]

\[ \nabla \psi \cdot (\hat{\nabla} \psi) = U^2 \]
What is the Pressure Along the Corner?

At $x = L$, flow is characterized by $u_0, p_0$.

Use Bernoulli's law...

$$\frac{1}{2} \rho u_0^2 + p_0 = \frac{1}{2} \rho u^2 + p(x, \gamma)$$

But $u^2 = \overline{u} \cdot \overline{u} = \frac{u_0^2}{L^2} (x^2 + \gamma^2)$ and along lower side ($\gamma = 0$)

$p(x, \gamma) = p_0 + \frac{1}{2} \rho \left( u_0^2 - u^2 \right)$

$= p_0 + \frac{1}{2} \rho u_0^2 \left( 1 - \frac{x^2}{L^2} \right)$

:. Pressure increases into corner

$p(0, 0) = p_0 + \frac{1}{2} \rho u_0^2$
Let's investigate the lift of a rotating cylinder by using a Java simulator.

http://www.grc.nasa.gov/WWW/K-12/airplane/cyl.html
Flow Around a Cylinder
(See Ch. 6-9)

\[ \nabla^2 \phi = 0 = \frac{1}{\lambda} \frac{2}{2\eta} \left( \frac{1}{2} \frac{\partial \phi}{\partial \eta} \right) + \frac{1}{\eta^2} \frac{\partial^2 \phi}{\partial \theta^2} \]

Boundary Conditions:
\[ \frac{\partial \phi}{\partial \eta} \bigg|_{\eta = \alpha} = 0 \]
\[ U_A = U_0 \cos \theta \quad \text{as} \quad \eta \to \infty \]
\[ U_B = -U_0 \sin \theta \quad \text{as} \quad \eta \to \infty \]

Solution:
\[ \phi(\eta, \theta) = f(\eta) \cos \theta \]
\[ \frac{1}{\lambda} \frac{2}{2\eta} \left( \frac{1}{2} \frac{\partial f(\eta)}{\partial \eta} \right) - \frac{f(\eta)}{\eta^2} = 0 \]

Let \( f(\eta) \propto \eta^{\alpha-1} \)
\[ \frac{\partial}{\partial \eta} (\alpha \eta^{\alpha-1}) - \frac{\alpha}{\eta} \eta^{\alpha-2} = 0 \]

Substituting
\[ \alpha (\alpha-1) + \alpha - 1 = 0 \Rightarrow \alpha = \pm 1 \]
Velocity Potential Solution

\[ \psi (r, \theta) = (c_1 r + \frac{c_2}{r}) \cos \theta \]

\[ \frac{2 \psi}{2^2} \bigg|_{r=a} = 0 = (c_1 - \frac{c_2}{a^2}) \cos \theta \Rightarrow c_2 = c_1 a^2 \]

As \( r \to \infty \)

\[ \mathcal{U}_n = \frac{2 \psi}{2^2} = c_1 \cos \theta = u_o \cos \theta \]

Thus,

\[ \psi (r, \theta) = u_o \left( r + \frac{a^2}{r} \right) \cos \theta \]

\[ \mathcal{U}_n (r, \theta) = u_o \left( 1 - \frac{a^2}{r^2} \right) \cos \theta \]

\[ \mathcal{U}_b (r, \theta) = -u_o \left( 1 + \frac{a^2}{r^2} \right) \sin \theta \]
What is the Pressure at the Surface of the Cylinder?

\[ u^2(\theta = 0) = U_\theta^2 = 4 U_0^2 \sin^2 \theta \]

**Bernoulli’s Law Gives**

\[ P(\theta = 0, \theta) = P_\infty + \frac{1}{2} \rho (U_0^2 - 4 U_0^2 \sin^2 \theta) \]

Lowest Pressure at \( \theta = 90^\circ, 270^\circ \)

Highest Pressure at \( \theta = 0, 180^\circ \)

Low Pressure \( = P_\infty - 2\rho U_0^2 \)

High Pressure \( = P_\infty + \frac{1}{2} \rho U_0^2 \)
Equations for Incompressible and Irrotational Flow are Linear

\[ \nabla \cdot \mathbf{u} = 0 \quad \nabla \times \mathbf{u} = 0 \]

So if \( \mathbf{u}_1 \) and \( \mathbf{u}_2 \) are solutions then also \( \mathbf{u}_3 = \mathbf{u}_1 + \mathbf{u}_2 \) is a solution.
A Solution with Circulation

**Rotating, Circulating Flow About the Cylinder**

**Axisymmetric Flow**

\[
\overline{U} = (0, u_\theta, 0)
\]

\[
u_\theta(\eta) = \frac{1}{\eta} \frac{2}{2\pi} \psi \quad \text{and} \quad \nabla^2 \phi = 0 = \frac{1}{\eta} \frac{2}{2\pi} \left( \eta \frac{2}{2\pi} \phi \right)
\]

**Solution**: \( u_\theta(\eta) = \frac{c_1}{\eta} \), \( \psi = c_1 \eta \)

**Define circulation about cylinder as**

\[
\Gamma = \oint_{C_1} \overline{U} \cdot d\overline{s} = \int_0^{2\pi} u_\theta \alpha \, d\theta = 2\pi c_1
\]

So \( u_\theta(\eta) = \frac{\Gamma}{2\pi \eta} \) where \( \Gamma = \text{circulation} \)
Add Circulation to Flow

Add circulation to flow past cylinder...

\[
\begin{align*}
U_x (\eta, \theta) &= U_0 \left(1 - \frac{\eta^2}{a^2}\right) \cos \theta \\
U_y (\eta, \theta) &= -U_0 \left(1 + \frac{\eta^2}{a^2}\right) \sin \theta + \frac{\Gamma}{2\pi \eta}
\end{align*}
\]

Bernoulli's Law at surface...

\[
\frac{1}{2} \rho u^2 = \frac{1}{2} \rho \left[ 2U_0 \sin \theta - \frac{\Gamma}{2\pi a} \right]^2
\]

Low Pressure

Lift

Higher pressure

Stagnation
Summary

• Continuity (incompressible flow and the Boussinesq approximation)
  \[ \nabla \cdot \mathbf{U} \approx 0 \n\]

• Force and acceleration (Euler equation)
  \[ \frac{\partial \mathbf{U}}{\partial t} + (\mathbf{U} \cdot \nabla) \mathbf{U} = -\frac{1}{\rho} \nabla P - \nabla \phi_g \]

• Vorticity (\( \nabla \times \mathbf{U} \), swirls!!)
  \[ \frac{d\mathbf{U}}{dt} = \frac{\partial \mathbf{U}}{\partial t} + \mathbf{\Omega} \times \mathbf{U} + \frac{1}{2} \nabla U^2 \]

• Bernoulli’s Principle (Lift and pressure)
  \[ \mathbf{U} \cdot \nabla \left( \frac{1}{2} U^2 + \frac{P}{\rho} + \phi_g \right) \approx 0 \]

• Visualizing flow (tracing lines...)

• 2D Euler flow (inviscid)

Important: B.P. can be used to find pressure from flow!