Lecture 3

- Velocity gradient tensor, strain, rotation
Simple Comments about Velocity Gradient Tensor

Rigio TRANSLATION is CHARACTERIZED AS

\[
\mathbf{x} = \mathbf{x}_0 + \mathbf{u} t
\]

\[
\frac{d\mathbf{x}}{dt} = \mathbf{u} = \text{constant}
\]

So \( \frac{\partial u_i}{\partial x_j} = 0 \)

Symmetric AND Anti-Symmetric PARTS

\[
\frac{\partial u_i}{\partial x_j} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) + \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} \right)
\]

\[
\varepsilon_{ij} + \gamma_{ij}
\]

\[
\varepsilon_{ij} + \frac{1}{2} R_{ij} \quad (R_{ij} = \text{Rotation Tensor})
\]
Stretching along one Axis

\[ \frac{d \Delta x_i}{dt} = \frac{\partial u_i}{\partial x_i} \Delta x_i \]

\[ \frac{\partial^2 \Delta x_i}{\partial t^2} = \varepsilon_{ii} = \begin{cases} \varepsilon_{11} & \text{if } \varepsilon_{11} > 0 \\ \varepsilon_{22} & \text{if } \varepsilon_{22} > 0 \\ \varepsilon_{33} & \text{if } \varepsilon_{33} > 0 \end{cases} \]

A cube...

\[ \frac{d}{dt} \Delta V = \frac{d}{dt} (\Delta x \Delta y \Delta z) = \varepsilon_{11} \Delta V + \varepsilon_{22} \Delta V + \varepsilon_{33} \Delta V = (\nabla \cdot \mathbf{u}) \Delta V \]

So

\[ (\nabla \cdot \mathbf{u}) \Delta V = \Delta V(t) \]

IF \( \nabla \cdot \mathbf{u} > 0 \), THEN expansion

IF \( \nabla \cdot \mathbf{u} = 0 \), volume is constant

IF \( \nabla \cdot \mathbf{u} < 0 \), THEN dilatation

Rotation

WHAT IF \( \frac{2u_i}{\partial x_j} = R_{ij} \) (ANTI- SYMMETRIC)

\[ = \frac{1}{2} R_{ij} \]

WHERE \( \omega_i = -\epsilon_{ijk} R_{jk} \)

\[ = -\frac{1}{2} \epsilon_{ijk} R_{jk} \]

THEN \( u_i = \epsilon_{ijk} \omega_j x_k + u_{ci} \)

\[ \mathbf{u} = \mathbf{\omega} \times \mathbf{x} + \mathbf{u}_c \] (Pure rotation)

\[ \Omega_{ij} = \frac{1}{2} \left( \frac{2u_i}{\partial x_j} - \frac{2u_j}{\partial x_i} \right) \]

\[ \frac{2u_i}{\partial x_j} = \epsilon_{ijk} \omega_j \frac{2x_k}{\partial x_m} = \epsilon_{ikm} \omega_j \]

SO THAT

\[ R_{ij} = \frac{1}{2} \left( \epsilon_{ikj} \omega_k - \epsilon_{ijk} \omega_k \right) = \epsilon_{ikj} \omega_k \] (check!)
Bending, Distorting, Shearing

Example Shearing
Lecture 4

- Navier-Stokes Equation

**Continuity**

**Conservation of Mass**

\[
\frac{\partial p}{\partial t} + \nabla \cdot (p \vec{u}) = 0
\]

\[
\frac{\partial \rho}{\partial t} + (\vec{u} \cdot \nabla) \rho = -\rho \nabla \cdot \vec{u}
\]

\[
\frac{\partial \rho}{\partial t} = -\Delta \rho + \rho \nabla \cdot \vec{u}
\]

\[
\frac{\partial p}{\partial t} = -\Delta p + \rho \nabla \cdot \vec{u}
\]

\[
\frac{\partial p}{\partial t} = -\Delta p + \rho \nabla \cdot \vec{u}
\]

\[
\rho \approx \rho_0 + \rho_0 \delta u
\]
Newton's Law

**Newton's Law for a Particle**

\[ \mathbf{F} = m \mathbf{a} \]

\[ \mathbf{F} = \frac{d}{dt} (m \mathbf{v}) \]

**Newton's Law for a Fluid:**

\[ \mathbf{F} = \frac{D}{Dt} (\rho \mathbf{u}) \]

\[ = \frac{\partial}{\partial t} (\rho \mathbf{u}) + \nabla \cdot (\rho \mathbf{u} \mathbf{u}) \]

\[ = \frac{\partial}{\partial t} (\rho \mathbf{u}) + \frac{\partial}{\partial x_j} (\rho \mathbf{u}_j \mathbf{u}_j) \]

\[ = \rho \frac{D \mathbf{u}}{Dt} \]

\( \Rightarrow \) **Conservation of Mass**

Momentum

\[ \rho \left( \frac{\partial}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} \right) = \rho \mathbf{F} + \nabla \cdot \mathbf{T} \]

\( \mathbf{T} = \text{STRESS TENSOR} \)

\( \text{usually symmetric} \)

\( \text{has normal stresses - pressure} \)

\( \text{has shear stresses - off diagonal} \)

**Gradients of stress produce force**

\( \tau_{ii} > 0 \) implies tensile stress

\( \tau_{ii} < 0 \) implies compressive stress

\( \tau_{ij} \ (i \neq j) \) are shear stresses
Models for Stress

- ISOTROPIC PRESSURE
  \[ \overline{\varepsilon} = -\rho \dot{\overline{\delta}} \quad \nabla \cdot \overline{\varepsilon} = -\nabla \rho \]

- MOVING FLUID WITH VISCOITY
  \[ \overline{\varepsilon} = -\rho \dot{\overline{\delta}} + \overline{\eta} \]  
  \( \overline{\eta} \) - VISCOUS STRESS

WHAT IS \( \overline{\eta} \)?

Stokesian Fluid

MATHEMATICIAN ISOTROPY AND STRESS SYMMETRY
(E.g., AIR, WATER BUT NOT MAGNETIZED PLASMA)

\[ \overline{\eta} = 2\mu \overline{\varepsilon} + \lambda (\sigma \cdot \overline{\varepsilon}) \overline{\delta} \]

\( \uparrow \) VISCOSITY
\( \uparrow \) BULK VISCOSITY

STOKES MODELED VISCOITY VIA KINETIC THEORY OF
MONOTONIC ATOMS AND SHOWN \( \lambda = -\frac{2}{3} \mu \).

THEN, STRESS TENSOR

\[ \overline{\varepsilon} = -\rho \dot{\overline{\delta}} + 2\mu \overline{\varepsilon} + \frac{2}{3} \lambda (\sigma \cdot \overline{\varepsilon}) \overline{\delta} \]
Navier-Stokes Equation

\[ p \left( \frac{\partial \tilde{u}}{\partial t} + (\tilde{u} \cdot \nabla) \tilde{u} \right) = -\nabla p + \rho \tilde{g} + \nabla \left[ 2 \kappa \tilde{\varepsilon} - \frac{\nu}{3} \mu (\nabla \cdot \tilde{u}) \tilde{\Delta} \right] \]

Assume \( \mu \) is independent of \( \tilde{x} \). Then,

\[ \nabla \cdot 2 \kappa \tilde{\varepsilon} = 2 \mu (\nabla \cdot \tilde{\varepsilon}) \]
\[ \varepsilon_{ij} = \frac{1}{2} \left( \frac{2 \kappa \frac{\partial u_i}{\partial x_j}}{2 \kappa \frac{\partial u_j}{\partial x_i}} \right) = \frac{1}{2} \nabla^2 \tilde{u} + \frac{1}{2} \nabla \left( \nabla \cdot \tilde{u} \right) \]

Navier-Stokes & Euler

\[ \frac{D \tilde{u}}{D t} = -\nabla p + \rho \tilde{g} + \begin{cases} 
\mu \left[ \nabla^2 \tilde{u} + \frac{1}{3} \nabla \left( \nabla \cdot \tilde{u} \right) \right] & \text{Navier-Stokes’ Equation} \\
\mu \nabla^2 \tilde{u} & \text{Incompressible N.S.} \\
0 & \text{Euler Equation} 
\end{cases} \]
Lecture 5

• "Equations of Fluid Dynamics"

Equations of Fluid Dynamics

(Conservation Laws)

Variables to characterize a simple fluid

\[ \rho(x,t) \quad \text{mass density} \]
\[ \mathbf{U}(x,t) \quad \text{flow velocity} \]
\[ P(x,t) \quad \text{pressure} \]
\[ T(x,t) \quad \text{temperature} \]

Six field variables, dynamics requires six equations of motion...

Conservation of mass

Newton's law / force / momentum

Conservation of energy

Equation of state (relating pressure to temperature)

Additionally, we can show that total entropy (disorder) must increase in time as required by second law of thermodynamics.
Navier-Stokes & Euler

\[
\rho \frac{D \mathbf{u}}{Dt} = -\nabla p + \rho \mathbf{g} + \left\{ \begin{array}{l}
\mu \left[ \nabla^2 \mathbf{u} + \frac{1}{3} \mathbf{I} (\nabla \cdot \mathbf{u}) \right] \quad \text{Navier-Stokes} \\
\mu \nabla^2 \mathbf{u} \quad \text{INCOMPRESSIBLE NAVIER-STOKES} \\
0 \quad \text{EULER EQUATION (IDEAL ФLUID) }
\end{array} \right.
\]

Navier-Stokes Eqs. is one of the most useful equations in applied physics: engineering, astrophysics, weather, oceanography, etc.

Magnetohydrodynamics includes N.S. plus Maxwell's Equations and buoyancy force is \( \mathbf{F} \times \mathbf{B} \) instead of \( \rho \mathbf{g} \).

Clay Prize: $1m if you can prove solutions to N.S. Equations are well-behaved and unique.

Mechanical Energy Density for a Stokes Fluid

\[
\rho \frac{D (\frac{1}{2} u^2)}{Dt} = \rho \mathbf{u} \cdot \mathbf{g} + \frac{2}{3} \eta (\nabla \cdot \mathbf{u}) - \nabla_j \epsilon_{ij} \eta_{ij}
\]

With \( \epsilon_{ij} = -\rho \delta_{ij} + 2\mu \epsilon_{ij} - \frac{2}{3} \mu (\nabla \cdot \mathbf{u}) \delta_{ij} \)

gives \( \nabla_j \epsilon_{ij} = -\rho \epsilon_{ii} + 2\mu \epsilon_{ij} \epsilon_{ji} - \frac{2}{3} \mu (\nabla \cdot \mathbf{u}) \epsilon_{ii} \)

Where \( \epsilon_{ii} = \frac{1}{2} \epsilon_{ij} = \nabla \cdot \mathbf{u} \)

Or?

\[
\rho \frac{D (\frac{1}{2} u^2)}{Dt} = \rho \mathbf{u} \cdot \mathbf{g} + \rho (\nabla \cdot \mathbf{u}) + \frac{2}{3} \left( \nabla \cdot \mathbf{u} \right) \left( \eta_{ii} \right) - \mu \left[ 2 \epsilon_{ij} \epsilon_{ij} - \frac{2}{3} (\nabla \cdot \mathbf{u})^2 \right]
\]

Work due to expansion

Viscous dissipation of mechanical work
Summary of Fluid Dynamical Equations

\begin{align*}
\frac{D \rho}{Dt} &= -\rho \nabla \cdot \mathbf{u} \\
\rho \frac{D \mathbf{u}}{Dt} &= \rho \mathbf{g} + \nabla \cdot \mathbf{T} \\
&= \rho \mathbf{g} - \nabla p + \mu \left[ \nabla^2 \mathbf{u} + \frac{1}{3} \nabla \left( \nabla \cdot \mathbf{u} \right) \right] \\
p &= \rho RT \\
\frac{D E}{Dt} &= \rho \mathbf{u} \cdot \frac{DT}{dt} = \nabla \cdot \mathbf{h} \nabla T - \rho \left( \nabla \cdot \mathbf{u} \right) + \mu \text{(dissipation)} \\
\frac{D \mathbf{h}}{Dt} &= \nabla \cdot \mathbf{h} \nabla T + \frac{DP}{Dt} + \mu \text{(dissipation)}
\end{align*}

\(p > 0, \ h > 0 \ \text{for} \ \partial S \geq 0\)

Lecture 6

- Bernoulli’s principle
- Co-rotating frame
Bernoulli's Equation

(Conservation of Energy)

\[ \rho \frac{D}{Dt} \left( E + \frac{1}{2} u^2 \right) = \rho \vec{f} \cdot \vec{u} + \nabla \cdot (\rho \vec{u} \vec{u}) - \nabla \cdot \vec{E} \]

**Bernoulli's Principle is very useful but it is an approximation valid when \( \mu = 0 \rightarrow 0 \): Viscosity and Thermodynamics can be ignored.**

So \( \nabla \cdot \vec{u} = 0 \rightarrow \nabla \cdot (\rho \vec{u}) = 0 \)

\[ \nabla \cdot (\rho \vec{u}) = - \nabla \cdot (\rho \vec{u}) \]

**Let** \( \vec{u} = - \nabla \psi_g \), \( \psi_g = \) gravitational potential.

**Then**

\[ \rho \frac{D}{Dt} \left( E + \frac{1}{2} u^2 \right) = - \rho \vec{u} \cdot \nabla \psi_g - \nabla \cdot (\rho \vec{u}) \]

So in steady state

\[ 0 \equiv - \rho \vec{u} \cdot \nabla \left[ E + \frac{\rho}{2} + \frac{1}{2} u^2 + \psi_g \right] \]

\( E + \frac{\rho}{2} + \frac{1}{2} u^2 + \psi_g = \) constant along streamlines

\( h = E + \frac{\rho}{2} = \) Enthalpy

---

Bernoulli's Equation

(Euler Equation)

\[ \rho \frac{D}{Dt} \vec{u} = \rho \vec{f} - \nabla \rho \]

\[ \rho \left( \frac{\partial u}{\partial t} + (\nabla \cdot \vec{u}) \vec{u} \right) = - \rho \nabla \psi_g - \nabla \rho \]

**Out** \( (\nabla \cdot \vec{u}) \vec{u} = - \nabla \left( \frac{1}{2} u^2 \right) + (\nabla \psi_g) \times \vec{u} \)

\( \vec{J} = \nabla \times \vec{u} = \) vorticity

So \( \frac{\partial u}{\partial t} + \nabla \left( \frac{1}{2} u^2 + \psi_g \right) + \frac{1}{\rho} \nabla \rho = - \vec{J} \times \vec{u} \)

**Out** \( h = E + \frac{\rho}{2} \)

\( \delta h = \delta P \), if \( S = \) constant \( \frac{\delta}{\delta x} \left( \frac{\delta h}{\delta P} \right) = \frac{1}{\rho} \frac{\delta P}{\delta P} = \frac{1}{\rho} \frac{\delta}{\delta x} \left( \frac{\delta h}{\delta P} \right) \)

**So** \( \frac{\partial h}{\partial x} = \frac{\delta}{\delta x} \left( \frac{\delta P}{\delta P} \right) = \frac{1}{\rho} \frac{\delta P}{\delta P} = \frac{1}{\rho} \frac{\delta}{\delta x} \left( \frac{\delta h}{\delta P} \right) \)

So \( \frac{\partial}{\partial t} \left( \frac{2u}{2} \right) + \nabla \left( \psi_g + \frac{1}{2} u^2 + \frac{\delta P}{\delta P} \right) = - \vec{J} \times \vec{u} \)

**If** \( \frac{\partial}{\partial t} \rightarrow 0 \), then \( \vec{J}, \vec{u} \) define a surface where \( \psi_g + \frac{1}{2} u^2 + \frac{\delta P}{\delta P} = \) constant
Incompressible Navier-Stokes in Co-Rotating Frame

\[
\frac{d\mathbf{u}}{dt} = -f + \frac{\nabla p}{\rho} + \left( \frac{\mu}{\rho} \right) \nabla^2 \mathbf{u}
\]

Co-rotating:

\[
\frac{d\mathbf{u}^r}{dt} = -f + \mathbf{u}^r \times \mathbf{u} - \frac{\nabla p}{\rho} - 2 \mathbf{u}^r \times \mathbf{u}^r + \left( \frac{\mu}{\rho} \right) \nabla^2 \mathbf{u}^r
\]

Lecture 7, 8

- Vorticity
- Kelvin’s Theorem
Vorticity Dynamics (Ch. 5)

\[ \mathbf{\Omega} = \text{vorticity, a vector field} \]

\[ = \nabla \times \mathbf{u} \quad (\text{thus}, \nabla \cdot \mathbf{\Omega} = 0 \quad \mathbf{\Omega} \text{ is solenoidal}) \]

\[ \nabla \times \left[ \frac{\partial \mathbf{u}}{\partial t} + \mathbf{\Omega} \times \mathbf{u} \right] = -\frac{1}{\rho} \nabla p - \nabla (\frac{1}{2} \mathbf{u}^2) + \mu \left( \nabla^2 \mathbf{u} + \frac{1}{3} \nabla (\nabla \cdot \mathbf{u}) \right) \]

\[ \frac{2 \mathbf{\Omega}}{2 t} + \nabla \times (\mathbf{\Omega} \times \mathbf{u}) = \mu \nabla^2 \mathbf{\Omega} = \mu \nabla^2 \frac{\mathbf{\Omega}}{2} \]

Vorticity Equation!

Kelvin's Theorem

\[ \frac{d}{dt} \int_S \mathbf{\Omega} \cdot d\mathbf{A} = 0 \quad (\text{as} \mu \to 0) \]

\[ \int_S \mathbf{\Omega} \cdot d\mathbf{A} = \int_S (\nabla \times \mathbf{u}) \cdot d\mathbf{A} \]

\[ = \oint_S \mathbf{u} \cdot d\mathbf{r} = C \]

Circulation of fluid is constant as \( \mu \to 0 \)

Vortex tube is a tube of constant vorticity flux.

Circulation within tube is constant as it moves with fluid (as \( \mu \to 0 \))
The Importance of Viscosity

INCOMPRESSIBLE EULER EQUATION

\[ \frac{2 \ddot{u}}{2t} + (\ddot{u}, \ddot{v}) \dot{u} = - \nabla \rho / \rho + \ddot{\rho} \]

\[ \nabla \cdot \dot{u} = 0 \]

LET \( \vec{r} = \nabla \times \dot{u} \). THEN

\[ (\ddot{u}, \ddot{v}) \dot{u} = \vec{r} \times \dot{u} + \frac{1}{2} \nabla \ddot{u} \]

\[ \frac{2 \ddot{u}}{2t} + \vec{r} \times \dot{u} = - \nabla \rho / \rho + \ddot{\rho} - \frac{1}{2} \nabla \ddot{u} \]

TAKE CQUAL OF THIS EQUATION

\[ \frac{2 \ddot{u}}{2t} + \nabla \cdot (\vec{r} \times \dot{u}) = 0 \quad (\text{if } \ddot{\rho} = - \nabla \rho) \]

IF \( \vec{r} = 0 \) at \( t=0 \), THEN \( \vec{r} = 0 \) FOREVER!

Simple Fluid Rotation

CYLINDRICAL COORDINATES:

**RIGID ROTATION**

\[ \vec{U}_0 = \omega \cdot \hat{\rho} \]

\[ \begin{bmatrix} \ddot{\rho} \\ \ddot{\varphi} \end{bmatrix} = \left( \nabla \times \vec{U} \right) = \frac{1}{2} \nabla \ddot{\rho} (\nabla \cdot \vec{U}) = 2 \omega \]

VORTICITY \( = 2 \cdot \) ROTATION

VORTICITY IS CONSTANT FOR EVERYWHERE

**RIGID ROTATION**

\[ \nabla \times \vec{r} = 0 \quad \text{NO VISCOS DISSIPATION} \]

**LINE VORTEX**

\[ \vec{U}_0 = \frac{\Pi}{\varphi \cdot \rho} \]

\[ \begin{bmatrix} \ddot{\rho} \\ \ddot{\varphi} \end{bmatrix} = \left( \nabla \times \vec{U} \right) = 0 \quad \text{NO VORTICITY WITHIN FLUID} \]

But "SINGULAR" LINE VORTEX AT ORIGIN

CIRCULATION \( = \oint \nabla \times \vec{U}_0 = \Pi \)

Since \( \vec{r} = 0 \), NO VISCOS DISSIPATION WITHIN FLUID.
Two Counter-Directed Line Vorticies

Two vorticies pointed in opposite directions travel together.

Rate of translation = $\frac{\Gamma}{2\pi h}$

Lectures 9, 10, & 11

- 2D Potential Flow
- Blasius Theorem
Potential Flow in 2D

**Incompressible**
\[ \nabla \cdot \mathbf{U} = 0 \]
\[ \mathbf{U} = -\nabla \varphi \]

**Irrtational**
\[ \nabla \times \mathbf{U} = 0 \]
\[ \mathbf{U} = \nabla \varphi \]

Then
\[ \nabla \cdot \nabla \varphi = \varphi_{xx} + \varphi_{yy} = 0 \]
\[ \nabla \times (\nabla \times \varphi) = 0 \]

\[ \mathbf{U} = \begin{pmatrix} \frac{2\varphi}{2y} \\ -\frac{2\varphi}{2x} \end{pmatrix} \]

**Velocity Potential**

**Stream Function (or Circulation Potential)**

Are equivalent flow descriptions.

If one is known, the other is known.

Irrotational Flow Consequences

- **Irrotational Flow Has No Closed Streamlines**
  since \( \mathbf{U} \) is always parallel to \( \nabla \varphi \) and
  \[ \oint \mathbf{U} \cdot d\mathbf{l} = 0 \]

- Since \( \nabla \times \mathbf{U} = 0 \), then velocity can be expressed as a gradient of a potential
  \[ \mathbf{U} = \nabla \varphi \quad \nabla \times \varphi = 0 \]

- **Euler’s Equation Can Be Simplified**...
  \[ \frac{\partial \mathbf{U}}{\partial t} + \nabla (\frac{1}{2} \mathbf{U}^2) - \mathbf{U} \times (\nabla \mathbf{U}) = -\nabla \rho \]
  \[ \nabla \cdot \left( \frac{\partial \mathbf{U}}{\partial t} + \frac{1}{2} \mathbf{U}^2 + \frac{\rho}{\rho} \right) = 0 \]

Thus, this function is constant/Encke's Distribution
in space/time, may be a function of time
\[ \frac{\partial \varphi}{\partial t} + \frac{1}{2} \mathbf{U}^2 + \frac{\rho}{\rho} = \text{constant in space} = f(x) \]
Complex Velocity Potential

When \( \nabla \times \mathbf{u} = \nabla \cdot \mathbf{u} = 0 \) in 2D, then we can define

\[
\Psi = \text{velocity potential}
\]

\[
\Psi = \text{stream function (vector potential)}
\]

where

\[
\nabla^2 \Psi = 0
\]

\[
\nabla^2 \Psi = 0
\]

So a **complex velocity potential** is a convenient mathematical technique ("trick") to solve Euler potential flow

\[
W(z) = \Psi + i \Psi
\]

Any analytic function \( W(z) \) is a solution for 2D potential flow!!!

Mass Source

\[
W(z) = \frac{M}{2\pi} \ln |z| = \frac{M}{2\pi} \ln (Re^i\theta) = \frac{M}{2\pi} \left[ \ln R + i \theta \right]
\]

\[
\Psi(\rho, \theta) = \frac{M}{2\pi} \ln \rho
\]

\[
\Psi(\rho, \theta) = \frac{M}{2\pi} \theta
\]

\[
\frac{dW}{dz^2} = \frac{M}{2\pi^2} = \frac{M}{2\pi} \frac{1}{(x^2 + y^2)} \Rightarrow U_x = \frac{M}{2\pi} \left( \frac{x}{x^2 + y^2} \right) \quad U_y = \frac{M}{2\pi} \left( \frac{-y}{x^2 + y^2} \right)
\]

\[
= \frac{M}{2\pi} \cos \theta = \frac{M}{2\pi} \sin \theta
\]

So

\[
U_x(\rho, \theta) = \frac{M}{2\pi} \rho
\]

\[
U_y(\rho, \theta) = 0
\]

![Consistent potential lines](image)
Line Vortex

\[ \omega(z) = -i \frac{\pi}{2\pi} \partial_z \phi = -i \frac{\pi}{2\pi} \left[ \partial_z \phi + i \theta \right] \]

\[ \psi(z, \theta) = \frac{\pi}{2\pi} \phi \]

\[ \psi(z, \theta) = -\frac{\pi}{2\pi} \partial_z \phi \]

\[ \frac{d\omega}{dz} = -i \frac{\pi}{2\pi} \phi = -i \frac{\pi}{2\pi} \left( x - iy \right) = -i \frac{\pi}{2\pi} (x - iy) \Rightarrow \frac{\pi}{2\pi} (x - iy) \]

\[ U_x = -\frac{\pi}{2\pi} \sin \theta \quad U_y = \frac{\pi}{2\pi} \cos \theta \]

On \( \gamma \)

\[ U_\alpha = 0 \quad U_\beta = \frac{\pi}{2\pi} \]

Blasius Theorem

\[ dD - i dL = (\text{Drag}) - i (\text{Lift}) \]

\[ = -p dy - i \rho \partial x \]

\[ = -i \rho \partial x^* \]

\[ \text{Common Conjugate} \]

So

\[ D - i L = -i \oint_c \rho \partial x^* \]

But Bernoulli gives \( \rho(z) \)

\[ \rho(z) = \rho_o + \frac{1}{2} \rho U^2 - \frac{1}{2} \rho (U_x^2 + U_y^2) \]

\[ = \rho_o + \frac{1}{2} \rho U^2 - \frac{1}{2} \rho (U_x + i U_y) (U_x - i U_y) \]

\[ D - i L = -i \oint_c \left[ e_o + \frac{1}{2} \rho \left( U_x + i U_y \right) \left( U_x - i U_y \right) \right] d\phi^* \]

\[ = i \oint_c \frac{1}{2} \rho \left( U_x - i U_y \right) \left( U_x + i U_y \right) d\phi^* \]

\[ = i \oint_c \frac{1}{2} \rho \left( U_x - i U_y \right) \left( U_x + i U_y \right) d\phi = i \oint_c \frac{1}{2} g \left( \frac{d\omega^2}{dz} \right) d\phi \]
Example: Flow Past a Cylinder

\[ \omega = U \frac{r^2}{2} + \frac{U a^2}{2} \]

\[ \frac{d\omega}{dz} = U - \frac{U a^2}{2} \]

\[ (\frac{d\omega}{dz})^2 = U^2 - 2Ua^2 + \frac{U^2 a^4}{2} \]

\[ \int (\frac{d\omega}{dz})^2 \, dz = 0 \quad \text{(no sources)} \quad \therefore \quad \theta = L = 0 \]

Flow Past a Rotating Cylinder

(Problem 6.9)

\[ \omega(\theta) = U_2 + \frac{U a^2}{2} + i \frac{\pi}{2\pi} \frac{\phi}{a} \left( \frac{r}{a} \right) \]

\[ S_{inc} \delta \left( \frac{r}{a} \right) = \frac{\phi}{\pi} \left( \frac{r}{a} \right) = \delta \left( \frac{r}{a} \right) + i \theta \]

\[ \omega(\theta) = (U_1 + \frac{U a^2}{2}) \cos \theta - \frac{\pi \theta}{2\pi} \]

\[ + i \left[ \pi \sin \theta \left( U_1 - \frac{U a^2}{2} \right) + \frac{\pi}{2\pi} \frac{\phi}{a} \left( \frac{a}{r} \right) \right] \]

\[ \frac{d\omega}{dz} = U - \frac{U a^2}{2} - \frac{i \pi}{2\pi} \]

**Note:** Flow vanishes (Stagnation Point) when

\[ \frac{d\omega}{dz} = 0 \quad \Rightarrow \quad U - \frac{U a^2}{2} = \frac{i \pi}{2\pi} \]

\[ \frac{U}{a} \quad \text{or} \quad \frac{U}{a} = \frac{1}{2\pi} \quad \Rightarrow \quad \left( \frac{U}{a} \right)^2 \quad \Rightarrow \quad \pi \frac{a}{U} = \frac{\pi}{2\pi} \]

**Prandtl's Theorem**

\[ \frac{d\omega}{dz} = \frac{d}{dz} \left[ \phi \left( \frac{dz}{d\theta} \right) \right] = i \int \frac{1}{2} \left( \frac{d\omega}{dz} \right)^2 \, dz = i \int \frac{1}{2} \left( \frac{d\omega}{dz} \right)^2 \, dz = \phi \frac{d\omega}{dz} \]

\[ = \phi \frac{d\omega}{dz} \]

\[ \therefore \quad \phi = \frac{1}{2} \frac{d\omega}{dz} \]
Lecture 12

- Dimensional analysis

Dimensional Variables

**Continuity**: \( \frac{\partial \rho}{\partial t} + u \cdot \nabla \rho = - \rho \nabla \cdot \mathbf{U} \)

**Navier-Stokes**: \( \frac{\partial \mathbf{U}}{\partial t} + (\mathbf{U} \cdot \nabla) \mathbf{U} = -\frac{1}{\rho} \nabla \rho + \frac{\mu}{\rho} \nabla^2 \mathbf{U} \)

Flow problems are characterized by:

- **Length Scale** = \( L^* \)
- **Flow Speed** = \( U^* \)

This sets a characteristic time \( t^* = L^*/U^* \).

Dimensionless variables:

\[ t' = \frac{t}{t^*} \quad \mathbf{\nabla}' = \frac{\mathbf{\nabla}}{L^*} \quad \mathbf{U}' = \frac{\mathbf{U}}{U^*} \]
Dimensionless Equations

Continuity
\[ \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{\partial}{\partial x} \mathbf{u} \]

Navier-Stokes
\[ \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla \mathbf{p} + \frac{1}{ho} \nabla \tau + \frac{1}{ho} \mathbf{f} \]

IF WE DEFINE A CHARACTERISTIC PRESSURE AS TWICE THE DYNAMIC PRESSURE
\[ p^* = \rho u^* \]

Then
\[ \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla \mathbf{p} + \frac{1}{ho} \nabla \left( \frac{\mu}{\rho} \nabla \mathbf{u} \right) \]

\[ \mathbf{f} \]

From the Numbers
Re

5. Nondimensional Parameters and Dynamic Similarity

Drag Force on Sphere

Figure 8.2 Drag coefficient for a sphere. The characteristic area is taken as \( A = \pi d^2 / 4 \). The reason for the sudden drop of \( C_D \) at \( Re \sim 5 \times 10^5 \) is the transition of the laminar boundary layer to a turbulent one, as explained in Chapter 10.
Drag Force on a Sphere

\[ C_d = \frac{(\text{Drag})}{\frac{1}{2} \rho U^2 A} \]

\[ A = \frac{1}{2} \pi d^2 \quad \text{for sphere} \]

\[ d = \text{diameter} \]

Low Re (Strong Viscosity)

\[ \text{Drag} \sim f(M, U, d) \]

\[ [M] \sim \frac{\text{Force}}{\text{Length}} \]

\[ \text{Drag} \sim M d U \]

Only dimensionally correct combination.

Thus

\[ \text{Drag} \sim \left( \frac{1}{\text{Re}} \right) \rho U^2 A \]

Two Limits:

- Strong Viscosity (No inertial effects)
- Small Viscosity (Only inertial)

High Re (Small Viscosity)

\[ \text{Drag} \sim \frac{f(p, U, d)}{\rho U^2 A} \]

\[ \text{Independent of } \text{Re} \]

\[ \text{Re} = \frac{\rho U}{\mu} \]

Lecture 13 & 14

- Viscosity
- Steady flow
Steady Laminar Flow in a Pipe

\[ \mathbf{u} \rightarrow \mathbf{u}_2 \rightarrow \mathbf{u} \]

**Navier-Stokes:**
\[ \rho \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + \mu \nabla^2 \mathbf{u} \]
\[ \mathbf{u} = (u, v, u_2) \quad u_2(n) \]

**Use Appendix 8, Cylindrical Coordinates**

\[ f : \quad \dot{\theta} = -\frac{1}{\mu} \left( \frac{2}{\mu} - 3 \right) \]

\[ \Theta : \quad \dot{\theta} = -\frac{1}{\mu} \frac{2}{\mu} \]  

Pressure is independent of \( \theta \)

\[ \rho \left( u_2 \right) u_2 = -\frac{2p}{3}\theta + \mu \nabla^2 u_2 \]

**No variation of \( u_2 \) along \( \Theta \)**

\[ \int \frac{2p}{3} = \frac{p}{3} \frac{2p}{\mu} + c_1 \]

Integrating twice:

\[ u_2 = \frac{p}{3\mu} \frac{2p}{\mu} + c_1 \sqrt{b} + c_2 \]

\( \text{This must vanish (why?)} \) so \( c_1 = 0 \)

**Steady Flow (Cont.)**

**Boundary Conditions:**  
\[ u_2 (n = a) = 0 \Rightarrow \text{"No slip" condition at pipe walls} \]

\[ u_2 (n = b) \text{ is well behaved} \]

\[ u_2 (n = b) = -\frac{1}{\mu} \left( \frac{2p}{3} (a^2 - b^2) \right) \]

**How much flow?**
\[ Q = \int_0^a 2\pi r \rho d\rho \quad u_2(\gamma) = -\frac{2p}{3\mu} \int_0^a r d\rho (a^2 - \gamma^2) \]

\[ = -\frac{4p}{3\mu} \frac{2p}{\mu} \]

Poisson's Law (like Ohm's Law) for pressure drop across pipe:

\[ \text{(Pressure drop)} = \text{(Length)} \times Q \left( \frac{3\mu}{4a^4} \right) \]

**Two Tubes with Equal Conductance**

\[ r = 1 \quad r = 2 \quad 8 \times L \]
Lecture 16, 17, 18

• Waves

Surface Gravity Wave

\[ \chi(x,t) = \text{vertical displacement of surface} \]

\[ h = \text{depth of fluid} \]

**Assumptions:**

- Incompressible
- Irrotational
- No viscosity
- Linear

**Boundary Conditions:**

- No vertical flow at bottom \((z = h)\)
- No "dynamic" pressure at top free surface.
  (The "dynamic" pressure is the pressure perturbation after the equilibrium hydrostatic pressure has been subtracted/removed.)
- \( \eta(x,t) \) represents the location of the top surface
- Uniform in \( y \)-direction
- Wave variation along \( x \)-direction, \( \Delta \)
Review: Surface Gravity Waves

**Step 1:** Assume a propagating disturbance

**Step 2:** Fluid response is irrotational, so \( \vec{u} = \nabla \phi \) (velocity potential)

**Step 3:** Fluid response is incompressible, so \( \nabla \cdot \vec{u} = \frac{\partial \phi}{\partial t} = 0 \)

**Step 4:** At surface, apply Bernoulli’s Principle (A.K.A. Navier-Stokes)

To obtain:

\[
\frac{\partial \phi}{\partial t} + \frac{\partial \phi}{\partial z} = 0 \quad \text{at} \quad z = 0
\]

**Step 5:** Apply boundary conditions:

- At bottom: \( \frac{\partial \phi}{\partial z} = 0 \) at \( z = -h \)
- At top: \( \phi = \frac{\partial \phi}{\partial z} = 0 \) at \( z = H \)

**Step 6:** Solve for \( \phi(x,z,t) \) by assuming the form:

\[
\phi(x,z,t) = f(z) \sin(kx - \omega t)
\]

From #3:

\[
f'' - \frac{k^2}{h^2} f = 0
\]

From #4, #5:

\[
\frac{\partial \phi}{\partial t} + g \frac{\partial \phi}{\partial z} = 0 \quad \text{at} \quad z = 0 \Rightarrow -\omega^2 f + g f' = 0
\]

---

**Shallow Water \( (kh \to 0) \)**

\[
\omega^2 = \frac{k^2}{h} g h \Rightarrow \text{non-dispersive}
\]

Phase velocity \( \frac{\omega}{k} = \sqrt{gh} \)

\[
U_x = \nu_0 \frac{\omega}{kh} \cos(kx - \omega t)
\]

\[
U_z = \nu_0 \omega \left(1 + \frac{z}{h}\right) \sin(kx - \omega t)
\]

\[
\left| \frac{|U_x|}{|U_z|} \right| = \frac{1}{\frac{h}{h}} \rightarrow 1
\]

Mostly horizontal flows back and forth

---

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Deep Water ($kh \to \infty$)

\[ \omega^2 = \kappa \Rightarrow \text{dispersive} \]

\[ \text{Phase-velocity} = \frac{\omega}{k} \sim \sqrt{\frac{\kappa}{\kappa}} \]

**Note:**

**Velocity Potential**

\[ \psi(x, z, t) = \eta_0 \left( \frac{\omega}{k} \right) e^{ikz} \sin(kx - \omega t) \quad (\text{for } z < 0) \]

\[ u_x = \eta_0 \left( \frac{\omega}{k} \right) e^{ikz} \cos(kx - \omega t) \]

\[ u_z = \eta_0 \left( \frac{\omega}{k} \right) e^{ikz} \sin(kx - \omega t) \]

\[ \frac{|u_z|}{|u_x|} \sim 0 \]

**Flow decreases exponentially to bottom... Approximation**

\[ \delta \approx h \]

---

**Wave Energy Density**

**Linear Euler Equation:**

\[ p_0 \frac{\partial u}{\partial t} = p_1 \bar{u} - \nabla p_1 \]

\[ \times \bar{u} \Rightarrow \quad p_0 \frac{\partial}{\partial t} \left( \frac{1}{2} u^2 \right) = -p_1 \bar{u} \cdot \bar{u} - \nabla \cdot \bar{p}_1 \]

\[ = -\rho \bar{u} \cdot \bar{g} - \nabla \cdot (\bar{u} \bar{p}_1) \]

**Wave Mechanical Energy**

**Wave Potential Energy**

**Wave Energy Converted Away**

**What is Potential Energy?**

\[ \frac{dE_{\text{pot}}}{dt} = \rho \bar{u} \cdot \bar{g} \quad \text{but} \quad \bar{u} \cdot \bar{g} = -\frac{\partial p_1}{\partial t} \frac{2p_0}{\kappa^2} \]

\[ = -\rho \bar{g} \frac{\partial p_1}{\partial t} \frac{1}{\kappa^2} \frac{1}{2} \left( \frac{p_0}{\kappa} \right) \]

\[ = \frac{g^2}{p_0} \frac{\partial p_1}{\partial t} \left( \frac{1}{2} \frac{p_0^2}{\kappa^2} \right) \]

\[ \therefore E_{\text{pot}} = \frac{g^2}{p_0} \frac{\partial p_1}{\partial t} \left( \frac{1}{2} \frac{p_0^2}{\kappa^2} \right) \]

**What is \( p_1 \)?**

\[ \frac{\partial p_1}{\partial t} = -\frac{\partial u_2}{\partial z} \frac{\partial p_0}{\partial t} \quad \text{but} \quad u_2 = \frac{2g}{\kappa^2} \Rightarrow \frac{\partial p_1}{\partial t} = -\frac{2g}{\kappa^2} \frac{\partial p_0}{\partial t} \]

Thus \( p_1 \propto \text{displacement} \Rightarrow \rho_1 = -\frac{\kappa^2 \rho_0}{2g} \)
Wave Kinetic Energy Density

**What is kinetic (mechanical) wave energy density?**

\[ E_{k,\omega} = \frac{1}{2} \rho_o \left( u_x^2 + u_z^2 \right) \]

\[ = \frac{1}{2} \rho_o \left( \left( \frac{h_x}{\lambda_x} \right)^2 + 1 \right) u_z^2 \]

But \( u_z \) is found from:

\[ \frac{2 \rho_u}{\lambda_x} + u_z \frac{2 \rho_o}{\lambda_x} = 0 \implies -j \omega \rho_u + u_z \frac{2 \rho_o}{\lambda_x} = 0 \]

or

\[ u_z = \frac{j \omega \rho_u}{(2 \rho_o/\lambda_x)} = -j \frac{\omega}{\lambda_x} \frac{g}{\rho_o} \rho_1 \]

So

\[ E_{k,\omega} = \frac{1}{2} \rho_o \left[ \left( \frac{h_x}{\lambda_x} \right)^2 + \right] \frac{\omega^2 g^2 \rho_1^2}{\lambda_x^2 \rho_o^2} \]

\[ = \frac{1}{2} \rho_1^2 \frac{g^2}{\rho_o - \lambda_x^2} = E_{pot} !!! \]

---

Wave Energy Flux

**Wave energy flux**:

\[ \overline{\omega_0} P_i = u_x \rho_i x^2 + u_z \rho_i z^2 \]

But \( \rho_i = \frac{\omega}{\lambda_x} \rho_o u_x \)

Therefore:

\[ \overline{\omega_0} P_i = \left( \frac{h_x}{\lambda_x} \right) \rho_o \left( \frac{h_z}{\lambda_z} \right) u_x^2 \hat{x} - \frac{\omega}{\lambda_x} \rho_o \left( \frac{h_z}{\lambda_z} \right) u_z^2 \hat{z} \]

\[ = \rho_o \ u_x^2 \frac{\omega}{\lambda_x} \left( \frac{h_x}{\lambda_x} \right) \left( \frac{h_z}{\lambda_z} \right) \]

But \( \overline{\omega_0} = \frac{\lambda_g}{\lambda_x^2} \left( \frac{h_x}{\lambda_x} \right) \left( \frac{h_z}{\lambda_z} \right) \)

so

\[ \overline{\omega_0} P_i = \rho_o \overline{\omega_0} u_x^2 \frac{\omega}{\lambda_x} \frac{h_z}{h_x} \left( \frac{\omega g \rho_1}{\rho_o} \right)^2 \]

\[ = \overline{\omega_0} \frac{\omega}{\lambda_x} \frac{h_z}{h_x} \frac{g^2 \rho_1^2}{\rho_o} \]

\[ = \rho_1^2 \overline{\omega_0} \frac{g^2}{\rho_o \lambda_x^2} = \overline{\omega_0} \left( E_{pot} + E_{k,\omega} \right) !!! \]
Example: Water Suspended in a Capillary Tube

\[
P_{\text{in}} - P_{\text{out}} = \frac{2\sigma}{R}
\]

But \( P_{\text{in}} = \rho g h \gg P_{\text{out}} \)

So \( h = \frac{2\sigma}{\rho g} \Rightarrow \frac{a^2}{12} = \text{maximum height of suspended water in capillary} \)

\[
\sqrt{\frac{2\sigma}{\rho g}} = a = \text{capillary constant}
\]

\[
\begin{align*}
\sigma &= 0.074 \text{ N/m} \\
\rho &= 1000 \text{ kg/m}^3 \text{ water} \\
\gamma &= 9.8 \text{ m/s}^2
\end{align*}
\]

\( a = 3.9 \text{ mm} \)

Gravity Waves with Surface Tension

Surface tension acts like a restoring force...

Pulls on surface to make \( \frac{\partial \eta}{\partial x} \rightarrow 0 \)

Surface waves:

\( \vec{u} = \nabla \varphi \quad \nabla^2 \varphi = 0 \quad \varphi(x, t) = f(z) \sin(\beta x - \omega t) \)

Bernoulli:

\[
\frac{\partial \varphi}{\partial t} + \frac{1}{2} \nabla^2 \varphi + \frac{(\rho_{\text{in}} - \rho_{\text{out}})}{\rho} + \gamma \eta = 0 \quad \text{at surface}
\]

Linear part:

\[
\frac{\partial \varphi}{\partial t} = \frac{\sigma \partial^2 \eta}{\rho} \left( \frac{\partial^2 \eta}{\partial x^2} - \gamma \eta \right) \quad \text{at } \varphi = 0
\]

Surface constant:

\[
\frac{\partial^2 \varphi}{\partial z^2} = \frac{2\gamma}{\lambda} \quad \text{at } z = 0
\]
Heavy Fluid on Top of Light Fluid

(Rayleigh-Taylor Instability)

\[ \rho_{\text{top}} \gg \rho_{\text{bottom}} \]

\[ \psi(x,t) = \rho(x,t) \cos(4x - \omega t) \]

Gravity Wave with Reversed Density Gradient or Reversed Direction of Gravity

\[ \nabla^2 \psi = 0 \quad \psi(x,0,0) = f(0) \sim (4x - \omega t) \]

With \[ f(0) = e^{-4t^2}(z^2 > 0) \]

Surface Conditions:

\[ \begin{align*}
- \frac{\partial \psi}{\partial t} &= \frac{2\eta}{2\epsilon} \\
\frac{\partial^2 \psi}{\partial x^2} &= +9\eta
\end{align*} \]

\[ \eta f = \omega \eta_0 \quad \xi \omega t = \sqrt{\eta_0} \xi \]

Surface Tension

Rayleigh-Taylor with Surface Tension

What is unstable growth rate with surface tension?

\[ \begin{align*}
\frac{2\psi}{\partial t} &= \frac{6 \eta}{\bar{\rho}} \frac{2\eta}{2x^2} + 9\eta \\
- \frac{2\psi}{\partial t} &= \frac{2\eta}{2\epsilon} \\
\omega^2 &= -\frac{\eta_0}{\epsilon} - \frac{6 \eta}{\bar{\rho} \xi^2}
\end{align*} \]

If \[ \frac{6 \eta}{\bar{\rho} \xi^2} > \eta, \text{ then Rayleigh-Taylor instability is stabilized!} \]

\[ \frac{6}{\bar{\rho}} \left( \frac{\pi}{L} \right)^2 > \eta \quad \text{or} \quad L < \pi \sqrt{\frac{\eta}{\bar{\rho} \xi^2}} \]
Equilibria can be Stable or Unstable

Four types of equilibria:

- **Stable**
- **Unstable**
- **Neutral**
- **Non-linear unstable**
Bénard Thermal Instability

Use "BOUSSINESQ" Approximation For Fluid Dynamics

Assume \( \nabla \cdot \mathbf{U} = 0 \)

And \( \rho(T) = \rho_0 \left[ 1 - \alpha(T - T_0) \right] \)

\( \alpha > 0 \) means hot fluid is less dense than cold fluid.

Hot fluid rises from bottom plate
Cold fluid falls from upper plate
Thermal diffusivity and viscosity damp convective fluid motion

Equations of Fluid Dynamics

Navier-Stokes:
\[
\rho_0 \frac{DU}{Dt} = -\nabla \rho - g \rho(T) \mathbf{e} + \mu \nabla^2 \mathbf{U}
\]

Thermal Diffusion:
\[
\frac{DT}{Dt} = \kappa \nabla^2 T
\]

Incompressible:
\( \nabla \cdot \mathbf{U} = 0 \)

Dynamical Variables:
\[
\bar{U}(x, y, z, t), \quad T = \bar{T}(z) + \hat{T}(x, y, z, t), \quad \bar{P}, \bar{P} \text{ are fluctuating variables}
\]
\[
\hat{P}(x, y, z, t), \quad \hat{T}, \hat{T} \text{ are static functions of hydrostatic equilibrium}
\]

Solution:

- Find Hydrostatic Relationship
- Solve Linearized Equations For Eigenvalues, \( \lambda \),
  and for "normal modes" that satisfy boundary conditions
Nonlinear Equations for Perturbations

Subtract Hydrostatic Equation from Equations for Fluid Dynamics to get Non-Linear Equations for Perturbations.

\[
\begin{align*}
\frac{2\dot{u}_x}{2\xi} + (\overline{u} \cdot \nabla) u_x &= -\frac{1}{8\xi} \frac{2\dot{y}}{2\xi} + 9\alpha \frac{\dot{T}}{T} + \gamma \nabla^2 u_x \\
\frac{2\dot{u}_x}{2\xi} + (\overline{u} \cdot \nabla) u_x &= -\frac{1}{8\xi} \frac{2\dot{y}}{2\xi} + \gamma \nabla^2 u_x \\
\frac{2\dot{u}_x}{2\xi} + (\overline{u} \cdot \nabla) u_x &= -\frac{1}{8\xi} \frac{2\dot{y}}{2\xi} + \gamma \nabla^2 u_y \\
\frac{2\dot{u}_x}{2\xi} + \frac{2\dot{u}_y}{2\xi} &= 0 \\
\frac{2\ddot{T}}{2\xi} + (\overline{u} \cdot \nabla) (\ddot{T} + \frac{T}{T}) &= \kappa \nabla^2 \ddot{T}
\end{align*}
\]

Unknown Functions: \( u_x, u_y, u_z, \overline{u}, \ddot{T} \)

5 Equations + 5 Unknowns

Two Coupled PDEs forms a
Linear Eigensystem

Perturbed Vertical Velocity:
\[
\frac{2}{2\xi} \nabla^2 u_x - \gamma \nabla^2 \nabla^2 u_x = 9\alpha \left( \nabla^2 - \frac{2\dot{z}}{2\xi} \right) \ddot{T}
\]

Perturbed Thermal Diffusion:
\[
\frac{2}{2\xi} \ddot{T} - \kappa \nabla^2 \ddot{T} = -\frac{\dot{u}_x}{\xi} \tilde{T} \quad (\nu \equiv \frac{\dot{u}_x}{\xi})
\]

Once \( u_x(x, y, z, \xi) \) and \( \ddot{T}(x, y, z, \xi) \) are known, we can find the forms for \( (\tilde{u}, u_x, u_y) \).
Kelvin-Helmholtz Instability

\[ u_1, \rho_1 \quad \rightarrow \quad x \]

\[ u_2, \rho_2 \]

**Kelvin-Helmholtz Instability** is when the destabilizing effect of shear overcomes the stabilizing effect of stratification.

Instability/Stability Condition

\[
\rho_1 (\omega - hu_1)^2 + \rho_2 (\omega - hu_2)^2 = \lambda g (\rho_2 - \rho_1)
\]

\[
\omega = \frac{\lambda (\rho_1 u_1 + \rho_2 u_2)}{\rho_1 + \rho_2} \pm \frac{\sqrt{\lambda g (\rho_2^2 - \rho_1^2) - \lambda^2 \rho_1 \rho_2 (u_1 - u_2)^2}}{\rho_1 + \rho_2}
\]

**Instability when**

\[
(u_1 - u_2)^2 > \frac{\lambda g (\rho_2^2 - \rho_1^2)}{\rho_1 \rho_2}
\]

**Stabilizing by Stratification**

**Instability by Shear**
Centrifugal Instability

- Occurs when centrifugal force overcomes viscous diffusion. (Similar to Bénard thermal instability when thermal buoyancy overcomes viscosity.)

\[ \text{IF } \lambda_1 = \lambda_2, \text{ Rigid Rotation} \]

\[ \text{IF } R_1^3 \lambda_1 = R_2^3 \lambda_2, \text{ Constant Circulation} \]

\[ \Gamma(\lambda) = 2\pi R U_0(\lambda) \quad \lambda (\lambda) = U_0(\lambda)/\lambda \]

\[ \Gamma = 2\pi \lambda^2 \lambda (\lambda) \]

Energetics of Fluctuations in Shear Flows

Energy (time-averaged) equation:

\[ \frac{\partial}{\partial t} \int \! \! \! \int dxdy \hat{u} \cdot \left[ \frac{\partial^2}{\partial x^2} + (\nabla \cdot \vec{u}) \cdot \nabla (\nabla \cdot \vec{u}) \right] \]

\[ = -\frac{1}{\mu} \nabla P + \nabla \cdot \hat{\sigma} \]

\[ + \text{Time Average} \]

\[ \frac{\partial}{\partial t} \int \! \! \! \int dxdy \frac{1}{2} (\hat{u})^2 = -\int \! \! \! \int dxdy \hat{u} \cdot \left[ \frac{\partial^2 \hat{u} \cdot \partial x}{\partial x} + (\nabla \cdot \vec{u}) \hat{u} + (\nabla \cdot \vec{u}) \hat{u} + \frac{1}{2} \hat{u} \cdot \hat{\sigma} \right] \]

After integrating by parts and using boundary conditions that \( \hat{u} \to 0 \) at walls

\[ \frac{\partial}{\partial t} \int \! \! \! \int dxdy \frac{1}{2} (\hat{u})^2 = -\int \! \! \! \int dxdy \left[ \frac{2\hat{u} \cdot \partial x}{\partial x} + \frac{1}{2} \left( \frac{\partial \hat{u}}{\partial x} \right)^2 \right] \]

\[ \text{Reynolds Stress} \]
Reynolds Stress May Destabilize Shear Flow

\[ \mathbf{\ddot{u}(y)} \]

- \[ \mathbf{\ddot{u}_x} \]
  - If \( \ddot{u}_y > 0 \), then slow fluid moves into faster region.
  - This slows down average flow in x-direction.
  - \( \ddot{u}_x \ddot{u}_y < 0 \)

- \[ \mathbf{\ddot{u}(y)} \]
  - If \( \ddot{u}_y < 0 \), then fast fluid moves into slower region.
  - This flow will increase average flow in x-direction.
  - \( \ddot{u}_x \ddot{u}_y < 0 \)

\[ - \int \rho_0 dy \ddot{u}_x \ddot{u}_y \frac{\partial \gamma}{\partial y} > 0 \]

And this causes fluctuations to increase in amplitude!!

Review of Waves/Instabilities

**Surface Gravity Waves (Rayleigh-Taylor Instability)**

\[ \nabla \cdot \mathbf{u} = 0 \quad \nabla \times \mathbf{u} = 0 \quad \mathbf{u} \rightarrow 0 \] (except at thin surface boundary layer)

Energy = Mechanical + Gravitational Potential

One interface

One (or none) boundary condition at bottom

**Internal Gravity (Buoyancy) Wave**

\[ \nabla \cdot \mathbf{u} = 0 \quad \nabla \times \mathbf{u} \neq 0 \]

Energy = Mechanical + Gravitational Potential

**Capillary Wave**

\[ \nabla \cdot \mathbf{u} = 0 \quad \nabla \times \mathbf{u} = 0 \] Surface tension

Energy = Mechanical + Tension + Gravity

**Sound Waves**

\[ \nabla \cdot \mathbf{u} \neq 0 \quad \nabla \times \mathbf{u} = 0 \]

Energy = Mechanical + Pressure/Compression
Review of Waves/Instabilities

**Denaro Thermal Instability**

\[ \nabla \cdot \vec{U} = 0 \]
\[ p(T) = p_0 \left[ 1 - \alpha(T - T_0) \right] \] (i.e. hot fluid is less dense)

**Thermal Conduction**

**Viscosity**

**Two Boundary Conditions**

**Prandtl Number**

\[ Pr = \frac{\nu}{\kappa} \]

**Rayleigh Number**

\[ Ra = \frac{g \Delta \rho L^4}{\nu \kappa} \]

**Kelvin–Helmholtz**

\[ \nabla \cdot \vec{U} = 0 \quad \nabla \times \vec{U} = 0 \]

**Interface Driven by Free Mechanical Energy of Shear Flow**

**Centrifugal Instability**

\[ \nabla \cdot \vec{U} = 0 \quad \nabla \times \vec{U} = 0 \]

**Viscosity + Rotation + Cylindrical Coordinates**

**Parallely Sheared Flow**

\[ \nabla \cdot \vec{U} = 0 \quad \nabla \times \vec{U} = 0 \]

**Viscosity + Ober-Summerfeld Eq.**

---

**Energetics of Fluctuations in Shear Flows**

**Energy (Time-Averaged) Equation**:

\[ \frac{\partial}{\partial t} \int dxdy \left[ \frac{2}{\rho} \frac{\partial \bar{U}}{\partial t} + \bar{U} \cdot \nabla \bar{U} \right] = -\frac{1}{\rho} \nabla p + \nabla \cdot \bar{\tau} \]

**Time Average**

\[ \frac{\partial}{\partial t} \int dxdy \bar{U} \cdot \left[ \frac{2}{\rho} \frac{\partial \bar{U}}{\partial t} + \bar{U} \cdot \nabla \bar{U} \right] = -\frac{1}{\rho} \nabla p + \nabla \cdot \bar{\tau} \]

**After Integrating at Parts**

**And Using Boundary Conditions That \( \bar{U} \to 0 \) at Walls**

\[ \frac{\partial}{\partial t} \int dxdy \bar{U} \cdot \left[ \frac{2}{\rho} \frac{\partial \bar{U}}{\partial t} + \bar{U} \cdot \nabla \bar{U} \right] = -\frac{1}{\rho} \nabla p + \nabla \cdot \bar{\tau} \]

**Reynolds Stress**

**Viscosity Dissipation**
How Big is the Reynolds Stress?

\[
- \frac{\partial \bar{u}}{\partial y} \frac{\partial \bar{u}}{\partial y} = \text{Time-averaged energy released by fluctuations in shear flow}
\]

\[
- \gamma \left( \frac{\partial \bar{u}}{\partial x} \right)^2 = \text{Time-averaged viscous dissipation of fluctuation energy}
\]

Turbulent "Eddy" viscosity \[ \text{Viscosity} \sim \frac{\bar{u}^2}{\gamma (\sigma/L)^2} = \frac{\bar{u}^2 L}{\gamma} = \text{Re} \]

At high Re, turbulent "eddy" viscosity is much, much bigger than real viscosity!

Three Size Scales for Turbulence

"Microscale"

\[ L_e \ll \text{Size of eddies when viscosity is important} \]

\[ \text{Re}(L_e) \sim 1 \]

"Inertial" or "Cascade"

\[ \sigma \ll \text{Mixing couples, fluctuation energy from large to small scales} \]

"System"

\[ L \ll \text{Most unstable length scale} \]

\[ \text{Large eddies, "stir" while pot} \]

Viscosity matters here

Energy is dissipated

\[ \rightarrow \text{Energy flows/transported from large to small (usually)} \rightarrow \]

\[ \rightarrow \text{gets smaller} \quad \bar{u}(L) \quad \text{gets larger} \rightarrow \]
Power-Law Spectrum of Fluctuation Intensity

At the relatively small scales represented by wavenumbers $K \approx 1^{-1}$, there is no direct interaction between the turbulence and the motion of the large, energy-containing eddies. This is because the small scales have been generated by a long series of small steps, losing information at each step. The spectrum in this range of large wavenumbers is nearly isotropic, as only the large eddies are aware of the directions of mean gradients. The spectrum here does not depend on how much energy is present at large scales (where most of the energy is contained), or the scales at which most of the energy is present. The spectrum in this range depends only on the parameters that determine the nature of the small-scale flow, so that we can write

$$S = S(K, e, v) \sim K^{-5/3}.$$ 

The range of wavenumbers $K \approx 1^{-1}$ is usually called the equilibrium range. The dissipating wavenumbers with $K \approx 1^{-1}$, beyond which the spectrum falls off very rapidly, form the high end of the equilibrium range (Figure 13.12). The lower end for which $1^{-1} \approx K \approx 1^{-1}$, is called the inertial subrange, as only the transfer of energy by inertial forces (vortex stretching) takes place in this range. Both production and dissipation are small in the inertial subrange. The production of energy by large eddies causes a peak of $S$ at a certain $K \approx 1^{-1}$, and the dissipation of energy causes a sharp drop of $S$ for $K \approx 1^{-1}$. The question is, how does $S$ vary with $K$ between the two limits in the inertial subrange?

**Figure 13.12** A typical wavenumber spectrum observed in the ocean, plotted on a log–log scale. The unit of $S$ is arbitrary, and the dots represent hypothetical data.

---

**Dimensional Analysis**

$$\left[ \frac{1}{\rho} \frac{dE}{dt} \right] \sim \{ \text{ENERGY DISSIPATED RATE PER UNIT MASS} \}
\sim \left[ \frac{L^2}{t^3} \right] \sim [L]^2 [t]^{-3}$$

$$[\Delta U] \sim \{ \text{MEAN (RMS) FLUCTUATING FLOW} \}
\sim [L] [t]^{-1}$$

$$[L] \sim \{ \text{SYSTEM SIZE} \}$$

$$[\nu] \sim \{ \text{VISCOITY, KINEMATIC} \}
\sim [L]^2 [t]^{-1}$$

**There are no other dimensional parameters!**
What is the magnitude of the fluctuation energy?

\[
[\varepsilon] \sim [L]^3 [T]^{-3} = [(\Delta u)]^2 [L]^3 [T]^{-3} \phi
\]

At large scale, turbulence cannot depend upon (real) viscosity.

\[\varepsilon_{00} \sim R_o Y\]

So \[2 = \alpha + \beta\] \quad \alpha = 3

\[-3 = -\alpha\] \quad \beta = -1

\[\varepsilon \propto (\Delta u)^3 / L\]

\textbf{Energy Spectrum of Fluctuation}

(Kolmogorov’s Dimensional Analysis)

- \[k = \frac{2\pi}{L}\] \textit{Wavenumber associated with fluctuation of size } \(L\)
- \[E(k) \propto k\] \textit{Mechanical energy (time-average)}
  \textit{At range } \(k \sim k_o \pm S_k\)

\[
\left[ E(k) \right] \sim \left[ \text{Energy (per unit mass)} / \text{wave number} \right] \sim \left[ \frac{L^2}{T^2} \right] \left[ L \right]
\]

\[
\sim \left[ \varepsilon \right]^\alpha \left[ \frac{L}{T} \right]^{\beta} \sim \left[ \frac{L^2}{T^2} \right]^{\alpha} \left[ L \right]^{-\beta}
\]

\[
3 = 2\alpha - \beta \quad \alpha = \frac{2}{3}
\]

\[
2 = 3\alpha \quad \beta = -\frac{5}{3}
\]

\[
E(k) \propto \varepsilon^{2/3} / S_{k}^{5/3}
\]

\[!!\]
Lecture 22

- Viscous Boundary Layer

Uniform Flow Across a Stationary Flat Plate
(Blasius, 1908)
What are the magnitudes of the terms in Navier-Stokes?

\[ u_y \sim u_\infty \frac{\delta}{L} \sim \frac{u_\infty}{\sqrt{Re}} \]

\[ |p - p_\infty| \sim \rho u_\infty^2 \quad \text{From Bernoulli's Principle} \]

We have:

\[ 2: \quad U_x \frac{2u_x}{x} + U_r \frac{2u_r}{y} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \gamma \frac{2u_x}{x^2} + \gamma \frac{2^3 u_x}{x^2} \]

\[ 3: \quad U_x \frac{2u_x}{x} + U_r \frac{2u_r}{y} = -\frac{1}{\rho} \frac{\partial p}{\partial y} + \gamma \frac{2u_r}{y^2} + \gamma \frac{2^3 u_r}{y^2} \]

\[ 4: \quad \frac{2u_x}{x} + \frac{2u_r}{y} = 0 \]

\[ 5: \quad D \approx -\frac{1}{\rho} \frac{\partial p}{\partial y} \]

Blasius Flat Plate Solution

- Since for uniform flow along a flat plate \( u_\infty = \text{constant} \), then \( p = \text{constant} \) along \( x \)

- Equations to solve

\[ U_x \frac{2u_x}{x} + U_r \frac{2u_r}{y} = \gamma \frac{2^3 u_x}{x^2} \]

\[ \frac{2u_x}{x} + \frac{2u_r}{y} = 0 \]

- Incompressible

\[ (\nabla \cdot \mathbf{U} = 0) \quad \mathbf{U} = -\frac{\gamma}{x} \times \nabla \psi = \begin{cases} \frac{\gamma}{2y} & u_x = \frac{2y}{2x} \\ \frac{\gamma}{2x} & u_r = -\frac{2y}{2x} \end{cases} \]

- PDE nonlinear 2D

\[ \frac{2\psi}{2y} \frac{2^2 \psi}{2x \cdot 2y} - \frac{2\psi}{2x} \frac{2^2 \psi}{2y^2} = \gamma \frac{2^3 \psi}{2y^2} \]
Boundary Conditions

\[ \frac{2y^2 + 2y^3}{2y^2 + 2y^3} = \frac{2y^4}{2y^3} = \gamma \]

For \( y \to \delta \), \( u \to u_\infty \) and \( y \to \infty \)

For \( \tau = 0 \) \( \overline{\tau} = 0 \) \( (u_y = u_T = 0) \frac{2y^4}{2y^3} = 0 \) as \( \tau = 0 \)

**Plasius Solution**

**Similarity:** \( u_x (x, y) = u_\infty g(\eta) \)

\[ \eta = \frac{y}{\delta(x)} \]

**General Shape of Boundary Does Not Change**

Along Boundary!

---

**What is \( f(\eta) \)?**

(continued...)

\[ \frac{2y^2 + 2y^3}{2y^2 + 2y^3} = \frac{2y^4}{2y^3} = \gamma \frac{2y^4}{2y^3} \]

\[ - u_\infty \frac{1}{\delta} \frac{\partial \delta}{\partial x} \eta f' f'' - u_\infty \frac{1}{\delta} \frac{\partial \delta}{\partial x} f'' \left[ f' \right] = u_\infty \frac{1}{\delta} f''' \]

or

\[ - (\frac{u_\infty \delta}{\gamma} \frac{\partial \delta}{\partial x}) f f'' = f''' \]

But \( \frac{u_\infty \delta}{\gamma} \frac{\partial \delta}{\partial x} = \frac{1}{2} \) or \( \delta = \sqrt{\frac{y}{u_\infty}} \)

Thus

\[ \frac{1}{2} f f' + f'' = 0 \]

\[ \frac{df}{d\eta} \to 1 \quad \eta \to \infty \quad (\gamma \to \infty) \]

\[ f(0) = \frac{df}{d\eta} \bigg|_0 = 0 \quad \text{as} \quad \eta \to \infty \]
Lecture 23

- Geophysical fluid dynamics

Geostrophic "Statics"

LET \( \frac{\partial}{\partial t} \to 0 \) "STATICS".

**Navier-Stokes in Rotating Frame...**

\[
\mathbf{\bar{u}} \cdot \nabla \mathbf{\bar{u}} + 2 \nabla \times \mathbf{\bar{u}} = \frac{1}{\rho_0} \frac{\partial p}{\partial t} + \frac{\partial p}{\partial \rho} \mathbf{\bar{v}} + \text{(viscosity)}
\]

SET Law \( \text{Rossby Number} \):

\[
\mathbf{R}_0 = \text{Rossby Number} = \frac{\text{nonlinear forces}}{\text{curvilinear force}} = \frac{u^2 / c}{f L}
\]

Ignoring \( \mathbf{\bar{u}} \cdot \nabla \mathbf{\bar{u}} \) and viscosity, then

\[
\mathbf{v} \cdot \nabla \mathbf{v} \gg \frac{u^2}{c L} \ll 1
\]

**Horizontal Force Balance Times:**

\[\mathbf{\hat{x}}: \quad -\mathbf{f} \mathbf{u}_x = -\frac{1}{\rho_0} \frac{\partial p}{\partial x} \quad \Rightarrow \quad \mathbf{u}_x = \frac{1}{\rho_0} \frac{\partial p}{\partial x} \]

\[\mathbf{\hat{y}}: \quad \mathbf{f} \mathbf{u}_y = -\frac{1}{\rho_0} \frac{\partial p}{\partial y} \quad \Rightarrow \quad \mathbf{u}_y = -\frac{1}{\rho_0} \frac{\partial p}{\partial y} \]

This is important!
Geostrophic Flows at Mid-latitudes

![Diagram of high and low pressure systems]

\[ \vec{U}_H = \text{HORIZONTAL FLOW} = \{ u_x, u_y \} \]

\[ \vec{v}_H = \text{HORIZONTAL PRESSURE GRADIENT} = \{ \frac{\partial p}{\partial x}, \frac{\partial p}{\partial y} \} \]

\[ \vec{U}_H \cdot \vec{v}_H = u_x \frac{\partial p}{\partial x} + u_y \frac{\partial p}{\partial y} \quad \text{but} \quad u_x = -\frac{1}{\rho_f} \frac{\partial p}{\partial y} \]

\[ u_y = \frac{1}{\rho_f} \frac{\partial p}{\partial x} \]

so

\[ = 0 \quad \text{Flow is perpendicular to pressure gradient.} \]

Dynamics of a Thin Layer
("Shallow" Fluid Equations)

\[ \eta(x, y, t) \]

\[ \rho = \rho g (\eta + h - z) \]

\[ \frac{\partial \eta}{\partial x} = \rho \frac{\partial \eta}{\partial x} \]

\[ \frac{\partial \eta}{\partial y} = \rho \frac{\partial \eta}{\partial y} \]

\[ \text{NOTE: THESE ARE (APPEAR) INDEPENDENT OF z. SO FLOW HORIZONTAL IS ALSO INDEPENDENT OF z.} \]

\[ \nabla \cdot \vec{u} = 0 = \frac{2u_x}{dx} + \frac{2u_y}{dy} + \frac{2u_z}{dz} \]

\[ \int_0^H \nabla \cdot (\vec{u} \cdot \vec{a}) = (\eta + h) \left( \frac{2u_x}{dx} + \frac{2u_y}{dy} \right) + u_x (H + \eta) - u_x (0) = 0 \]

\[ \text{but} \quad u_x (H + \eta) = \frac{\partial \eta}{\partial x} = \frac{\partial \eta}{\partial x} = \frac{2\eta}{dx} + u_x \frac{\partial \eta}{\partial x} + u_y \frac{\partial \eta}{\partial y} \]

Thus,

\[ \frac{\partial \eta}{\partial x} + \int u_x \frac{\partial \eta}{\partial x} + (H + \eta) \frac{2u_y}{dy} + \int u_y \frac{\partial \eta}{\partial y} + (H + \eta) \frac{2u_z}{dz} = 0 \]
Euler Equation for Rotating Thin Fluid

\[
\begin{align*}
\frac{\partial u_x}{\partial t} + (u_x v_y) u_x - f v_x &= -g \frac{\partial \eta}{\partial x} \\
\frac{\partial u_y}{\partial t} + (u_x v_y) u_y + f v_x &= -g \frac{\partial \eta}{\partial y}
\end{align*}
\]

Integrated Continuity:

\[
\frac{\partial h}{\partial t} + \frac{2}{2x} (u_x h) + \frac{2}{2y} (u_y h) = 0 \quad (h = H + \lambda)
\]

1. Linearized, these equations define Rossby waves.
2. Combining the full nonlinear terms, prove conservation of potential vorticity.

Conservation of "Potential" Vorticity

\[
\begin{align*}
\frac{2 \eta \lambda}{2x} + u_x v_y \eta_x + (f + \lambda)(\frac{2u_x}{2x} + \frac{2u_y}{2y}) + \beta u_y &= 0 \\
\frac{\partial \eta}{\partial t} - \frac{1}{h} \frac{\partial h}{\partial t} \frac{h}{\partial x} + \frac{\partial}{\partial t} &= 0
\end{align*}
\]

Therefore

\[
\frac{\partial \eta}{\partial t} - \frac{1}{h} (f + \lambda) \frac{\partial h}{\partial t} + \frac{\partial}{\partial t} = 0
\]

or

\[
\frac{D}{D \tau} \left( \frac{\eta \lambda + f}{h} \right) = 0
\]

\( f = \text{planetary vorticity} \)
\( \lambda = \text{relative vorticity} \)

Potential vorticity = \( \frac{\eta \lambda + f}{h} \) = constant
Physics of Rossby Waves

As a fluid element moves in latitude, $N_2$, relative vorticity must change accordingly.
- $N_2$ increasing as fluid moves southward
- $N_2$ decreasing as fluid moves northward.

Rossby Wave Equation

1) Linearized equation for potential vorticity:
   \[ \frac{D}{Dt} \left( \frac{N_2 + f}{\beta} \right) = \frac{3N_2}{2\beta} + u_\theta \beta - \frac{f}{\beta} \frac{2\theta}{\beta} = 0 \]

2) Geostrophic flow:
   \[ \begin{align*}
   u_\theta &= -\frac{f}{\beta} \frac{2\theta}{\beta y} \\
   u_\theta &= +\frac{f}{\beta} \frac{2\theta}{\beta y} \\
   N_2 &= \frac{2u_\theta}{\beta} - \frac{2\theta}{\beta} \\
   &= \frac{g}{f} \left( \frac{2\theta}{\beta} + \frac{2\theta}{\beta y} \right)
   \end{align*} \]

Contents:

\[ \frac{gH}{f} \frac{2}{2\beta} \left( \frac{2\theta}{\beta x} + \frac{2\theta}{\beta y} \right) + \frac{gH}{f} \frac{2\theta}{\beta x} \beta - \frac{f}{\beta} \frac{2\theta}{\beta x} = 0 \]

Define $c = \sqrt{gH}$ = Speed of gravity wave (shallow)

\[ \Lambda = \frac{c}{f} = \text{Rossby Radius} \]

Then:

\[ \frac{2}{2\beta} \left( \frac{2\theta}{\beta x} + \frac{2\theta}{\beta y} \right) - \frac{1}{\Lambda^2} \frac{2\theta}{\beta} + \frac{2\theta}{\beta x} = 0 \]
Final Exam Questions...

1. Bernoulli’s
2. Dimensional analysis, Potential flow, Scaling
3. Complex velocity potential
4. Linearized disturbances (Gravity Waves)
5. Linearized disturbances (Boussinseq Eq)
6. Motion of a line vortex
7. Applying Blasius theorem, drag