APPH 4200
Physics of Fluids

Review

23 Lectures

> 700 pages of text

Final Exam Questions...

1. Bernoulli's
2. Dimensional analysis, Potential flow, Scaling
3. Complex velocity potential
4. Motion of a line vortex, complex velocity-potential
5. Linearized disturbances: waves (Boussinseq Eq)
6. Linearized disturbances (Gravity Waves/Instabilities)
7. Viscosity, high-Re flow, drag, boundary layers

1 Problem 1 (40 points)

Fig. 1 above shows the geometry of fluid flow directed through a tube connected to a circular plate. The fluid passes through the tube, then radially outward in an axisymmetric way. At a small distance, h, beneath the circular plate, is a rigid disk that is placed on a surface. The radius of the tube is a, and the radius of the plate is R.

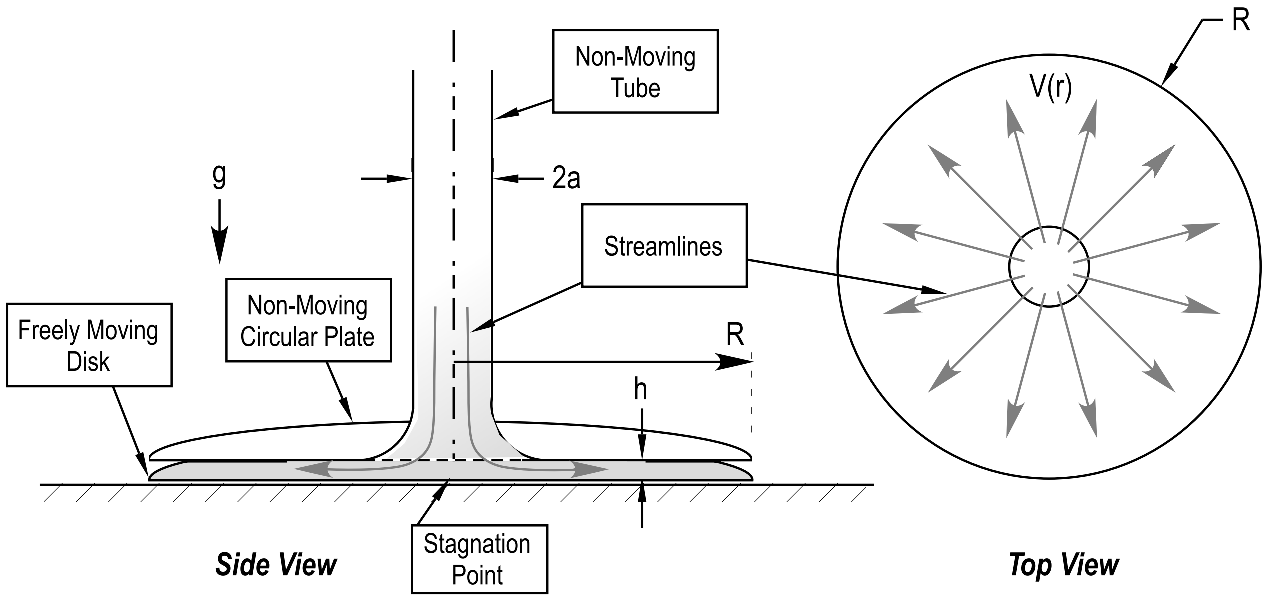

Figure 1: For Question 1. Diagram of a tube connected to a circular plate located just above a disk. When the fluid flow is sufficiently fast, the net force on the disk is upward.

1 When the fluid flow is small, then the disk feels a net force directed downward. When the fluid flow is sufficiently fast, then the disk feels a net "lift" that can exceed the gravitational force downward.

Your problem is to compute the net force on the disk due to the fluid flow and derive a condition for disk levitation, *i.e.* when the lift due to the rapid radial flow of fluid exceeds the gravitation force.

The flow in this apparatus is actually complicated, especially at the junction between the tube and the circular disk. However, you can simplify your analysis with some assumptions. Assume that $R \gg a$, and that $h \ll a$. Assume that the fluid flow is inviscid, and ignore the pressure variations due to the gravitation force on the fluid (*i.e.* terms like $\rho g h$.) Then, it is reasonable to ignore the variation of fluid flow as a function of radius within the tube and as a function of height between the circular plate and the disk. You can analyze the flow along streamlines (indicated by the arrows in Fig. 1) and along the surface of the disk in order to estimate the fluid flow and the fluid pressure.

Let V be the fluid velocity at the inlet to the tube and let P_0 be the static pressure outside the apparatus and within the fluid at point where the tube connects to the circular disk. *Notice:* at the center of the disk is a *stagnation point*, where the fluid flow vanishes. This is a point of *maximum* pressure. (Do you know why?)

In order to calculate the net force on the disk, you must derive an expression for the pressure on the top surface of the disk, $P(r)$. Follow a streamline along the centerline of the tube, to the center of the disk (*i.e.* the stagnation point), and then radially outward along the top surface of the disk. The flow velocity is purely radial, $U_r(r)$, as indicated in the figure.

(a) Explain why the center of the disk is the location of maximum pressure?

(b) Find an expression for the radial fluid flow along the disk, $U_r(r)$, as a function of radius. You will need expressions fro $r < a$ and for $r > a$.

(b) What is the pressure, $P(r)$, along the disk? Plot your solution graphically.

(c) Show that regions with "negative" relative pressure exist when $h < a/2$.

(d) Integrate your expression for $P(r)$ and compute the total *downward* force on the disk.

(e) Derive an expression for the condition when the net fluid force vanishes. Extra: also derive an expression for when the net levitation force exceeds the gravitation force for the disk (defining M to be the mass of the rigid disk.)

2

2 Problem 2 (20 Points)

(a) Describe the inviscid flow of fluid containing a single line vortex when the fluid is far from any boundaries.
(b) Describe the inviscid flow of fluid containing a single line vortex when the line vortex is near a flat boundary (and the direction of vorticity is tangent to the surface of the boundary.)

3

3 Problem 3 (40 Points)

In this problem, you are to (i) derive the dispersion relation for shallow water waves, and (ii) and describe the dynamical properties of the waves. Gravity is pointing downward, in the $-\hat{z}$-direction, and the wave is propagating in the $\hat{x}$-direction. You may do this in any way you wish, but you may also use the steps below to aid your derivation.

(a) Assume that the surface of the water varies as $\eta(x,t) = \eta_0 \cos(kx - \omega t)$, where $kH \ll 1$, and where H is the unperturbed depth of the water and $k = 2\pi/\lambda$, is the horizontal wavenumber. (The wavelength is λ.) The perturbed height of the fluid is $h(x,t) = H + \eta(x,t)$.

Furthermore, the fluid is incompressible, inviscid, and irrotational.

With these assumptions, the vertical velocity will take the approximate form of

$$U_z(x,z,t) \approx C_1 z \sin(kx - \omega t)$$

with C_1 a constant. Note that U_z vanishes at the bottom of the fluid layer at $z = 0$.

Prove this relationship and find C_1 in terms of η_0 by noting that the relationship between the surface displacement and the vertical velocity is

$$U_z(x, z = h, t) \equiv \frac{Dh}{Dt} = \frac{\partial \eta}{\partial t} + U_x \frac{\partial \eta}{\partial x}$$

3

Prove this relationship and find C_1 in terms of η_0 by noting that the relationship between the surface displacement and the vertical velocity is

$$U_z(x, z = h, t) \equiv \frac{Dh}{Dt} = \frac{\partial \eta}{\partial t} + U_x \frac{\partial \eta}{\partial x}$$

(b) By integrating the continuity relation (conservation of mass), find the horizontal velocity.

(c) Use Bernoulli's Principle for a time-dependent irrotational fluid and derive the dispersion relation for shallow water waves.

(d) Draw graphical pictures of the streamlines and pathlines of the flow.

(e) Find the time-averaged mechanical energy of the wave integrated over a wavelength.

(f) Find the time-averaged gravitational potential energy of the wave integrated over a wavelength.

(g) Viscosity causes the mechanical energy to (slowly) dissipate. Assuming the viscosity is very small, compute the rate of dissipation of wave mechanical energy due to fluid viscosity. (Ignore any dissipation caused by the boundaries.)

(h) Using your answers to Parts e, f, and g, estimate the rate by which the amplitude of the shallow water wave damps due to weak viscosity.

1

In 2008, a team of Japanese researchers created traffic jams in controlled experiments on a circular test track. While the mathematical theory behind these traffic jams was developed more than 15 years ago, this was the first time they were studied experimentally.

Traffic jams are like nonlinear waves that can be modeled using a continuity equation. (As someone who has been commuting in New York City for nearly 25 years, I understand how traffic jams occur: If one car brakes, the driver just behind must also, and a shockwave, or jam, can travel backwards through traffic as a periodic bunching cars.)

The Japanese researchers put 22 cars on a 230-meter single-lane circular track. See Fig. 1. They asked drivers to drive steadily at 30 kilometers per hour. At first, traffic moved freely. But small fluctuations soon appeared in the distances between cars, breaking down the free flow, until finally a cluster of several vehicles was forced to slow nearly to a complete stop.

This cluster spread backwards through the traffic like a wave. Every time a vehicle at the front of the cluster was able to escape at up to 40 km/h, another vehicle joined the back of the jam. The propagation speed of the jam was roughly −20 km/h and it travelled backwards through the ring of vehicles. This is about the same speed as similiar traffic "waves" observed on highways.

In this problem, you are to examine *linear* waves of a "traffic fluid". The fluid obeys a continuity equation:

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot \rho V(\rho)$$

(1 cont)

where the velocity, $V(\rho)$, of traffic flow depends upon the density of cars on the road, ρ, according to the relationship:

$$V(\rho) = V_{max}\,(1 - \rho/\rho_{max})$$

This means: with a very small number of cars on the road, $\rho \to 0$, traffic moves at the speed limit, $V \approx V_{max}$. But, as the number of cars per length of road approaches a maximum, $\rho \to \rho_{max}$, then $V \approx 0$.

In the experiment, traffic "waves" are seen to propagate *backwards* at -20 km h^{-1}. See Fig. 1.

Questions:

- What is the velocity of *linear* waves for this "traffic fluid"? Your answer should be a wave disturbance propagating at a speed, $C(\rho)$, that depends upon the traffic density, ρ. [*Hint*: Let $\rho(x,t) = \rho_0 + \tilde{\rho}(x,t)$, where $|\tilde{\rho}|/\rho_0 \ll 1$.]

- If the maximum speed is 40 km h^{-1}, and if traffic flows at 30 km h^{-1} at $\rho = 95$ cars/km, what must be the value of ρ_{max} such that "traffic waves" move backwards at -20 km h^{-1}?

(1 cont)

Traffic jams without bottlenecks—experimental evidence for the physical mechanism of the formation of a jam

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Figure 2. A snapshot of the experiment on a circular road. The circumference is 230 m, and the number of vehicles is 22. A 360-degree video camera is situated at the center of the circle for measurement.

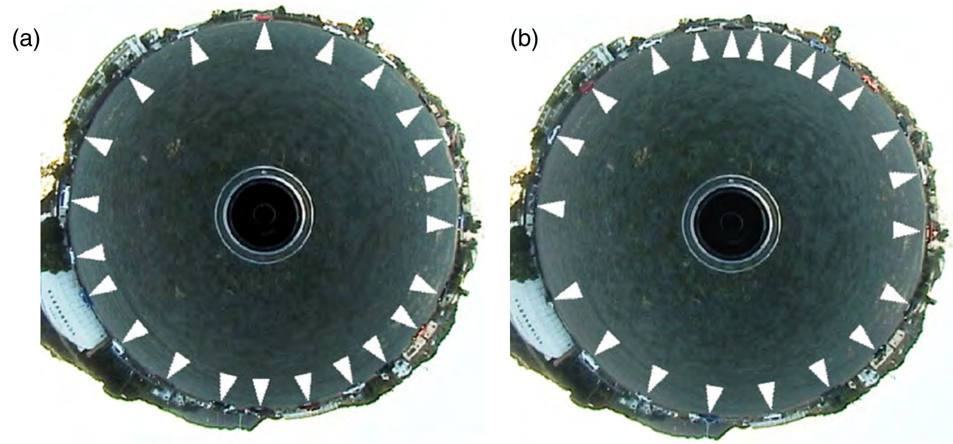

Figure 3. The vehicles move along the circle. The pictures are taken by the 360-degree video camera. We show the position of each vehicle by a triangle: (a) The snapshot at the initial stage shows that the vehicles move as the free flow. (b) The snapshot 3 min later shows that a jam has been formed. A jam consists of five vehicles, which is seen at the top of the circle at this moment. This picture corresponds to figure 2. (A jam is observed on the upper right corner of the circle.)

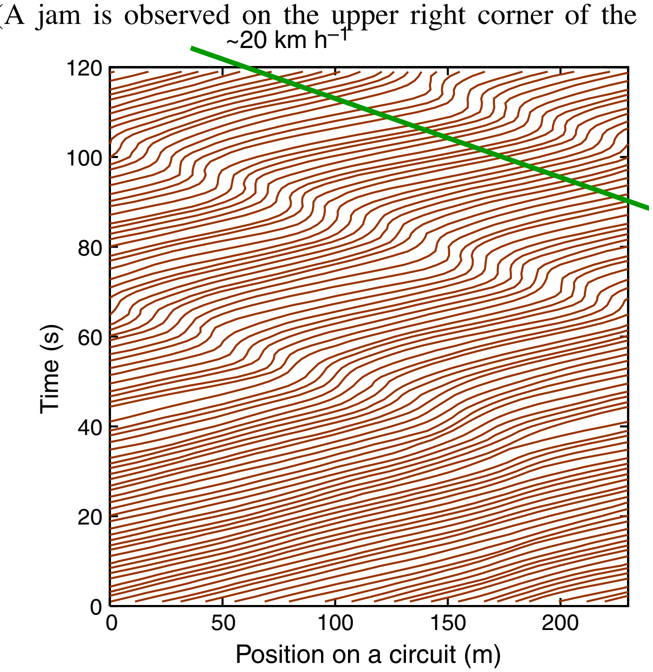

Figure 4. The traces of individual vehicles on the circular road (0 m-point is identical to 230 m-point) in the period of 2 min during the emergence of a jam. The data are captured from the movie taken by the 360-degree video camera (for example, pictures in figure 3). The jam cluster is clearly observed and it moves backward at a velocity of roughly 20 km h^{-1} and maintains its motion.

2

Consider two-dimensional, incompressible, irrotational Eulerian fluid flow. In this case, we can represent the flow with a complex velocity potential, $w(z) = \Phi(z) + I\Psi(z)$, where $z \equiv x + iy$. $\mathbf{U} = \nabla\Phi$ and $\mathbf{U} = -\hat{z} \times \nabla\Psi$. $\Phi(x, y)$ is the velocity potential, and $\Psi(x, y)$ is the streamfunction.

- Describe the flow, both mathematically and with a sketch, when $w(z) = z + 1/z$.
- Can this flow represent the flow around an object? If so, what is the relative pressure along the surface of the object and what is the total force exerted by the flow onto the object?

3

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  Furthermore, the fluid is incompressible, inviscid, and irrotational.

  With these assumptions, the vertical velocity will take the approximate form of

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  with C_1 a constant. Note that U_z vanishes at the bottom of the fluid layer at $z = 0$.

  Prove this relationship and find C_1 in terms of η_0 by noting that the relationship between the surface displacement and the vertical velocity is

  $$U_z(x, z = h, t) \equiv \frac{Dh}{Dt} = \frac{\partial \eta}{\partial t} + U_x \frac{\partial \eta}{\partial x}$$

- By integrating the continuity relation (conservation of mass), find the horizontal velocity.

- Use Bernoulli's Principle for a time-dependent irrotational fluid and derive the dispersion relation for shallow water waves.

- Draw graphical pictures of the streamlines and pathlines of the flow.

4

In this problem, you are to calculate the height of the jet of water, Δh, that extends above the upper surface of fluid in "Heron's Fountain." See Fig. 2.

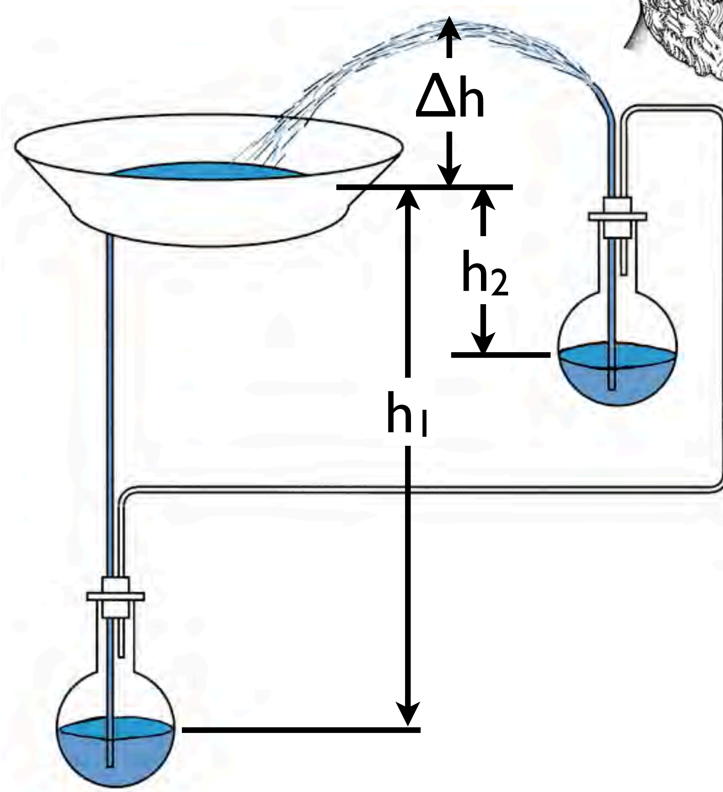
Heron's fountain consists of an upper fluid basin, and two sealed containers. Let h_1 be the distance from the upper fluid level to the fluid in the lower sealed contained. Let h_2 be the distance from the upper fluid level to the fluid level in the middle sealed container.

What is Δh?

Hero (or Heron) of Alexandria (c. 10–70 AD) was an ancient Greek mathematician and engineer who was active in his native city of Alexandria. He is considered the greatest experimenter of antiquity. Hero invented a steam-powered device called an *aeolipile* (hence sometimes called a "Hero engine"), a windwheel, pumps, syringe, and the "vending machine".

He also invented the self-powered "fountain", which is the subject of Problem #4.

Two examples of "Heron's Fountain" built in the mid 1800's



5

Assume the velocity in the laminar boundary layer of flat plate is given by

$$u_x(x, y) \approx U_0\, g(\eta)$$

where $\eta(x, y) \equiv y/\delta(x)$. y is the distance above the plate, and $\delta(x)$ is the boundary layer thickness that increases along the plate in the direction of fluid flow. $g(\eta)$ is a "similarity solution" of the Navier-Stokes equation within the laminar boundary layer.

- Estimate the thickness of boundary layer, $\delta(x)$, by deriving a condition for the existence of similarity-solution to the viscous boundary layer equations:

  $$\begin{aligned} u_x \frac{\partial u_x}{\partial x} + u_y \frac{\partial u_x}{\partial y} &= \nu \frac{\partial^2 u_x}{\partial y^2} \\ \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} &= 0 \end{aligned}$$

  Show all of your algebra. Your condition should have the form

  $$\delta \frac{d\delta}{dx} \frac{U_0}{\nu} = constant$$

  Defining a Reynolds's number along the plate as $Re_x \equiv xU_0/\nu$, how does $\delta(x)$ depend upon Re_x?

- Let $g(\eta) = \sin(\pi\eta/2)$. What is the drag, or "skin friction", or local wall shear stress, on the plate per unit surface area? Your answer should indicate the scaling of the wall stress as a function of U_0, ρ (the mass density), the viscosity, $\mu = \rho\nu$, and the length, x, along the plate.

- For a 1 m long plate immersed in kerosene flowing at 0.5 m/s with $\nu = 2.29 \times 10^{-6}$ m^2/s, estimate the thickness δ near the middle of the plate.

2010 Final

- Bernoulli's Principle
- Poiseuille Flow, viscous drag
- Dimensionless scaling
- Surface gravity waves and wave breaking
- Vorticity equation

(Score: 63 ± 16 from 100)

1 Problem 1 (20 points)

- Why do flags flap in the wind?
- A large, cylindrical bucket filled with water has a small, circular hole in the bottom. Approximately how long will it take for the bucket to empty one-half of the initial water?
- As the water exits from the hole, it forms a stream. The diameter of the stream *decreases* as it falls from the bucket. Estimate the width of the stream as a function of height below the bucket.

2 Problem 2 (20 points)

Consider viscous flow between two infinite parallel plates separated by a distance h. A pressure gradient, dP/dz, drives the flow.

- Calculate the fluid's velocity profile between the plates.
- What is the stress tensor, σ_{ik}, in the fluid?
- Show that there will be a drag force exerted on each plate per unit area in the direction of the flow equal to

$$F_z = -\frac{1}{2}h\frac{dP}{dz}$$

- If one of the plates was moving in the direction of the flow at constant speed, U_0, how does the drag force on each plate change?

3 Problem 3 (20 points)

- Typically, human blood is three to four times more viscous than water.

  Suppose you wanted to understand blood flow in a tube but, for various reasons, you wanted to use the flow of air in a tube as a scale model of the blood flow in an artery.

  How would you go about designing the scale model, and how would you choose the pressures and flow rates in the experimental model?

- The Prandtl number is the ratio of the kinematic viscosity, ν, to the thermal diffusivity, κ.

  Show that the flow in two different fluids behave the same way provided that they have the same Reynolds numbers and the same Prandtl numbers.

4 Problem 4 (20 points)

- When ocean waves run from deep to shallow water, the wave amplitude increases. This is called "wave shoaling".

  The wave frequency and flux of wave energy density are conserved as they waves move to the shore.

  Compute the relationship between the depth of the water and the oncoming wave amplitude during wave shoaling.

- Wave breaking generally occurs when a wave's crest become comparable to the wave's wavelength. Describe the conditions when you would expect wave breaking to occur for incoming water waves with 10 foot (3 m) wavelength.

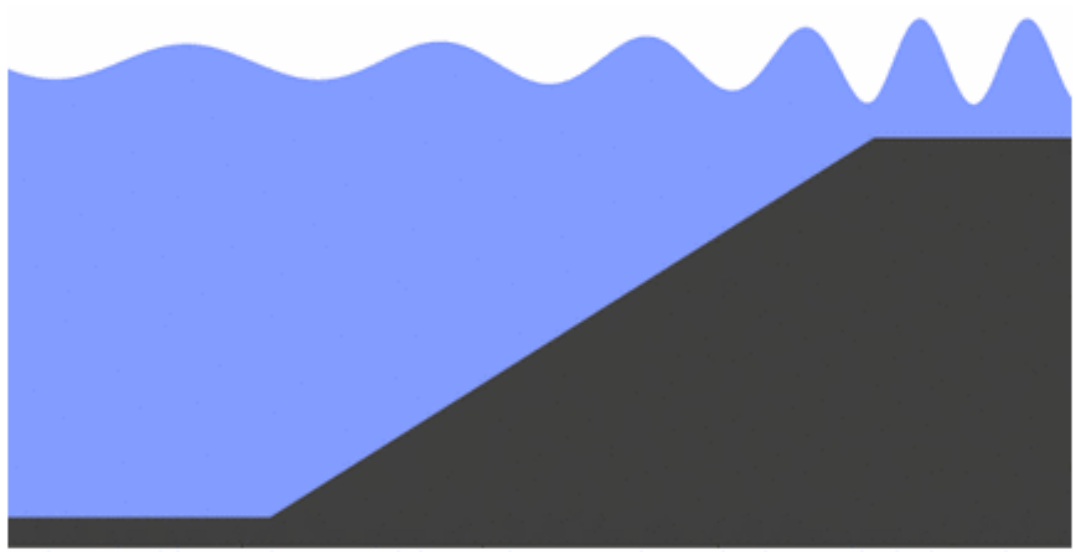


Figure 1: For Question 4: A schematic of incoming water waves at a beach.

5 Problem 5 (20 points)

Consider fluid flow in a two-dimensional plane.

With the vorticity defined as $\vec{\Omega} \equiv \nabla \times \mathbf{U}$, show that the Navier-Stokes equation and the equation for continuity imply

$$\frac{D\vec{\Omega}}{Dt} = \nu \nabla^2 \vec{\Omega}$$

2011 Final

- Stokes' Flow
- Poiseuille Flow, viscous drag
- Surface gravity waves and wave breaking
- 2D Potential Flow

(Score: 61 ± 13 from 100)

1 Problem 1 (25 points)

In 1851, George Stokes derived an approximate expression for the viscous drag force on a sphere. Stoke's law is

$$F_d = 3\pi\mu dU$$

where μ is the fluid dynamic viscosity, d is the diameter of the sphere, and U is the sphere's velocity through the fluid.

- Consider eight small water droplets falling through air at 0.1 m/s. What happens to the water velocity when the eight droplets combine into a single drop?
- What are the conditions required for the validity of Stoke's law?
- As the velocity or diameter of the sphere increases sufficiently, the settling velocity (or terminal velocity) is *slower* than the velocity predicted by Stoke's law. Explain.

2 Problem 2 (25 points)

Consider steady viscous flow between two infinite parallel plates separated by a distance h. A pressure gradient, dP/dz, drives the flow.

- Calculate the fluid's velocity profile between the plates.
- What is the stress tensor, σ_{ik}, in the fluid?
- Make an *estimate* of the time required for the flow between the two plates to reach steady conditions, if, at $t = 0$, the flow was everywhere zero and the pressure gradient was applied instantly at $t = 0^+$?

3 Problem 3 (25 points)

Consider a fluid obeying the Boussinesq equations, with both viscosity and thermal diffusivity neglected:

$$\frac{d\rho}{dt} = 0, \tag{1}$$

$$\rho_0 \frac{d\mathbf{u}}{dt} = -\nabla p - \rho g \hat{z}, \tag{2}$$

where ρ_0 is a reference density, $\hat{z}$ is the unit vector in the vertical (z) direction, and as usual

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla$$

is the total derivative.

A motionless basic state exists in which the density has a constant gradient in the z direction only, $\partial \overline{\rho}/\partial z =$ constant, $\partial \overline{\rho}/\partial y = \partial \overline{\rho}/\partial x = 0$, where the overbar indicates the basic state.

1. Find the pressure field in the basic state.

2. Linearize the equations of motion to find equations for small amplitude perturbations about the basic state. To simplify the analysis, you may assume that the perturbations are independent of y and have no velocity component in the y direction.

3. Assume that the pressure gradients associated with the perturbations can be neglected. Then show that the system is unstable if the basic state density increases with z, and neutral, with oscillatory solutions which neither grow nor decay, if the density decreases with z. For the latter case, find the frequency of oscillations.

4 Problem 4 (25 points)

Consider two-dimensional, incompressible, irrotational Eulerian fluid flow. In this case, we can represent the flow with a complex velocity potential, $w(z) = \Phi(z) + i\Psi(z)$, where $z \equiv x + iy$. $\mathbf{U} = \nabla\Phi$ and $\mathbf{U} = -\hat{z} \times \nabla\Psi$. $\Phi(x, y)$ is the velocity potential, and $\Psi(x, y)$ is the streamfunction.

- Describe the flow, both mathematically and with a sketch, when $w(z) = z^2$.
- Can this flow represent the flow past an object?
- If so, write an expression for the total force applied to the object by the flow.

2012 Final Exam

- Time-dependent viscous flow
- Sound waves
- Steady viscous flow
- Bernoulli's principle

(Score: 41 ± 34 from 100)

1 Problem 1 (25 points)

Describe the linear oscillations of a viscous fluid above a large plate oscillating back-and-forth.

[*Hint:* This is one of the few exact solutions to the Navier-Stokes equation, first solved by George Stokes.]

2 Problem 2 (25 points)

Derive the linear dispersion relation for sound waves in a homogeneous, boundless gas.

You may assume the pressure and density of the gas are related by an equation of state

$$\frac{D}{Dt}\left(\frac{P}{\rho^{\gamma}}\right) = 0$$

where γ is an adiabatic constant.

3 Problem 3 (25 points)

What is the total flow per unit width of a viscous fluid down an inclined plane? The thickness of the fluid layer is h, and the plane is inclined by an angle θ with respect to horizontal.

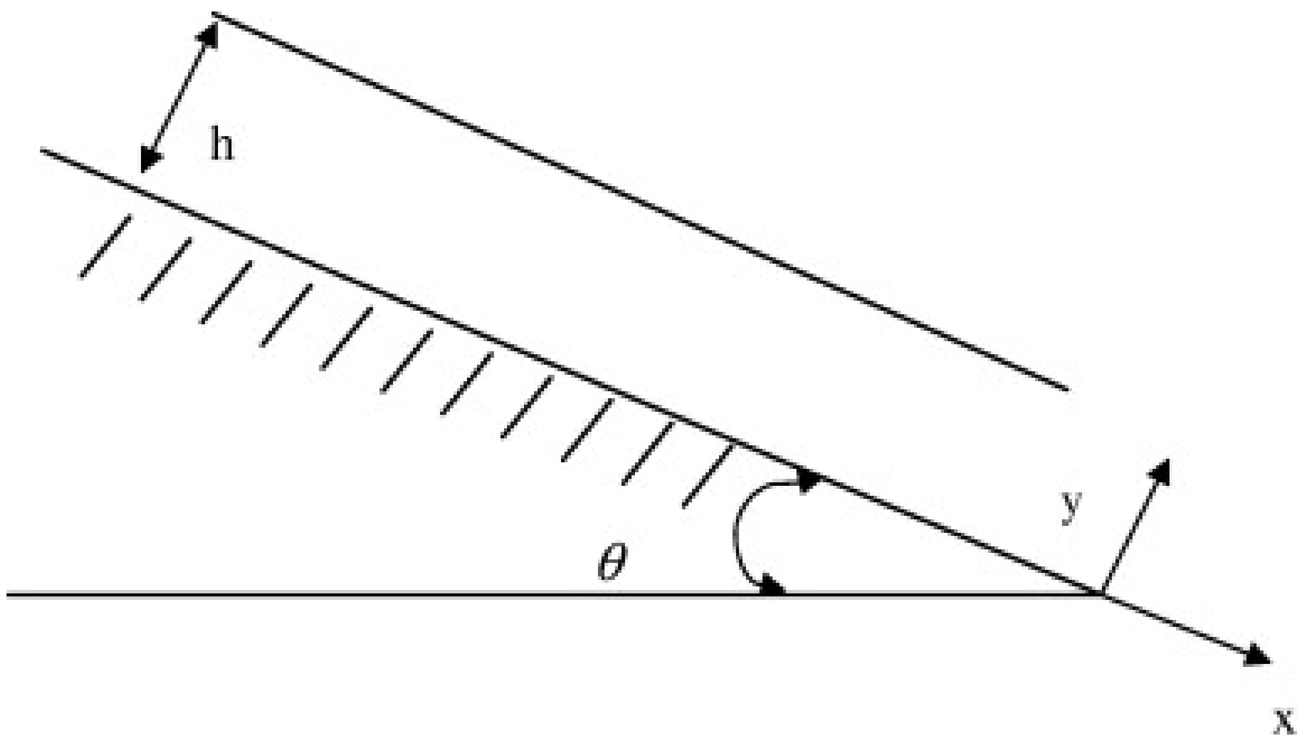


Figure 1: For Question 3: Steady flow down an inclined plane.

4 Problem 4 (25 points)

In the problem, you are to analyze the inviscid steady laminar flow of fluid in a wide channel of depth h. As the fluid moves through the channel, it encounters a small gradual "bump" of height Δh. The total height at this point in the channel can either *increase* or *decrease* depending upon the Froude number of the fluid, $Fr \equiv U/\sqrt{gh}$, and the relative height of the bump, $\delta \equiv \Delta h/h$.

Questions:

- Derive an equation for the *relative* thickness of the fluid layer over the bump relative to the channel fluid depth, h, in terms of the dimensionless quantities, δ and Fr.
- Show that there can be two possible solutions to the channel thickness over the bump.
- Show that when the bump gets too large, then there is no solution possible for steady, inviscid, continuous flow over the bump.

[*Hint:* You do not need to derive exact answers for your analysis. Instead, try to construct a graphical proof.]

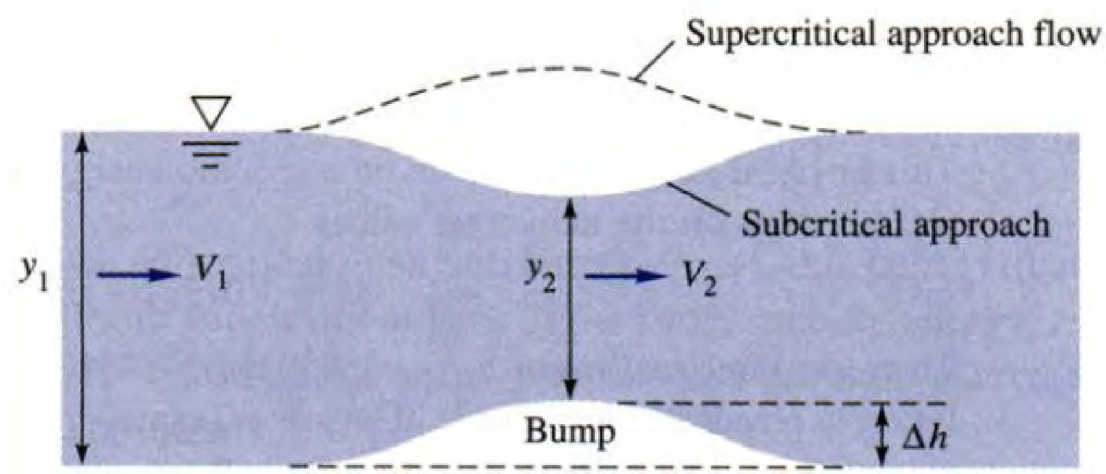


Figure 2: For Question 4: Steady flow over a small bump.

2013 Final Exam

- Viscous drag (and Linear stability)
- Viscous boundary layers
- Poiseuille Flow
- Deep water waves (and viscosity!)

(Score: 52 ± 14 from 100)

1 Problem 1 (25 points)

Consider an annulus of *incompressible* fluid between two concentric cylinders, which are able to rotate at different speeds. Let $R_1 < R_2$, and let Ω_1 and Ω_2 be the angular rates of rotation for each cylinder.

- What is the radial profile of the equilibrium fluid rotation speed, $u_\theta(r)$?
- What is the torque (from viscous drag) on the outer cylinder per unit axial length?
- What is the torque on the inner cylinder per unit axial length?
- Ignoring the effects of viscous dissipation, under what conditions will the fluid be unstable, or stable? This is the so-called "Rayleigh's Inviscid Criterion" for the centrifugal instability.

  Express your stability condition in terms of R_1, R_2, Ω_1, and Ω_2.

2 Problem 2 (25 points)

Consider the viscous flow (*i.e.* Poiseuille Flow) between two fixed plates located at $y = \pm h/2$. The fluid has density, ρ, and viscosity, μ.

- When $-\nabla P = A$ (the pressure gradient is a constant), find the velocity profile between the two plates, $u_x(y)$.

- When $-\nabla P = B\cos(\omega t)$, the pressure gradient oscillates in time. Show that, in this case, the fluid velocity has the form:

$$u_x(y,t) = \Re\left\{\frac{i}{\omega}\frac{B}{\rho}\,e^{-i\omega t}\left[1 - \frac{\cos\beta y}{\cos\beta h/2}\right]\right\}$$

  The constant, β, is a complex number that depends upon ω. What is β?

- For oscillating flow, the viscous scale length $\delta \equiv \sqrt{2\mu/\omega\rho} = \sqrt{2\nu/\omega}$. When the pressure oscillates slowly, $\delta \gg h$. When the pressure oscillates quickly, $\delta \ll h$.

  Describe what the flow profile looks like for the two limits: when the pressure oscillates slowly and when it oscillates quickly. Discuss.

3 Problem 3 (25 points)

Consider the equal separation of a viscous fluid from one to two circular pipes. See figure below.

- Show that the axisymmetric viscous flow in a circular pipe with constant pressure gradient is described with an axial flow, which is fastest in the center of the pipe, and has a radial profile equal to $u_z(r) = (-dP/dz)\,(R^2 - r^2)/4\mu$.

- You are to choose the radii of the pipes so that the viscous stress on the walls of the smaller tubes equals the viscous stress on the wall of the large tube.

  What is the ratio of R_2/R_1?

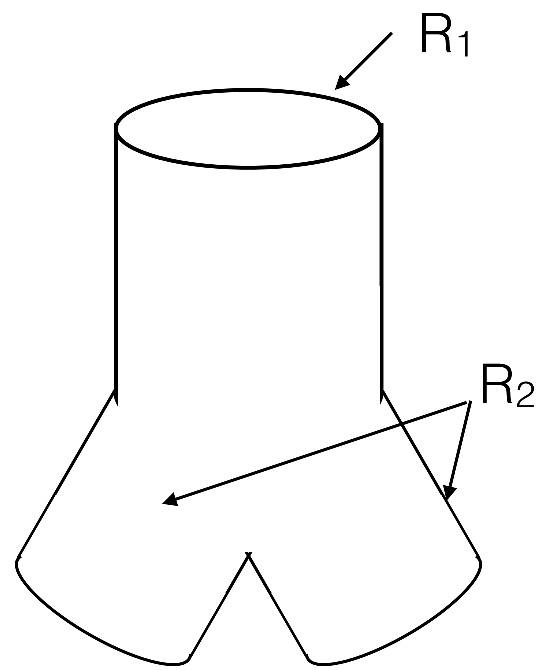


Figure 1: For Question 3: Division of flow from one to two pipes.

4 Problem 4 (25 points)

In the problem, you are to examine the effects of viscous on surface gravity waves.

Assume "deep water" waves where the fluid perturbations vanish as $z \to -\infty$. Take the vertical displacement of the top fluid surface to vary as $\eta = \eta_0 \cos(kx - \omega t)$. Assume the gravity and fluid density are uniform, and $\mathbf{g} = -g\hat{z}$ and the fluid incompressible.

- Derive an expression for the fluid's velocity potential, $\mathbf{U} = \nabla\phi(x, z, t)$, in the limit when the viscosity can be neglected. (This is when $k^2\nu \ll \omega$.)

  Show that the vertical variation of the velocity perturbations decrease exponentially from the surface as e^{kz}, for $z < 0$.

- Estimate the damping rate of the surface wave (for the case of very small viscosity) by taking the ratio of the rate of dissipation per wavelength to the wave energy per wavelength. Express your answer in terms of the viscous scale length, $\delta \equiv \sqrt{2\nu/\omega}$.

(continued...)

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- When the viscous is large, the vertical profile of the velocity perturbations change. Write down the linear equations of motion for surface waves for the case when viscosity can not be neglected.

- *Optional; Extra-Credit.* These equations can be solved if you assume that the velocity perturbations are represented by the weighted sum of two vertical variations.

  Assume that the complex phasor for the horizontal velocity can be expressed as $u_x(x, z, t) = \exp(-i\omega t + ikx)\,(Ae^{kz} + Be^{mz})$, where A and B are complex constants and where m is related to k and the viscous scale length, $\delta = \sqrt{2\nu/\omega}$, where $m^2 = k^2 - 2i/\delta^2$. With this assumption, a self-consistent solution to the Navier-Stokes Equation is possible with the pressure given as

  $$P/\rho = -gz + A\frac{\omega}{k}\, e^{-i\omega t + ikx}\, e^{kz}\ (\text{for } z < 0).$$

  With u_z determined by $\nabla \cdot \mathbf{u} = 0$, these dynamical forms satisfy the viscous boundary conditions at the free surface ($z = \eta \simeq 0$) are the absence of stress. This means the shear and normal vertical stresses must vanish at the surface.

  By eliminating the complex amlitudes, A and B, the dispersion relation for gravity waves on the surface of a viscous fluid can be derived.

  What is this dispersion relation? Discuss the solution at as δ becomes small compared with $1/k$, the velocity component proportional to B becomes unimportant.